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Research Article

Geometric Lattice Structure of Covering-Based Rough Sets through Matroids

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Covering-based rough set theory is a useful tool to deal with inexact, uncertain, or vague knowledge in information systems. Geometric lattice has been widely used in diverse fields, especially search algorithm design, which plays an important role in covering reductions. In this paper, we construct three geometric lattice structures of covering-based rough sets through matroids and study the relationship among them. First, a geometric lattice structure of covering-based rough sets is established through the transversal matroid induced by a covering. Then its characteristics, such as atoms, modular elements, and modular pairs, are studied. We also construct a one-to-one correspondence between this type of geometric lattices and transversal matroids in the context of covering-based rough sets. Second, we present three sufficient and necessary conditions for two types of covering upper approximation operators to be closure operators of matroids. We also represent two types of matroids through closure axioms and then obtain two geometric lattice structures of covering-based rough sets. Third, we study the relationship among these three geometric lattice structures. Some core concepts such as reducible elements in covering-based rough sets are investigated with geometric lattices. In a word, this work points out an interesting view, namely, geometric lattice, to study covering-based rough sets.

1. Introduction

Rough set theory [1] was proposed by Pawlak to deal with granularity in information systems. It is based on equivalence relations. However, the equivalence relation is rather strict, hence the applications of the classical rough set theory are quite limited. For this reason, rough set theory has been extended to generalized rough set theory based on tolerance relation [2], similarity relation [3], and arbitrary binary relation [4–8]. Through extending a partition to a covering, we generalize rough set theory to covering-based rough set theory [9–11]. Because of its high efficiency in many complicated problems such as attribute reduction and rule learning in incomplete information/decision, covering-based rough set theory has been attracting increasing research interest [12, 13].

Lattice is suggested by the form of the Hasse diagram depicting it. In mathematics, a lattice is a partially ordered set in which any two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet). They encode the algebraic behavior of the entailment relation and such basic logical connectives as "and" (conjunction) and "or" (disjunction), which results in adequate algebraic semantics for a variety of logical systems. Lattice, especially geometric lattice, is one of the most important algebraic structures and is used extensively in both theoretical and applicable fields, such as data analysis, formal concept analysis [14–16], and domain theory [17].

Matroid theory [18, 19] borrows extensively from linear algebra theory and graph theory. There are dozens of equivalent ways to define a matroid. Significant definitions of a matroid include those in terms of independent sets, bases, circuits, closed sets or flats and rank functions, which provide well-established platforms to connect with other theories. In applications, matroids have been widely used in many fields such as combinatorial optimization, network flows, and algorithm design, especially greedy algorithm design [20, 21]. Some works on the connection between rough sets and matroids have been conducted [22–25].

In this paper, we pay attention to geometric lattice structures of covering based-rough sets through matroids. First, a geometric lattice structure in covering-based rough sets is generated by the transversal matroid induced by a covering. Moreover, we study the characteristics of the geometric lattice structure, such as atoms, modular elements, and modular pairs. We also point out a one-to-one correspondence between this type of geometric lattices and transversal matroids in the context of covering-based rough sets. Second, generally, covering upper approximation operators are not necessarily closure operators of matroids. Then we present three sufficient and necessary conditions for two types of covering upper approximation operators to be closure operators of matroids and exhibit representations of corresponding special matroids. We study the properties of these matroids and their closed-set lattices which are also geometric lattices. Third, we study the relationship among these three geometric lattices through corresponding matroids. Furthermore, some core concepts such as reducible and immured elements in covering-based rough sets are studied by geometric lattices.

The rest of this paper is organized as follows. In Section 2, we recall some fundamental concepts related to covering-based rough sets, lattices, and matroids. Section 3 establishes a geometric lattice structure of covering-based rough sets through the transversal matroid induced by a covering. In Section 4, we present two geometric lattice structures of covering-based rough sets through two types of upper approximation operators. Section 5 studies the relationship among these three geometric lattice structures. This paper is concluded and further work is pointed out in Section 6.

2. Preliminaries

In this section, we review some basic concepts of matroids, lattices, and covering-based rough sets.

2.1. Matroids

Matroid theory borrows extensively from the terminology of linear algebra theory and graph theory, largely because it is the abstraction of various notions of central importance in these fields, such as independent set, base, and rank function. We introduce the concept of matroid, first.

Definition 2.1 (Matroid [19]). A matroid is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions.

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$, where $|X|$ denotes the cardinality of X.

Let $M = M(E, \mathcal{I})$ be a matroid. The members of $\mathcal{I}$ are the independent sets of M. A set in $\mathcal{I}$ is maximal, in the sense of inclusion, is called a base of the matroid M. If $A \notin \mathcal{I}$, A is called a dependent set of the matroid M. In the sense of inclusion, a minimal dependent subset of E is called a circuit of the matroid M. If $\{a\}$ is a circuit, we call $\{a\}$ a loop. Moreover, if $\{a, b\}$ is a circuit, then a and b are said to be parallel. A matroid is called a simple matroid if it has no loops and no parallel elements. The rank function of a matroid is a function $r_M : 2^E \to N$ defined by $r_M(X) = \max\{|I| : I \subseteq X, I \in \mathcal{I}\}(X \subseteq E)$. For each $X \subseteq E$, we say $\mathrm{cl}_M(X) = \{a \in E : r_M(X) = r_M(X \cup \{a\})\}$ is the closure of X in M. When there is no confusion, we use the symbol $\mathrm{cl}(X)$ for short. X is called a closure set if $\mathrm{cl}(X) = X$.

The rank function of a matroid, directly analogous to a similar theorem of linear algebra, has the following proposition.

Proposition 2.2 (Rank axiom [19]). *Let E be a set. A function $r_M : 2^E \to N$ is the rank function of a matroid on E if and only if it satisfies the following conditions.*

(R1) *For all $X \in 2^E$, $0 \leq r_M(X) \leq |X|$.*

(R2) *If $X \subseteq Y \subseteq E$, then $r_M(X) \leq r_M(Y)$.*

(R3) *If $X, Y \subseteq E$, then $r_M(X \cup Y) + r_M(X \cap Y) \leq r_M(X) + r_M(Y)$.*

The following proposition is the closure axiom of a matroid. It means that a operator satisfies the following four conditions if and only if it is the closure operator of a matroid.

Proposition 2.3 (Closure axiom [19]). *Let E be a set. A function $cl_M : 2^E \to 2^E$ is the closure operator of a matroid M on E if and only if it satisfies the following conditions.*

(1) *If $X \subseteq E$, then $X \subseteq cl_M(X)$.*

(2) *If $X \subseteq Y \subseteq E$, then $cl_M(X) \subseteq cl_M(Y)$.*

(3) *If $X \subseteq E$, $cl_M(cl_M(X)) = cl_M(X)$.*

(4) *If $X \subseteq E, x \in E$ and $y \in cl_M(X \cup \{x\}) - cl_M(X)$, then $x \in cl_M(X \cup \{y\})$.*

Transversal theory is a branch of a matroid theory. It shows how to induce a matroid, namely, transversal matroid, from a family of subsets of a set. Hence, the transversal matroid establishes a bridge between collections of subsets of a set and matroids.

Definition 2.4 (Transversal [19]). Let S be a nonempty finite set and $J = \{1, 2, \ldots, m\}$. $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ denotes a family of subsets of S. A transversal or system of distinct representatives of $\{F_1, F_2, \ldots, F_m\}$ is a subset $\{e_1, e_2, \ldots, e_m\}$ of S such that $e_i \in F_i$ for all i in J. If for some subset K of J, X is a transversal of $\{F_i : i \in K\}$, then X is said to be a partial transversal of $\{F_1, F_2, \ldots, F_m\}$.

Example 2.5. Let $S = \{1,2,3,4\}$, $F_1 = \{2,3\}$, $F_2 = \{4\}$ and $F_3 = \{2,4\}$. For $\mathcal{F} = \{F_1, F_2, F_3\}$, $T = \{2,3,4\}$ is a transversal of $\mathcal{F}$ because $2 \in F_3$, $3 \in F_1$ and $4 \in F_2$. $T' = \{2,4\}$ is a partial transversal of $\mathcal{F}$ because there exists a subset of $\mathcal{F}$, that is, $\{K_1, K_2\}$, such that T' is a transversal of it.

The following proposition shows what kind of matroids are transversal matroid.

Proposition 2.6 (Transversal matroid [19]). *Let $\mathcal{F} = \{F_i : i \in J\}$ be a family subsets of E. $M(\mathcal{F}) = (E, \mathcal{I}(\mathcal{F}))$ is a matroid, where $\mathcal{I}(\mathcal{F})$ is the family of all partial transversals of $\mathcal{F}$. One calls $M(\mathcal{F}) = (E, \mathcal{I}(\mathcal{F}))$ the transversal matroid induced by $\mathcal{F}$.*

2.2. Lattices

Let $(P, \leq)$ be an ordered set and $a, b \in P$. We say that a is covered by b (or b covers a) if $a < b$ and there is no element c in P with $a < c < b$. A chain in P from x_0 to x_n is a subset $\{x_0, x_1, \ldots, x_n\}$ of P such that $x_0 < x_1 < \cdots < x_n$. The length of such a chain is n, and the chain is maximal if x_i covers x_{i-1} for all $i \in \{1, 2, \ldots, n\}$. If, for every pair $\{a, b\}$ of elements of P with $a < b$, all maximal chains from a to b have the same length, then P is said to satisfy the Jordan-Dedekind chain condition. The height $h_P(y)$ of an element y of P is the maximum length of a chain from 0 to y. A poset $(\mathcal{L}, \leq)$ is a lattice if $a \vee b$ and $a \wedge b$ exist for all $a, b \in \mathcal{L}$. Suppose $\mathcal{L}$ is a lattice with zero element 0. If a covers 0, then $a \in \mathcal{L}$ is called an atom of $\mathcal{L}$. Moreover, the atoms of $\mathcal{L}$ are precisely the elements of height one. It is not difficult to check that every finite lattice has a zero and the one. A finite lattice $\mathcal{L}$ is called semimodular if it satisfies the Jordan-Dedekind chain condition and for every pair $\{x, y\}$ of elements of $\mathcal{L}$, the equality $h_{\mathcal{L}}(x) + h_{\mathcal{L}}(y) \geq h_{\mathcal{L}}(x \vee y) + h_{\mathcal{L}}(x \wedge y)$ holds. A geometric lattice is a finite semimodular lattice in which every element is a join of atoms.

Next, we introduce the modular element and modular pair which are important concepts of lattices.

Definition 2.7 (see [17]). Let $\mathcal{L}$ be a lattice and $a, b \in \mathcal{L}$.

(ME) For all $x, z \in \mathcal{L}$, $x \geq z$ implies $x \wedge (a \vee z) = (x \wedge a) \vee z$, then a is called a modular element of $\mathcal{L}$.

(MP) For all $z \in \mathcal{L}$, $b \geq z$ implies $b \wedge (a \vee z) = (b \wedge a) \vee z$, then (a, b) is called a modular pair of $\mathcal{L}$.

As we know, if a is a modular element of $\mathcal{L}$, then (a, b) is a modular pair of $\mathcal{L}$ for all $b \in \mathcal{L}$, which roots in an important result of lattices. For a semimodular lattice, modular pair has close relation with height function.

Lemma 2.8 (see [17]). *Let $\mathcal{L}$ be a semimodular lattice, then (a, b) is a modular pair if and only if $h_{\mathcal{L}}(a \vee b) + h_{\mathcal{L}}(a \wedge b) = h_{\mathcal{L}}(a) + h_{\mathcal{L}}(b)$ for all $a, b \in \mathcal{L}$.*

2.3. Closed-Set Lattice of a Matroid

If M is a matroid and $\mathcal{L}(M)$ denotes the set of all closed sets of M ordered by inclusion, then $(\mathcal{L}(M), \subseteq)$ is a lattice. In addition to that, the operations join and meet of it are, respectively, defined as $X \wedge Y = X \cap Y$ and $X \vee Y = \mathrm{cl}_M(X \cup Y)$ for all $X, Y \in \mathcal{L}(M)$. The zero of $\mathcal{L}(M)$

is $\mathrm{cl}_M(\emptyset)$, while the one is E. The following lemma gives another definition of a geometric lattice from the viewpoint of matroid. In fact, the set of all closed sets of a matroid ordered by inclusion is a geometric lattice.

Lemma 2.9 (see [19]). *A lattice $\mathcal{L}$ is geometric if and only if it is the lattice of closed sets of a matroid.*

The following lemma establishes the relation between the rank function of a matroid and the height function of the closed-set lattice of the matroid.

Lemma 2.10 (see [19]). *Let M be a matroid. $h_{\mathcal{L}(M)}(X) = r_M(X)$ for all $X \in \mathcal{L}(M)$.*

2.4. Covering-Based Rough Sets

In this subsection, we introduce some concepts of covering-based rough sets used in this paper.

Definition 2.11 (Covering and partition). Let E be a universe of discourse, $\mathcal{C}$ a family of subsets of E, and none of subsets in $\mathcal{C}$ be empty. If $\cup\mathcal{C} = E$, then $\mathcal{C}$ is called a covering of E. Any element of $\mathcal{C}$ is called a covering block. If $\mathcal{P}$ is a covering of E and it is a family of pairwise disjoint subsets of E, then P is called a partition of E.

It is clear that a partition of E is certainly a covering of E, so the concept of a covering is an extension of the concept of a partition.

Let E be a finite set and R be an equivalent relation on E. R will generate a partition $E/R = \{Y_1, Y_2, \ldots, Y_m\}$ of E, where $Y_1, Y_2, \ldots, Y_m$ are the equivalence classes generated by R. For all $X \subseteq E$, the lower and upper approximations of X, are, respectively, defined as follows:

$$R_*(X) = \bigcup\left\{Y_i \in \frac{E}{R} : Y_i \subseteq X\right\},$$
$$R^*(X) = \bigcup\left\{Y_i \in \frac{E}{R} : Y_i \bigcap X \neq \emptyset\right\}. \tag{2.1}$$

Next, we introduce certain important concepts of covering-based rough sets, such as minimal description, indiscernible neighborhood, neighborhood, reducible element, and approximation operators.

Definition 2.12 (Minimal description [26]). Let $\mathcal{C}$ be a covering of E and $x \in E$:

$$Md_{\mathcal{C}}(x) = \{K \in \mathcal{C} : x \in K \wedge (\forall S \in \mathcal{C} \wedge x \in S \wedge S \subseteq K \Longrightarrow K = S)\}, \tag{2.2}$$

is called the minimal description of x. When the covering is clear, we omit the lowercase $\mathcal{C}$ in the minimal description.

Definition 2.13 (Indiscernible neighborhood and neighborhood [27, 28]). Let $(E, \mathcal{C})$ be a covering approximation space and $x \in E$.

$\cup\{K : x \in K \in \mathcal{C}\}$ is called the indiscernible neighborhood of x and denoted as $I_{\mathcal{C}}(x)$.

$\cap\{K : x \in K \in \mathcal{C}\}$ is called the neighborhood of x and denoted as $N_{\mathcal{C}}(x)$. When the covering is clear, we omit the lowercase $\mathcal{C}$.

Definition 2.14 (A reducible covering [29]). Let $\mathcal{C}$ be a covering of a domain E and $K \in \mathcal{C}$. If K is a union of some sets in $\mathcal{C} - \{K\}$, we say K is a reducible element in $\mathcal{C}$; otherwise K is an irreducible element in $\mathcal{C}$. If every element in $\mathcal{C}$ is irreducible, we say $\mathcal{C}$ is irreducible; otherwise $\mathcal{C}$ is reducible.

Definition 2.15 (Reduct [29]). For a covering $\mathcal{C}$ of a universe E, when we remove all reducible elements from $\mathcal{C}$, the set of remaining elements is still a covering of E, and this new irreducible covering has not reducible element. We call thus new covering a *reduct* of $\mathcal{C}$ and it is denoted by $reduct(\mathcal{C})$.

Definition 2.16 (Immured element [27]). Let $\mathcal{C}$ be a covering of E and K an element of $\mathcal{C}$. If there exists another element K' of $\mathcal{C}$ such that $K \subset K'$, we say that K is an immured element of covering $\mathcal{C}$.

Definition 2.17 (Exclusion [27]). Let $\mathcal{C}$ be a covering of E. When we remove all immured elements from $\mathcal{C}$, the set of all remaining elements is still a covering of E, and this new covering has no immured element. We called this new covering an *exclusion* of $\mathcal{C}$, and it is denoted by $exclusion(\mathcal{C})$.

The second type of covering rough set model was first studied by Pomykała in [30]. While the sixth type of covering-based upper approximation operator was first defined in [31].

Definition 2.18. Let $\mathcal{C}$ be a covering of E. The covering upper approximation operators $\mathrm{SH}, \mathrm{XH} : 2^E \to 2^E$ are defined as follows: For all $X \in 2^E$,

$$\mathrm{SH}(X) = \cup\{K \in \mathcal{C} : K \cap X \neq \emptyset\} = \cup\{I(x) : x \in X\},$$
$$\mathrm{XH}(X) = \{x : N(x) \cap X \neq \emptyset\}.$$

$\mathrm{SH}_{\mathcal{C}}$ and $\mathrm{XH}_{\mathcal{C}}$ are called the second and the sixth covering upper approximation operators with respect to the covering $\mathcal{C}$, respectively. When there is no confusion, we omit $\mathcal{C}$ at the lowercase.

3. A Geometric Lattice Structure of Covering-Based Rough Sets through Transversal Matroid

As we know, if M is a matroid and $\mathcal{L}(M)$ denotes the set of all closed sets of M ordered by inclusion, then $\mathcal{L}(M)$ is a geometric lattice. In this section, we study the properties such as atoms, modular elements and modular pairs of this type of geometric lattice through transversal matroid induced by a covering. We also study the structure of matroid induced by the geometric lattice. It is interesting to find that there is a one-to-one correspondence between this type of geometric lattices and transversal matroids in the context of covering-based rough sets.

Let E be a nonempty finite set and $\mathcal{C}$ a covering of E. As shown in Proposition 2.6, $M(\mathcal{C}) = (E, \mathcal{I}(\mathcal{C}))$ is the transversal matroid induced by covering $\mathcal{C}$. $\mathcal{L}(M(\mathcal{C}))$ is the set of all closed sets of $M(\mathcal{C})$. Especially, $\mathcal{L}(M(\mathcal{P}))$ is the set of all closed sets of the transversal matroid induced by partition $\mathcal{P}$. Based on Lemma 2.9, we know $\mathcal{L}(M(\mathbf{C}))$ and $\mathcal{L}(M(\mathcal{P}))$ are geometric lattices.

The theorem below connects a covering $\mathbf{C}$ with $\mathrm{cl}_{M(\mathbf{C})}(\emptyset)$. In fact, $\emptyset \in \mathcal{L}(M(\mathcal{F}))$ if and only if $\mathcal{F}$ is a covering.

Theorem 3.1. *Let E be a set and $M(\mathcal{F})$ a transversal matroid induced by a family of $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ subsets of E, where $F_i \neq \emptyset$ $(1 \leq i \leq m)$. $cl_{\mathcal{F}}(\emptyset) = \emptyset$ if and only if $\mathcal{F}$ is a covering of E.*

Proof. "$\Leftarrow$": According to the definition of transversal matroid, any partial transversal is an independent set of transversal matroid. Since $\mathcal{F}$ is a covering, any single-point set is an independent set. Based on the definition of closure operator of a matroid, we have $\mathrm{cl}_{M(\mathcal{F})}(\emptyset) = \emptyset$.

"$\Rightarrow$": Since $\mathrm{cl}_{M(\mathcal{F})}(\emptyset) = \emptyset$, any single-point set is an independent set, that is, for all $x \in E$, there exists $i_x \in \{1, 2, \ldots, m\}$ such that $x \in F_{i_x} \subseteq E$. Hence, $E = \bigcup_{x\in E}\{x\} \subseteq \bigcup_{x\in E} F_{i_x} \subseteq \bigcup_{i=1}^{m} F_i \subseteq E$. Thus $\bigcup_{i=1}^{m} F_i = E$. For all $i \in \{1, 2, \ldots, m\}$, $F_i \neq \emptyset$ and $\bigcup_{i=1}^{m} F_i = E$, hence $\mathcal{F}$ is a covering. □

Theorem 3.1 indicates that the zero of $\mathcal{L}(M(\mathbf{C}))$ is $\emptyset$. The following lemma presents the form of the atoms of $\mathcal{L}(M(\mathbf{C}))$. In fact, the closure of any single-point set is an atom of the lattice.

Lemma 3.2. *Let $\mathbf{C}$ be a covering of E. For all $x \in E$, $cl_{M(\mathbf{C})}(\{x\})$ is an atom of $\mathcal{L}(M(\mathbf{C}))$.*

Proof. Since $\mathbf{C}$ is a covering, we know any single-point set is an independent set. Thus for all $x \in E$, $r_{M(\mathbf{C})}(\mathrm{cl}_{M(\mathbf{C})}(\{x\})) = r_{M(\mathbf{C})}(\{x\}) = 1$. As we know, the atoms of a lattice are precisely the elements of height one. Combining with Lemma 2.10 and the fact that $\mathrm{cl}_{M(\mathbf{C})}(\{x\})$ is a closed set, we know $\mathrm{cl}_{M(\mathbf{C})}(\{x\})$ is an atom of $\mathcal{L}(M(\mathbf{C}))$. □

Lemma 3.2 does not establish the concrete form of $\mathrm{cl}_{M(\mathbf{C})}(\{x\})$. In order to solve that problem, we first define two sets as follows.

Definition 3.3. Let $\mathbf{C} = \{K_1, K_2, \ldots, K_m\}$ be a covering of a finite set $E = \{x_1, x_2, \ldots, x_n\}$. We define the following.

(i) $A = \{K_i - \bigcup_{j=1, j\neq i}^{m} K_j : K_i - \bigcup_{j=1, j\neq i}^{m} K_j \neq \emptyset,\ i \in \{1, 2, \ldots, m\}\} = \{A_1, A_2, \ldots, A_s\}$.

(ii) $B = E - \bigcup_{i=1}^{s} A_i$.

Remark 3.4. For all $i \in \{1, 2, \ldots, s\}$ and $x \in A_i$, there exists only one block such that x belongs to it, and there exist at least two blocks such that y belongs to them for all $y \in B$.

The following two propositions establish two characteristics of A and B.

Proposition 3.5. *Let $\mathbf{C}$ be a covering of E. $\{A_1, A_2, \ldots, A_s\} \cup \{\{x\} : x \in B\}$ forms a partition of E.*

Proof. Let $P = \{A_1, A_2, \ldots, A_s\} \cup \{\{x\} : x \in B\}$. According to Definition 3.3, we know $(\bigcup_{i=1}^{s} A_i) \cup (\cup\{\{x\} : x \in B\}) = E$. Now we need to prove for all $P_1, P_2 \in P$, $P_1 \cap P_2 = \emptyset$. According to the definition of A, if $P_1, P_2 \in A$, then $P_1 \cap P_2 = \emptyset$. If $P_1, P_2 \in \{\{x\} : x \in B\}$, then $P_1 \cap P_2 = \emptyset$ because P_i and P_j are different single-points. If $P_1 \in A$ and $P_2 \in \{\{x\} : x \in B\}$, then $P_1 \cap P_2 = \emptyset$ because $B \cap (\bigcup_{k=1}^{s} A_k) = \emptyset$, $P_1 \subseteq \bigcup_{k=1}^{s} A_k$ and $P_2 \subseteq B$. □

Proposition 3.6. *$\mathcal{C}$ is a partition of E if and only if $B = \emptyset$.*

Proof. According to the definition of A and B, the necessity is obvious. Now we prove the sufficiency. If $\mathcal{C}$ is not a partition, then there exist $K_i, K_j \in \mathcal{C}$ such that $K_i \cap K_j \neq \emptyset$. Thus there exists $x \in E$ such that $x \in K_i \cap K_j$, that is, there exist at least $K_i, K_j \in \mathcal{C}$ such that x belongs to them, hence $x \in B$. That contradicts the assumption that $B = \emptyset$. □

Based on Lemma 3.2 and Definition 3.3, we can establish the concrete form of the atoms of lattice $\mathcal{L}(M(\mathcal{C}))$.

Theorem 3.7. *Let $\mathcal{C}$ be a covering of E. $\{A_1, A_2, \ldots, A_s\} \cup \{\{x\} : x \in B\}$ is the set of atoms of lattice $\mathcal{L}(M(\mathcal{C}))$.*

Proof. According to the definition of A_i, we may as well suppose $A_i = K_h - \bigcup_{j=1,j\neq h}^{m} K_j$. Based on $\mathcal{C}$ being a covering and the definition of transversal matroid, we know any single-point set is an independent set, thus for all $x \in A_i$, $\{x\}$ is an independent set. For all $y \in A_i$ and $y \neq x$, we know $x, y \in K_h$ and $x, y \notin K_j$ for all $1 \le j \le m$ and $j \neq h$, thus x and y cannot be chosen from different blocks in the covering $\mathcal{C}$. That shows that $\{x, y\}$ is not an independent set according to the definition of transversal matroid. Hence, $\{x\}$ is a maximal independent set included in A_i, that is, $r_{M(\mathcal{C})}(A_i) = 1$. Next, we need to prove A_i is a closed set. Since $A_i \subseteq \mathrm{cl}_{M(\mathcal{C})}(A_i)$, we need to prove $\mathrm{cl}_{M(\mathcal{C})}(A_i) \subseteq A_i$, that is, $y \notin A_i$ implies $y \notin \mathrm{cl}_{M(\mathcal{C})}(A_i)$. If $y \notin A_i$, based on the fact that $\mathcal{C}$ is a covering and the definition of A_i, then there exists $j \neq h$ such that $y \in K_j$. Thus $\{x, y\}(x \in A_i)$ is an independent set. That implies $y \notin \mathrm{cl}_{M(\mathcal{C})}(A_i)$, thus $\mathrm{cl}_{M(\mathcal{C})}(A_i) = A_i$. Hence, $A_i \in \mathcal{L}(M(\mathcal{C}))$. Combining with $r_{M(\mathcal{C})}(A_i) = 1$, we know for all $1 \le i \le m$, A_i is an atom of lattice $\mathcal{L}(M(\mathcal{C}))$.

According to the definition of transversal matroid and the fact that $\mathcal{C}$ is a covering, any single-point set is an independent set. Thus for all $x \in B$, $r_{M(\mathcal{C})}(\{x\}) = 1$. For all $y \in E$ and $y \neq x$, if $y \in B$, then there exist at least two blocks containing y according to the definition of B. We may as well suppose $y \in K_k, K_t$ and $x \in K_l, K_p$, where $\{K_k, K_t\}$ may be the same as $\{K_l, K_p\}$. Based on this, $\{x, y\}$ is an independent set. This implies $y \notin \mathrm{cl}_{M(\mathcal{C})}(\{x\})$. If $y \notin B$, then we may as well suppose $y \in A_i$, thus $y \in K_h$ for the definition of A_i, where K_h may be the same with K_l or K_p. Based on this, x and y can be chosen from different blocks in covering $\mathcal{C}$, thus $\{x, y\}$ is an independent set. That implies $y \notin \mathrm{cl}_{M(\mathcal{C})}(\{x\})$. From above discussion, we have $\mathrm{cl}_{M(\mathcal{C})}(\{x\}) = \{x\}$. Hence, $\{x\} \in \mathcal{L}(M(\mathcal{C}))$ for all $x \in B$. Combining with $r_{M(\mathcal{C})}(\{x\}) = 1$, we know $\{x\}$ is an atom of lattice $\mathcal{L}(M(\mathcal{C}))$ for all $x \in B$.

Next, we will prove the set of atoms of lattice $\mathcal{L}(M(\mathcal{C}))$ cannot be anything but $\{A_1, A_2, \ldots, A_s\} \cup \{\{x\} : x \in B\}$. According to Lemma 3.2, we know $\{\mathrm{cl}_{M(\mathcal{C})}(\{x\}) : x \in E\}$ is the set of atoms of lattice $\mathcal{L}(M(\mathcal{C}))$. Similar to the proof of the second part, we know that if $x \in B$ then $\mathrm{cl}_{M(\mathcal{C})}(\{x\}) = \{x\}$. If $x \notin B$, then x belongs to one of elements in A. We may as well suppose $x \in A_i$. Combining A_i is an atom with $\emptyset \subseteq \mathrm{cl}_{M(\mathcal{C})}(\{x\}) \subseteq \mathrm{cl}_{M(\mathcal{C})}(A_i) = A_i$, we have $\mathrm{cl}_{M(\mathcal{C})}(\{x\}) = A_i$. Hence, $\{A_1, A_2, \ldots, A_s\} \cup \{\{x\} : x \in B\}$ is the set of atoms of lattice $\mathcal{L}(M(\mathcal{C}))$. □

The proposition below connects simple matroid and the cardinal number of A_i. In fact, a matroid is simple if and only if $|A_i| = 1$ for all $A_i \in A$.

Lemma 3.8. *Let $\mathcal{C}$ be a covering of E. For all $A_i \in A$, if $|A_i| \ge 2$, then x and y are parallel in $M(\mathcal{C})$ for all $x, y \in A_i$.*

Proof. According to the definition of A_i, we may as well suppose $A_i = K_h - \bigcup_{j=1, j\neq h}^{m} K_j$, where $1 \leq h \leq m$. For all $x, y \in A_i$, then $x, y \in K_h$, and for all $j \in \{1, 2, \ldots, m\}$ and $j \neq h$, we have $x \notin K_j$ and $y \notin K_j$. Thus $\{x, y\}$ is not an independent set. Based on the definition of transversal matroid and the fact that $\mathcal{C}$ is a covering, any single-point set is an independent set. Thus $\{x\}$ or $\{y\}$ is an independent set. Hence, x, y are parallel in $M(\mathcal{C})$. □

Proposition 3.9. *Let $\mathcal{C}$ be a covering of E. $M(\mathcal{C})$ is a simple matroid if and only if $|A_i| = 1$ for all $A_i \in A$.*

Proof. "⇒": Since $M(\mathcal{C})$ is a simple matroid, it does not contain parallel elements. If there exists $A_i \in A$ such that $|A_i| \neq 1$, then $|A_i| \geq 2$ because $A_i \neq \emptyset$. According to Lemma 3.8, we know for all $x, y \in A_i$, x, y are parallel which contradicts the assumption that $M(\mathcal{C})$ is a simple matroid. Hence, for all $A_i \in A$, $|A_i| = 1$.

"⇐": According to the definition of parallel element, if $|A_i| = 1$ for all $A_i \in A$, then $M(\mathcal{C})$ does not contain parallel elements; otherwise, we may as well suppose x, y are parallel, then there exists only one block which contains x, y. Hence, there exists $A_i \in A$ such that $x, y \in A_i$, that is, $|A_i| \geq 2$. This contradicts the fact that $|A_i| = 1$ for all $A_i \in A$. Based on the definition of transversal matroid and the fact that $\mathcal{C}$ is a covering, any single-point set is an independent set, thus $M(\mathcal{C})$ does not contain loops. Hence, $M(\mathcal{C})$ does not contain parallel elements and loops which implies that $M(\mathcal{C})$ is a simple matroid. □

When a covering degenerates into a partition, we also have the above results.

Corollary 3.10. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a partition of E. $\mathcal{P}$ is the set of atoms of lattice $\mathcal{L}(M_{\mathcal{P}})$.*

Corollary 3.11. *Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a partition of E. $M(\mathcal{P})$ is a simple matroid if and only if $|P_i| = 1$ for all $P_i \in P$.*

For a geometric lattice $\mathcal{L}(M(\mathcal{C}))$, any closure of single-point is an atom of it. However, the closure of any two elements of E may not be a element which covers some atoms of this lattice. The following proposition shows in what condition $\mathrm{cl}_{M(\mathcal{C})}(\{x, y\})$ covers certain atoms of lattice $\mathcal{L}(M(\mathcal{C}))$.

Proposition 3.12. *Let $\mathcal{C}$ be a covering of E. For all $x, y \in E$, $cl_{M(\mathcal{C})}(\{x, y\})$ covers $cl_{M(\mathcal{C})}(\{x\})$ if and only if there does not exist $A_i \in A$ such that $x, y \in A_i$.*

Proof. "⇐": For all $x, y \in E$, $\{x\} \subseteq \{x, y\}$, then $\mathrm{cl}_{M(\mathcal{C})}(\{x\}) \subseteq \mathrm{cl}_{M(\mathcal{C})}(\{x, y\})$ and $1 = r_{M(\mathcal{C})}(\mathrm{cl}_{M(\mathcal{C})}(\{x\})) \leq r_{M(\mathcal{C})}(\mathrm{cl}_{M(\mathcal{C})}(\{x, y\})) = r_{M(\mathcal{C})}(\{x, y\}) \leq |\{x, y\}| = 2$. Now we need to prove $r_{M(\mathcal{C})}(\mathrm{cl}_{M(\mathcal{C})}(\{x, y\})) = 2$. If $r_{M(\mathcal{C})}(\mathrm{cl}_{M(\mathcal{C})}(\{x, y\})) = r_{M(\mathcal{C})}(\{x, y\}) = 1 = r_{M(\mathcal{C})}(\{x\})$, then $\{x, y\} \notin \mathcal{I}(\mathcal{C})$, that is, there is only one block contains x, y. It means that there exists $A_i \in A$ such that $x, y \in A_i$. That contradicts the hypothesis. Hence, $r_{M(\mathcal{C})}(\mathrm{cl}_{M(\mathcal{C})}(\{x, y\})) = 2$, that is, $\mathrm{cl}_{M(\mathcal{C})}(\{x, y\})$ covers $\mathrm{cl}_{M(\mathcal{C})}(\{x\})$.

"⇒": For all $x, y \in E$, if there exists $A_i \in A$ such that $x, y \in A_i$, then there is only one block contains x, y, thus $x, y \notin \mathcal{I}(\mathcal{C})$, hence $\{x, y\} \subseteq \mathrm{cl}_{M(\mathcal{C})}(\{x\})$. That implies $\mathrm{cl}_{M(\mathcal{C})}(\{x, y\}) \subseteq \mathrm{cl}_{M(\mathcal{C})}(\{x\})$ which contradicts the assumption that $\mathrm{cl}_{M(\mathcal{C})}(\{x, y\})$ covers $\mathrm{cl}_{M(\mathcal{C})}(\{x\})$. □

The modular element and the modular pair are core concepts in lattice. As we know, if a is a modular element of $\mathcal{L}$, then (a, b) is a modular pair of $\mathcal{L}$ for all $b \in \mathcal{L}$, which roots in an important result of lattices. The following theorem shows the relationship among modular element, modular pair and rank function of a matroid in lattice $\mathcal{L}(M)$.

Theorem 3.13. *Let M be a matroid and $\mathcal{L}(M)$ the set of all closed sets of M.*

(1) *For all $X, Y \in \mathcal{L}(M)$, (X, Y) is a modular pair of $\mathcal{L}(M)$ if and only if $r_M(X\cup Y)+r_M(X\cap Y) = r_M(X) + r_M(Y)$.*

(2) *For all $X \in \mathcal{L}(M)$, X is a modular element of $\mathcal{L}(M)$ if and only if $r_M(X\cup Y)+r_M(X\cap Y) = r_M(X) + r_M(Y)$, for all $Y \in \mathcal{L}(M)$.*

Proof. (1) According to Lemmas 2.8 and 2.10, we know (X, Y) is a modular pair of $\mathcal{L}(M)$ if and only if $r_M(X) + r_M(Y) = r_M(X \vee Y) + r_M(X \wedge Y) = r_M(\mathrm{cl}_M(X \cup Y)) + r_M(X \cap Y) = r_M(X \cup Y) + r_M(X \cap Y)$.

(2) It comes from the definition of modular element and (1). □

Let $\{A_i : i \in \Gamma\}$ be the set of atoms of lattice $\mathcal{L}(M(\mathcal{C}))$, where Γ denotes the index set. The following theorem shows the relationship among atoms, modular pairs, and modular elements of the lattice.

Theorem 3.14. *Let $\mathcal{C}$ be a covering of E. For all $i, j \in \Gamma$.*

(1) *(A_i, A_j) is a modular pair of $\mathcal{L}(M(\mathcal{C}))$.*

(2) *A_i is a modular element of $\mathcal{L}(M(\mathcal{C}))$.*

Proof. (1) Since $\mathcal{C}$ is a covering, $\mathrm{cl}_{M(\mathcal{C})}(\emptyset) = \emptyset$. A_i and A_j are atoms, so $A_i \cap A_j = \emptyset$. According to Theorem 3.13, we need to prove $r_{M(\mathcal{C})}(A_i\cup A_j)+r_{M(\mathcal{C})}(A_i\cap A_j) = r_{M(\mathcal{C})}(A_i)+r_{M(\mathcal{C})}(A_j)$, that is, $r_{M(\mathcal{C})}(A_i \cup A_j) = 2$. According to the submodular inequality of $r_{M(\mathcal{C})}$, we have $r_{M(\mathcal{C})}(A_i \cup A_j)+r_{M(\mathcal{C})}(A_i\cap A_j) \le r_{M(\mathcal{C})}(A_i)+r_{M(\mathcal{C})}(A_j)$, that is, $1 \le r_{M(\mathcal{C})}(A_i\cup A_j) \le 2$. If $r_{M(\mathcal{C})}(A_i\cup A_j) = 1 = r_{M(\mathcal{C})}(A_i)$, then $A_j \subseteq \mathrm{cl}_{M(\mathcal{C})}(A_i) = A_i$ which contradicts that $A_i \cap A_j = \emptyset$.

(2) A_i is a modular element of $\mathcal{L}(M(\mathcal{C}))$ if and only if $r_{M(\mathcal{C})}(A_i \cup A) + r_{M(\mathcal{C})}(A_i \cap A) = r_{M(\mathcal{C})}(A_i) + r_{M(\mathcal{C})}(A)$ for all $A \in \mathcal{L}(M(\mathcal{C}))$.

Case 1. If A_i and A are comparable, that is, $A_i \subseteq A$, then $r_{M(\mathcal{C})}(A_i \cup A) + r_{M(\mathcal{C})}(A_i \cap A) = r_{M(\mathcal{C})}(A_i) + r_{M(\mathcal{C})}(A)$.

Case 2. If A_i and A are not comparable, there are two cases. One is that A is an atom of $\mathcal{L}(M(\mathcal{C}))$, the other is that A is not an atom of $\mathcal{L}(M(\mathcal{C}))$. If A is an atom of $\mathcal{L}(M(\mathcal{C}))$, then we obtain the result from (1). If A is not an atom of $\mathcal{L}(M(\mathcal{C}))$, then $A \cap A_i = \emptyset$. Hence, $r_{M(\mathcal{C})}(A) \le r_{M(\mathcal{C})}(A\cup A_i) \le r_{M(\mathcal{C})}(A) + 1$. If $r_{M(\mathcal{C})}(A\cup A_i) = r_{M(\mathcal{C})}(A)$, then $A_i \subseteq \mathrm{cl}_{M(\mathcal{C})}(A) = A$ which contradicts that $A_i \cap A = \emptyset$. Hence, $r_{M(\mathcal{C})}(A \cup A_i) = r_{M(\mathcal{C})}(A) + 1$.

In a word, for all $A \in \mathcal{L}(M(\mathcal{C}))$, $r_{M(\mathcal{C})}(A_i \cup A) + r_{M(\mathcal{C})}(A_i \cap A) = r_{M(\mathcal{C})}(A_i) + r_{M(\mathcal{C})}(A)$, that is, A_i is a modular element of $\mathcal{L}(M(\mathcal{C}))$ for all $i \in \Gamma$. □

When a covering degenerates into a partition, it is not difficult for us to obtain the following result.

Corollary 3.15. *Let $\mathcal{P}$ be a partition of E. For all $P_i, P_j \in \mathcal{P}$:*

(1) *(P_i, P_j) is a modular pair of $\mathcal{L}(M(\mathcal{P}))$.*

(2) *P_i is a modular element of $\mathcal{L}(M(\mathcal{P}))$.*

The following lemma shows how to induce a matroid by a lattice. In fact, if a function f on a lattice is nonnegative, integer-valued, submodular and $f(\emptyset) = 0$, then it can determine a matroid.

Lemma 3.16 (see [19]). *Let $\mathcal{L}_E$ be a lattice of subsets of a set E such that $\mathcal{L}_E$ is closed under intersection, and contains $\emptyset$ and E. Suppose that f is a nonnegative, integer-valued, submodular function on $\mathcal{L}_E$ for which $f(\emptyset) = 0$. Let $\mathcal{I}(\mathcal{L}_E, f) = \{X \subseteq E : f(X) \geq |X \cap T|$, for all $T \in \mathcal{L}_E\}$. $\mathcal{I}(\mathcal{L}_E, f)$ is the collection of independent sets of a matroid on E.*

According to the definition of $\mathcal{L}(M(\mathcal{C}))$, we find that $\mathcal{L}(M(\mathcal{C}))$ is closed under intersection, and contains $\emptyset$ and E. Moreover, the rank function of $M(\mathcal{C})$ is a nonnegative, integer-valued, submodular function on $\mathcal{L}(M(\mathcal{C})$ for which $r_{M(\mathcal{C})}(\emptyset) = 0$. Similar to Lemma 3.16, we can obtain the following theorem.

Theorem 3.17. *Let $\mathcal{C}$ be a covering of E. We define $\mathcal{I}(\mathcal{L}(M(\mathcal{C})), r_{M(\mathcal{C})}) = \{X \subseteq E : r_{M(\mathcal{C})}(Y) \geq |X \cap Y|$, for all $Y \in \mathcal{L}(M(\mathcal{C}))\}$, then $M(E, \mathcal{I}(\mathcal{L}(M(\mathcal{C}), r_{M(\mathcal{C})}))$ is a matroid.*

For any given matroid M, we know that for all $X \subseteq E$, X is an independent set of M if and only if $r_M(X) = |X|$. Moreover, based on the properties of rank function, we have $r_M(X) \leq |X|$. Hence, X is an independent set of M if and only if $r_M(X) \geq |X|$ for all $X \subseteq E$.

Lemma 3.18. *Let M be a matroid. X is an independent set of M if and only if for all closed set Y of M, $r_M(Y) \geq |X \cap Y|$.*

Proof. "$\Rightarrow$": Since for all closed set Y, $X \cap Y \subseteq X$ and X is an independent set, $X \cap Y$ is an independent set of M according to the independent set axiom of a matroid. Hence, we have $r_M(Y) \geq r_M(X \cap Y) \geq |X \cap Y|$.

"$\Leftarrow$": For all closed set Y, $r_M(Y) \geq |X \cap Y|$. Especially, for $Y = \mathrm{cl}_M(X)$, we have $r_M(X) = r_M(\mathrm{cl}_M(X)) = r_M(Y) \geq |X \cap Y| = |X|$. Hence, X is an independent set of matroid M. □

What is the relation between the two matroids induced by a covering and a geometric lattice, respectively? In order to establish the relation between them, we first denote $r_{\mathcal{L}(M(\mathcal{C}))}$ as the rank function of $M(E, \mathcal{I}(\mathcal{L}(M(\mathcal{C}), r_{M(\mathcal{C})}))$ on E. The following theorem shows there is a one-to-one correspondence between geometric lattices and transversal matroids in the context of covering-based rough sets.

Theorem 3.19. *Let $\mathcal{C}$ be a covering of E. For all $X \subseteq E$, $\mathcal{I}(\mathcal{L}(M(\mathcal{C}), r_{M(\mathcal{C})}) = \mathcal{I}(M(\mathcal{C}))$ and $r_{\mathcal{L}(M(\mathcal{C}))}(X) = r_{M(\mathcal{C})}(X)$.*

Proof. According to Lemma 3.18, we know that $\mathcal{I}(\mathcal{L}(M(\mathcal{C}), r_{M(\mathcal{C})}) = \{X \subseteq E : r_{M(\mathcal{C})}(Y) \geq |X \cap Y|$, for all $Y \in \mathcal{L}(M(\mathcal{C}))\}$ and $\mathcal{I}(\mathcal{C}) = \{X \subseteq E : r_{M(\mathcal{C})}(X) = |X|\}$ are equivalent, that is, $M(E, \mathcal{I}(\mathcal{L}(M(\mathcal{C}), r_{M(\mathcal{C})}))$ and $M(E, \mathcal{I}(\mathcal{C}))$ are equivalent. So does $r_{\mathcal{L}(M(\mathcal{C}))}(X) = r_{M(\mathcal{C})}(X)$ for all $X \subseteq E$. □

When a covering degrades into a partition, we can obtain a matroid $M(\mathcal{P}) = (E, \mathcal{I}(P))$, where $\mathcal{I}(\mathcal{P}) = \{X \subseteq E$: for all $P_i \in \mathcal{P}, |X \cap P_i| \leq 1\}$ and $r_{M(\mathcal{P})}(X) = \max\{|I| : I \subseteq X, I \in \mathcal{I}(\mathcal{P})\} = |\{P_i \in \mathcal{P} : P_i \cap X \neq \emptyset\}|$ for all $X \subseteq E$. As we know, for all $X, Y \in \mathcal{L}(M)$, $X \vee Y = \mathrm{cl}_M(X \cup Y)$. If the matroid is $M(\mathcal{P})$, then $X \vee Y = X \cup Y$.

Lemma 3.20 (see [25]). *If $\mathcal{P}$ is a partition of E and M is the matroid, then $cl_M = R^*$ for all $X \subseteq E$.*

Lemma 3.21. *Let $\mathcal{P}$ be a partition of E. For all $X, Y \in \mathcal{L}(M(\mathcal{P}))$, $X \vee Y = X \cup Y$.*

Proof. $X \vee Y = \mathrm{cl}_{M(\mathcal{P})}(X \cup Y) = R^*(X \cup Y) = R^*(X) \cup R^*(Y) = \mathrm{cl}_{M(\mathcal{P})}(X) \cup \mathrm{cl}_{M(\mathcal{P})}(Y) = X \cup Y$. □

Based on the above two lemmas, we can obtain the following proposition.

Proposition 3.22. *Let $\mathcal{P}$ be a partition of E. For all $X \subseteq E$, $\mathcal{I}(\mathcal{L}(M(\mathcal{P})), r_{M(\mathcal{P})}) = \mathcal{I}(M(\mathcal{P}))$ and $r_{\mathcal{L}(M(\mathcal{P}))}(X) = r_{M(\mathcal{P})}(X)$.*

Proof. We need to prove only $\mathcal{I}(\mathcal{L}(M(\mathcal{P})), r_{M(\mathcal{P})}) = \mathcal{I}(M(\mathcal{P}))$. For all $X \in \mathcal{I}(\mathcal{L}(M(\mathcal{P})), r_{M(\mathcal{P})})$, then $r_{M(\mathcal{P})}(Y) \geq |X \cap Y|$ for all $Y \in \mathcal{L}(M(\mathcal{P}))$. Since $P_i \in \mathcal{L}(M(\mathcal{P}))$, $|X \cap P_i| \leq r_{M(\mathcal{P})}(P_i) = 1$. Thus $X \in \mathcal{I}(M(\mathcal{P}))$. Hence, $\mathcal{I}(\mathcal{L}(M(\mathcal{P}), r_{M(\mathcal{P})}) \subseteq \mathcal{I}(M(\mathcal{P}))$. According to Lemmas 2.9 and 3.21, for all $Y \in \mathcal{L}(M(\mathcal{P}))$, there exists $K \subseteq \{1, 2, \ldots, m\}$ such that $Y = \bigvee_{i \in K} P_i = \bigcup_{i \in K} P_i$. Thus $r_{M(\mathcal{P})}(Y) = |K|$ and $|X \cap Y| = |X \cap (\bigcup_{i \in K} P_i)| = |\bigcup_{i \in K}(X \cap P_i)| = \sum_{i \in K} |X \cap P_i|$. For all $X \in \mathcal{I}(M(\mathcal{P}))$, $|X \cap P_i| \leq 1$. Hence, $|X \cap Y| = \sum_{i \in K} |X \cap P_i| \leq |K| = r_{M(\mathcal{P})}(Y)$, that is, $\mathcal{I}(M(\mathcal{P})) \subseteq \mathcal{I}(\mathcal{L}(M(\mathcal{P})), r_{M(\mathcal{P})})$. □

4. Two Geometric Lattice Structures of Covering-Based Rough Sets through Approximation Operators

A geometric lattice structure of covering-based rough sets is established through the transversal matroid induced by a covering, and its characteristics including atoms, modular elements, and modular pairs are studied in Section 3. In this section, we study matroidal structures and the geometric lattice structures from the viewpoint of covering upper approximation operators. The conditions of two types of upper approximation operators to be matroidal closure operators are obtained, and the properties of the matroids and their geometric lattice structures induced by the operators are also established.

Pomykała first studied the second type of covering rough set model [30]. Zhu and Wang studied the axiomatization of this type of upper approximation operator and the relationship between it and the *Kuratowski* closure operator in [27]. First, we give some properties of this operator.

Proposition 4.1. *Let $\mathbf{C}$ be a covering of E. SH has the following properties:*

(1) $SH(\emptyset) = \emptyset$,

(2) $X \subseteq SH(X)$ *for all* $X \subseteq E$,

(3) $SH(X \cup Y) = SH(X) \cup SH(Y)$ *for all* $X, Y \subseteq E$,

(4) $x \in SH(\{y\}) \Leftrightarrow y \in SH(\{x\})$ *for all* $x, y \in E$,

(5) $X \subseteq Y \subseteq E \Rightarrow SH(X) \subseteq SH(Y)$,

(6) *for all* $x, y \in E$, $y \in SH(X \cup \{x\}) - SH(X)$, *then* $x \in SH(X \cup \{y\})$.

Proof. (1)–(5) were proven in [30, 32, 33]. Here we prove only (6). According to (3), we know $\mathrm{SH}(X \cup \{x\}) - \mathrm{SH}(X) = \mathrm{SH}(X) \cup \mathrm{SH}(\{x\}) - \mathrm{SH}(X) = \mathrm{SH}(\{x\}) - \mathrm{SH}(X)$. If $y \in \mathrm{SH}(X \cup \{x\}) - \mathrm{SH}(X)$, then $y \in \mathrm{SH}(\{x\})$. According to (4) and (5), we have $x \in \mathrm{SH}(\{y\}) \subseteq \mathrm{SH}(X \cup \{y\})$. □

We find that the idempotent of SH is not valid, so what is the condition that guarantees it holds for SH? We have the following conclusion.

Proposition 4.2. *Let $\mathbf{C}$ be a covering of E. For all $X \subseteq E$, $SH(SH(X)) = SH(X)$ if and only if $\{I(x) : x \in E\}$ induced by $\mathbf{C}$ forms a partition of E.*

Proof. "$\Leftarrow$": According to (2), (5) of Proposition 4.1, we have $\mathrm{SH}(X) \subseteq \mathrm{SH}(\mathrm{SH}(X))$. Now we prove $\mathrm{SH}(\mathrm{SH}(X)) \subseteq \mathrm{SH}(X)$. For all $x \in \mathrm{SH}(\mathrm{SH}(X))$, there exists $y \in \mathrm{SH}(X)$ such that $x \in I(y)$. Since $y \in \mathrm{SH}(X)$, there exists $z \in X$ such that $y \in I(z)$. According to the definition of $I(y)$, we know $y \in I(y)$, thus $I(z) \cap I(y) \neq \emptyset$. For $\{I(x) : x \in E\}$ forms a partition, $I(z) = I(y)$. Since $x \in I(y)$, $x \in I(z)$, that is, $x \in \mathrm{SH}(X)$, thus $\mathrm{SH}(\mathrm{SH}(X)) \subseteq \mathrm{SH}(X)$.

"$\Rightarrow$": In order to prove $\{I(x) : x \in E\}$ forms a partition, we need to prove that for all $x, y \in E$, if $I(x) \cap I(y) \neq \emptyset$, then $I(x) = I(y)$. If $I(x) \cap I(y) \neq \emptyset$, then there exists $z \in I(x) \cap I(y)$. For $\mathrm{SH}(\mathrm{SH}(\{x\})) = \cup\{I(u) : u \in I(x)\}$ and $z \in I(x)$, then $I(z) \subseteq \mathrm{SH}(\mathrm{SH}(\{x\})) = \mathrm{SH}(\{x\}) = I(x)$. Based on the definition of $I(z)$ and $z \in I(x)$, we have $x \in I(z)$, thus $I(x) \subseteq \mathrm{SH}(\mathrm{SH}(\{z\})) = \mathrm{SH}(\{z\}) = I(z)$. Hence, $I(x) = I(z)$. Similarly, we can obtain $I(y) = I(z)$, thus $I(x) = I(z) = I(y)$. □

The following theorem establishes a necessary and sufficient condition for SH to be a closure operator.

Theorem 4.3. *Let $\mathcal{C}$ be a covering of E. SH is a closure operator of a matroid if and only if $\{I(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E.*

Proof. It comes from Propositions 4.1 and 4.2 and (2), (5), and (6) of Proposition 2.3. □

For a given covering $\mathcal{C}$ of E, we may as well suppose the set of indiscernible neighborhoods of E as $\{I(x) : x \in E\} = \{I(x_1), I(x_2), \ldots, I(x_s)\}$ where $x_1, x_2, \ldots, x_s \in E$.

Definition 4.4. Let $\mathcal{C}$ be a covering of E. We define $\mathcal{I}' = \{I \subseteq E : |I \cap I(x_i)| \leq 1,\ \text{for all } i \in \{1, 2, \ldots, s\}\}$.

As we know, if $\{I(x) : x \in E\} = \{I(x_1), I(x_2), \ldots, I(x_s)\}$ forms a partition of E, then $M(E, \mathcal{I}')$ is a matroid and SH is the closure operator of a matroid. Thus SH can determine a matroid, and the independent sets of the matroid induced by it are established as follows:

$$\mathcal{I}_{\mathrm{SH}}(\mathcal{C}) = \{I \subseteq E : \forall x \in I,\ x \notin \mathrm{SH}(I - \{x\})\}. \tag{4.1}$$

The following proposition shows $M(E, \mathcal{I}_{\mathrm{SH}}) = M(E, \mathcal{I}')$ under the condition that $\{I(x) : x \in E\}$ forms a partition of E.

Proposition 4.5. *Let $\mathcal{C}$ be a covering of E. If $\{I(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E, then $M(E, \mathcal{I}_{SH}(\mathcal{C}))$ is a matroid and $\mathcal{I}_{SH}(\mathcal{C}) = \mathcal{I}'$.*

Proof. Let $\mathcal{I}_{\mathrm{cl}} = \{I \subseteq E : \text{for all } x \in I, x \notin \mathrm{cl}(I - \{x\})\}$. we know that if an operator cl satisfies (1)–(4) of Proposition 2.3, $M(E, \mathcal{I}_{\mathrm{cl}})$ is a matroid. $\{I(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition, hence, $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$ is a matroid. Since $\mathrm{SH}(I) = \bigcup_{y \in I} I(y)$, $\mathrm{SH}(I - \{x\}) = \bigcup_{y \in I - \{x\}} I(y)$. According to the definition of $I(x)$, we know $x \in I(x)$. On one hand, for all $I \in \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$, we know that for all $x \in I$, $x \notin \mathrm{SH}(I - \{x\})$, that is, for all $y \in I$ and $y \neq x$, $x \notin I(y)$. If $I \notin \mathcal{I}'$, that is, there exists $1 \leq i \leq s$ such that $|I \cap I(x_i)| \geq 2$, then we may as well suppose there exist u, v such that $u, v \in I(x_i)$ and $u, v \in I$. Since $u \in I(u), v \in I(v)$ and $\{I(x) : x \in E\}$ forms a partition, $I(u) = I(v) = I(x_i)$. Based on that, we know there exists $u \in I$ and $u \neq v$ such that $u \in I(v)$, that implies contradiction. Hence, $I \in \mathcal{I}'$, that is, $\mathcal{I}_{\mathrm{SH}}(\mathcal{C}) \subseteq \mathcal{I}'$. On the other hand, if $I \notin \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$, then there exists $x \in I$ such that $x \in \mathrm{SH}(I - \{x\}) = \bigcup_{y \in I - \{x\}} I(y)$. That implies that there exists $y \in I$ and $y \neq x$ such that $x \in I(y)$. Since $x \in I(x)$ and $\{I(x) : x \in U\}$

forms a partition, $I(x) = I(y)$. Thus $x, y \in I \cap I(x)$, that implies $|I \cap I(x)| \geq 2$, that is, $I \notin \mathcal{I}'$. Hence, $\mathcal{I}' \subseteq \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$. □

We denote the rank function of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$ by r_{SH}. Then some properties of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$ are established in the following proposition.

Proposition 4.6. *Let $\mathcal{C}$ be a covering of E. If $\{I(x) : x \in E\} = \{I(x_1), I(x_2), \ldots, I(x_s)\}$ induced by $\mathcal{C}$ forms a partition of E, then*

(1) *X is a base of $M(E, \mathcal{I}_{SH}(\mathcal{C}))$ if and only if $|X \cap I(x_i)| = 1$ for all $i \in \{1, 2, \ldots, s\}$. Moreover, $M(E, \mathcal{I}_{SH}(\mathcal{C}))$ has $|I(x_1)||I(x_2)| \cdots |I(x_s)|$ bases.*

(2) *For all $X \subseteq E$, $r_{SH}(X) = |\{I(x_i) : I(x_i) \cap X \neq \emptyset,\ i = 1, 2, \ldots, s\}|$.*

(3) *X is a dependent set of $M(E, \mathcal{I}_{SH}(\mathcal{C}))$ if and only if there exists $I(x_i) \in \{I(x) : x \in E\}$ such that $|I(x_i) \cap X| > 1$.*

(4) *X is a circuit of $M(E, \mathcal{I}_{SH}(\mathcal{C}))$ if and only if there exists $I(x_i) \in \{I(x) : x \in E\}$ such that $X \subseteq I(x_i)$ and $|X| = 2$.*

Proof. (1) According to the definition of base of a matroid, we know that X is a base of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C})) \Leftrightarrow X$ is a maximal independent set of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C})) \Leftrightarrow |X \cap I(x_i)| = 1$ for all $i \in \{1, 2, \ldots, s\}$ because $\mathcal{I}_{\mathrm{SH}}(\mathcal{C}) = \mathcal{I}'$. Since X is a base of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$ and $I(x_1), I(x_2), \ldots, I(x_s)$ are different, $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$ has $|I(x_1)||I(x_2)| \cdots |I(x_s)|$ bases.

(2) According to the definition of rank function, we know $r_{\mathrm{SH}}(X) = |B_X| = |\{I(x_i) : |B_X \cap I(x_i)| = 1\}| \leq |\{I(x_i) : X \cap I(x_i) \neq \emptyset\}|$, where B_X is a maximal independent set included in X. Now we just need to prove the inequality $|\{I(x_i) : |B_X \cap I(x_i)| = 1\}| < |\{I(x_i) : X \cap I(x_i) \neq \emptyset\}|$ does not hold; otherwise, there exists $1 \leq i \leq s$ such that $I(x_i) \cap X \neq \emptyset$ and $I(x_i) \cap B_X = \emptyset$ because $B_X \in \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$. Thus there exists $e_i \in I(x_i) \cap X$ such that $B_X \cup \{e_i\} \subseteq X$ and $B_X \cup \{e_i\} \in \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$. That contradicts the assumption that B_X is a maximal independent set included in X. Hence, $r_{\mathrm{SH}}(X) = |\{I(x_i) : I(x_i) \cap X \neq \emptyset,\ i = 1, 2, \ldots, s\}|$.

(3) According to the definition of dependent set, we know that X is a dependent set $\Leftrightarrow X \notin \mathcal{I}_{\mathrm{SH}}(\mathcal{C}) \Leftrightarrow$ there exists $i \in \{1, 2, \ldots, s\}$ such that $|X \cap I(x_i)| > 1$.

(4) "$\Rightarrow$": As we know, a circuit is a minimal dependent set. X is a circuit of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$, then there exists $i \in \{1, 2, \ldots, s\}$ such that $|I(x_i) \cap X| = 2$. Now we just need to prove $|X| = 2$; otherwise, we may as well suppose $X = \{x, y, z\}$ where $x, y \in I(x_i) \cap X$. Thus we can obtain $|(X - \{z\}) \cap I(x_i)| = 2$, that is, $X - \{z\} \notin \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$. That contradicts the minimality of circuit. Combining $|X| = 2$ with $|I(x_i) \cap X| = 2$, we have $X \subseteq I(x_i)$.

"$\Leftarrow$": Since $|X| = 2$, we may as well suppose $X = \{x, y\}$, and $|X \cap I(x_i)| = 2$ because there exists $I(x_i) \in \{I(x) : x \in E\}$ such that $X \subseteq I(x_i)$, thus X is a dependent set. For all $j \in \{1, 2, \ldots, s\}$, $|\{x\} \cap I(x_j)| \leq 1$ and $|\{y\} \cap I(x_j)| \leq 1$ which implies $\{x\}$ and $\{y\}$ are independent sets, hence X is a circuit of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$. □

For a covering $\mathcal{C}$ of E, we denote $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$ as the set of all closed sets of $M(E, \mathcal{I}_{\mathrm{SH}}(\mathcal{C}))$. When $\{I(x) : x \in E\} = \{I(x_1), I(x_2), \ldots, I(x_s)\}$ forms a partition of E, then for all $X, Y \in \mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$, $X \wedge Y = X \cap Y$, $X \vee Y = \mathrm{SH}(X \cup Y) = \mathrm{SH}(X) \cup \mathrm{SH}(Y) = X \cup Y$ and $\mathrm{SH}(\emptyset) = \emptyset$.

Proposition 4.7. *Let $\mathcal{C}$ be a covering of E. If $\{I(x) : x \in E\} = \{I(x_1), I(x_2), \ldots, I(x_s)\}$ forms a partition of E, then*

(1) *$I(x_1), I(x_2), \ldots, I(x_s)$ are all atoms of $\mathcal{L}(M_{SH})$.*

(2) *For all $x, y \in E$ and $x \neq y$, there does not exist $I(z) \in \{I(x_1), I(x_2), \ldots, I(x_s)\}$ such that $x, y \in I(z)$ if and only if $SH(\{x, y\})$ covers $SH(\{x\})$ or $SH(\{y\})$.*

(3) *For all $i, j \in \{1, 2, \ldots, s\}$, $(I(x_i), I(x_j))$ is a modular pair of $\mathcal{L}_{SH}(M(\mathcal{C}))$.*

(4) *For all $i \in \{1, 2, \ldots, s\}$, $I(x_i)$ is a modular element of $\mathcal{L}_{SH}(M(\mathcal{C}))$.*

Proof. (1) comes from Corollary 3.10, Theorem 4.3 and Proposition 4.5. Based on Proposition 3.12 and Theorem 4.3, we can obtain (2). According to Corollary 3.15, Theorem 4.3 and Proposition 4.5, it is easy to obtain (3) and (4). □

Based on Theorem 4.3, we know that a necessary and sufficient condition for SH to be a closure operator of a matroid is that $\{I(x) : x \in E\}$ forms a partition of E. The following two propositions show what kind of coverings can satisfy that condition.

Lemma 4.8. *Let $\mathcal{C}$ be a covering of E and $K \in \mathcal{C}$. If K is an immured element, then $I(x)$ is the same in $\mathcal{C}$ as in $\mathcal{C} - \{K\}$.*

Proof. If $x \notin K$, then $I_{\mathcal{C}}(x) = I_{\mathcal{C}-\{K\}}(x)$. If $x \in K$, then $I_{\mathcal{C}}(x) = \bigcup_{x \in K'} K' \cup K$. Since K is an immured element, there exists $x \in K'$ such that $K \subseteq K'$. Thus $I_{\mathcal{C}}(x) = \bigcup_{x \in K'} K' \cup K = \bigcup_{x \in K'} K' = I_{\mathcal{C}-\{K\}}(x)$. Hence, $I(x)$ is the same in $\mathcal{C}$ as in $\mathcal{C} - \{K\}$. □

Proposition 4.9. *Let $\mathcal{C}$ be a covering of E. If $exclusion(\mathcal{C})$ is a partition of E, then $\{I(x) : x \in E\}$ induced by $\mathcal{C}$ also forms a partition of E.*

Proof. Since $exclusion(\mathcal{C})$ is a partition of E, $\{I(x) : x \in E\}$ induced by $exclusion(\mathcal{C})$ forms a partition. Suppose $\{K_1, K_2, \ldots, K_s\}$ is the set of all immured elements of $\mathcal{C}$. According to Lemma 4.8, we know for all $x \in E$, $I(x)$ is the same in $exclusion(\mathcal{C})$ as in $exclusion(\mathcal{C}) \cup \{K_1\}$. Thus $\{I(x) : x \in E\}$ induced by $exclusion(\mathcal{C}) \cup \{K_1\}$ forms a partition of E. And the rest may be deduced by analogy, we know that for all $x \in E$, $I(x)$ is the same in $exclusion(\mathcal{C})$ as in $\mathcal{C}$, thus $\{I(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E. □

The proposition below establishes a necessary and sufficient condition for $\{I(x) : x \in E\}$ forms a partition of E from the viewpoint of coverings.

Proposition 4.10. *Let $\mathcal{C}$ be a covering of E. $\{I(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E if and only if $\mathcal{C}$ satisfies (TRA) condition: For all $x, y, z \in E$, $x, z \in K_1 \in \mathcal{C}$, $y, z \in K_2 \in \mathcal{C}$, there exists $K_3 \in \mathcal{C}$ such that $x, y \in K_3$.*

Proof. "⇐": For all $x, y \in E$, $I(x) \cap I(y) = \emptyset$ or $I(x) \cap I(y) \neq \emptyset$. If $I(x) \cap I(y) \neq \emptyset$, then there exists $z \in I(x)$ and $z \in I(y)$. According to the definition of $I(x)$ and $I(y)$, there exist K_1, K_2 such that $x, z \in K_1$ and $y, z \in K_2$. According to the hypothesis, we know there exists $K_3 \in \mathcal{C}$ such that $x, y \in K_3$. Now we need to prove only $I(x) = I(y)$. For all $u \in I(x)$, there exists $K \in \mathcal{C}$ such that $u, x \in K$. Since $x, y \in K_3$, there exists $K' \in \mathcal{C}$ such that $u, y \in K'$, that is, $u \in I(y)$, thus $I(x) \subseteq I(y)$. Similarly, we can prove $I(y) \subseteq I(x)$. Hence, $I(x) = I(y)$, that is, $\{I(x) : x \in E\}$ forms a partition of E.

"$\Rightarrow$": For all $x, y, z \in E$, $x, z \in K_1 \in \mathcal{C}$ and $y, z \in K_2 \in \mathcal{C}$, we can obtain $z \in I(x)$ and $z \in I(y)$. That implies $I(x) \cap I(y) \neq \emptyset$. Since $\{I(x) : x \in E\}$ forms a partition of E, $I(x) = I(y)$. Thus there exists $K_3 \in \mathcal{C}$ such that $x, y \in K_3$. □

The following theorem presents a necessary and sufficient condition for SH to be a closure operator of a matroid from the viewpoint of coverings.

Theorem 4.11. *Let $\mathcal{C}$ be a covering of E. $\mathcal{C}$ satisfies (TRA) condition if and only if SH induced by $\mathcal{C}$ is a closure operator of a matroid.*

Proof. It comes from Theorem 4.3 and Proposition 4.10. □

The sixth type of covering-based upper approximation operator was first defined in [31]. Xu and Wang introduced this type of covering-based rough set model and studied the relationship between it and binary relation based rough set model. The following proposition gives some properties of this covering upper approximation operator.

Proposition 4.12 (see [26]). *Let $\mathcal{C}$ be a covering of E. XH has the following properties:*

(1) $XH(E) = E$,

(2) $XH(\emptyset) = \emptyset$,

(3) $X \subseteq XH(X)$ *for all* $X \subseteq E$,

(4) $XH(X \cup Y) = XH(X) \cup XH(Y)$ *for all* $X, Y \subseteq E$,

(5) $XH(XH(X)) = XH(X)$ *for all* $X \subseteq E$,

(6) *for all* $X \subseteq Y \subseteq E \Rightarrow XH(X) \subseteq XH(Y)$.

From the above proposition, we find that XH does not satisfy the (4) of Proposition 2.3. The following proposition establishes a necessary and sufficient condition for XH to satisfy the condition.

Proposition 4.13. *Let $\mathcal{C}$ be a covering of E. For all $x, y \in E$ and $X \subseteq E$, XH satisfies*

$$y \in XH\left(X \bigcup \{x\}\right) - XH(X) \Longrightarrow x \in XH\left(X \bigcup \{y\}\right) \tag{4.2}$$

if and only if $\{N(x) : x \in E\}$ forms a partition of E.

Proof. $\Rightarrow$: For all $x, y \in E$, if $N(x) \cap N(y) \neq \emptyset$, then there exists $z \in N(x) \cap N(y)$. Let $X = \emptyset$. According to (2) of Proposition 4.12, we know that if $y \in \mathrm{XH}(\{x\})$ then $x \in \mathrm{XH}(y)$, that is, if $x \in N(y)$ then $y \in N(x)$. Since $z \in N(x)$, $N(z) \subseteq N(x)$. According to the assumption, we also have $x \in N(z)$, that is, $N(x) \subseteq N(z)$. Thus $N(x) = N(z)$. Similarly, $z \in N(y)$, we have $N(z) = N(y)$. Thus $N(x) = N(z) = N(y)$. Hence, $\{N(x) : x \in E\}$ forms a partition of E.

$\Leftarrow$: Since $\mathrm{XH}(X \cup Y) = \mathrm{XH}(X) \cup \mathrm{XH}(Y)$ for all $X, Y \subseteq U$, $y \in \mathrm{XH}(X \cup \{x\}) - \mathrm{XH}(X) = \mathrm{XH}(X) \cup \mathrm{XH}(\{x\}) - \mathrm{XH}(X) = \mathrm{XH}(\{x\}) - \mathrm{XH}(X)$. Now we prove $x \in \mathrm{XH}(\{y\})$. Since $y \in \mathrm{XH}(\{x\})$, $x \in N(y)$. Because the fact that $\{N(x) : x \in E\}$ forms a partition of E, $x \in N(x)$ and $x \in N(y)$, we have $N(x) = N(y)$, thus $y \in N(x)$, that is, $x \in \mathrm{XH}(\{y\})$. Hence $x \in \mathrm{XH}(\{y\}) \subseteq \mathrm{XH}(X \cup \{y\})$. □

The following theorem establishes a necessary and sufficient condition for XH to be a closure operator of a matroid.

Theorem 4.14. *Let $\mathcal{C}$ be a covering of E. $\{N(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E if and only if XH is a closure operator of a matroid.*

Proof. It comes from (3), (5), and (6) of Propositions 4.12 and 4.13. □

For convenience, for a given covering $\mathcal{C}$ of E, we may as well suppose the set of all neighborhoods as $\{N(x) : x \in E\} = \{N(x_1), N(x_2), \ldots, N(x_t)\}$ where $x_1, x_2, \ldots, x_t \in E$.

Definition 4.15. Let $\mathcal{C}$ be a covering of E. We define $\mathcal{I}'' = \{I \subseteq E : |I \cap N(x_i)| \leq 1, \text{ for all } i \in \{1, 2, \ldots, t\}\}$.

Theorem 4.14 indicates that if $\{N(x) : x \in E\}$ forms a partition of E, then XH is a closure operator of a matroid. Hence, XH can determine a matroid, and the independent sets of the matroid induced by it are established as follows:

$$\mathcal{I}_{\mathrm{XH}}(\mathcal{C}) = \{I \subseteq E : \forall x \in I, x \notin \mathrm{XH}(I - \{x\})\}. \tag{4.3}$$

Similar to the case of SH, we can obtain the following results.

Proposition 4.16. *Let $\mathcal{C}$ be a covering of E. If $\{N(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E, then $M(E, \mathcal{I}_{XH}(\mathcal{C}))$ is a matroid and $\mathcal{I}_{XH}(\mathcal{C}) = \mathcal{I}''$.*

r_{XH} denotes the rank function of $M(E, \mathcal{I}_{\mathrm{XH}}(\mathcal{C}))$ and $\mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$ denotes the set of all closed sets of $M(E, \mathcal{I}_{\mathrm{XH}}(\mathcal{C}))$. Then we can obtain the following proposition.

Proposition 4.17. *Let $\mathcal{C}$ be a covering of E. If $\{N(x) : x \in E\} = \{N(x_1), N(x_2), \ldots, N(x_t)\}$ induced by $\mathcal{C}$ forms a partition of E, then*

(1) *X is a base of $M(E, \mathcal{I}_{XH}(\mathcal{C}))$ if and only if $|X \cap N(x_i)| = 1$ $(1 \leq i \leq t)$, and $M(E, \mathcal{I}_{XH}(\mathcal{C}))$ has $|N(x_1)||N(x_2)| \cdots |N(x_s)|$ bases.*

(2) *For all $X \subseteq E$, $r_{XH}(X) = |\{N(x_i) : N(x_i) \cap X \neq \emptyset,\ 1 \leq i \leq t\}|$.*

(3) *X is a circuit of $M(E, \mathcal{I}_{XH}(\mathcal{C}))$ if and only if there exists $N(x_i) \in \{N(x) : x \in E\}$ such that $X \subseteq N(x_i)$ and $|X| = 2$.*

(4) *X is a dependent set of $M(E, \mathcal{I}_{XH}(\mathcal{C}))$ if and only if there exists $N(x_i) \in \{N(x) : x \in E\}$ such that $|N(x_i) \cap X| > 1$.*

(5) *$\{N(x_1), N(x_2), \ldots, N(x_t)\}$ is the set of all atoms of lattice $\mathcal{L}_{XH}(M(\mathcal{C}))$.*

(6) *For all $x, y \in E$ and $x \neq y$, there does not exist $N(z) \in \{N(x_1), N(x_2), \ldots, N(x_t)\}$ such that $x, y \in N(z)$ if and only if $XH(\{x, y\})$ covers $XH(\{x\})$ or $XH(\{y\})$.*

(7) *For all $i, j \in \{1, 2, \ldots, t\}$, $(N(x_i), N(x_j))$ is a modular pair of lattice $\mathcal{L}_{XH}(M(\mathcal{C}))$.*

(8) *For all $i \in \{1, 2, \ldots, t\}$, $N(x_i)$ is a modular element of lattice $\mathcal{L}_{XH}(M(\mathcal{C}))$.*

The proof of Propositions 4.16 and 4.17 is similar to that of Propositions 4.5, 4.6, and 4.7, respectively. So we omit the proofs of them. Similar to the case of SH, we also study what kind of coverings can make $\{N(x) : x \in E\}$ form a partition of E. This paper establishes only two kinds of coverings. There are some coverings which satisfy the condition appear in [34, 35].

Lemma 4.18. *Let $\mathcal{C}$ be a covering on E and K be reducible in $\mathcal{C}$. For all $x \in U$, $N(x)$ is the same in $\mathcal{C}$ as in $\mathcal{C} - \{K\}$.*

Proof. For all $x \in E$, $Md(x)$ is the same for covering $\mathcal{C}$ and covering $\mathcal{C} - \{K\}$, so $N(x) = \cap Md(x)$ is the same for the covering $\mathcal{C}$ and covering $\mathcal{C} - \{K\}$. □

Proposition 4.19. *Let $\mathcal{C}$ be a covering of E. If $reduct(\mathcal{C})$ is a partition of E, then $\{N(x) : x \in E\}$ induced by $\mathcal{C}$ is also a partition of E.*

Proof. Since $reduct(\mathcal{C})$ is a partition of E, $\{N(x) : x \in E\}$ induced by $reduct(\mathcal{C})$ forms a partition of E. Suppose $\{K_1, K_2, \ldots, K_s\}$ is the set all reducible elements of $\mathcal{C}$. According to Lemma 4.18, we know that for all $x \in E$, $N(x)$ is the same in $reduct(\mathcal{C})$ as in $reduct(\mathcal{C}) \cup \{K_1\}$, thus $\{N(x) : x \in E\}$ induced by $reduct(\mathcal{C}) \cup \{K_1\}$ forms a partition of E. And the rest may be deduced by analogy, then we can obtain for all $x \in E$, $N(x)$ is the same in $reduct(\mathcal{C})$ as in $\mathcal{C}$, thus $\{N(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E. □

The following proposition establishes a sufficient condition for $\{N(x) : x \in E\}$ to be a partition of E from the viewpoint of coverings.

Proposition 4.20. *Let $\mathcal{C}$ be a covering of E. If $\mathcal{C}$ satisfies (EQU) condition: For all $K \in \mathcal{C}$, for all $x, y \in K$, the number of blocks which contain x is equal to that of blocks which contain y, then $\{N(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition of E.*

Proof. For all $x, y \in E$, if $N(x) \cap N(y) \neq \emptyset$, then there exists $z \in E$ such that $z \in N(x)$ and $z \in N(y)$, that is, the blocks which contain x also contain z and the blocks which contain y also contain z. Hence, there exist K_i, K_i' such that $x, z \in K_i$ and $y, z \in K_i'$. Without loss of generality, we suppose $\{K_1, K_2, \ldots, K_s\}$ is the set of all blocks which contain x and $\{K_1', K_2', \ldots, K_t'\}$ is the set of all blocks which contain y. Since the number of blocks which contain z is equal to that of blocks which contain x, $\{K_1, K_2, \ldots, K_s\}$ is the set of all blocks which contain z, thus $\{K_1', K_2', \ldots, K_t'\} \subseteq \{K_1, K_2, \ldots, K_s\}$. Hence, $N(x) \subseteq N(y)$. Similarly, we can prove $N(y) \subseteq N(x)$. Hence, $N(x) = N(y)$, that is, $\{N(x) : x \in E\}$ forms a partition of E. □

Based on Theorem 4.14, Propositions 4.19 and 4.20, we can obtain the following two corollaries.

Corollary 4.21. *Let $\mathcal{C}$ be a covering of E. If $reduct(\mathcal{C})$ is a partition of E, then XH is a closure operator of a matroid.*

Corollary 4.22. *Let $\mathcal{C}$ be a covering on E. If $\mathcal{C}$ satisfies (EQU) condition, then XH is a closure operator of a matroid.*

5. Relationships among Three Geometric Lattice Structures of Covering-Based Rough Sets

In Section 3, the properties of the geometric lattice induced by a covering have been studied by the matroid $M(E, \mathcal{I}(\mathcal{C}))$. Section 4 presents three sufficient and necessary conditions for two types of covering upper approximation operators to be closure operators of matroids. Moreover, we exhibit two types of matroidal structures through closure axioms, and then obtain two geometric lattice structures of covering-based rough sets. In this section, we study the relationship among above three types of geometric lattices through corresponding matroids. We also discuss the reducible element and the immured element's influence on the relationship among this three types of matroidal structures and geometric lattice structures.

The following proposition shows the relationship between $\mathcal{I}_{\mathrm{SH}}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$, and the relationship between $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$ and $\mathcal{L}(M(\mathcal{C}))$.

Proposition 5.1. *Let $\mathcal{C}$ be a covering of E. If SH induced by $\mathcal{C}$ is a closure operator, then $\mathcal{I}_{SH}(\mathcal{C}) \subseteq \mathcal{I}(\mathcal{C})$ and $\mathcal{L}_{SH}(M(\mathcal{C})) \subseteq \mathcal{L}(M(\mathcal{C}))$.*

Proof. Since SH induced by $\mathcal{C}$ is a closure operator, $\{I(x) : x \in E\} = \{I(x_1), I(x_2), \ldots, I(x_s)\}$ forms a partition of E. For all $I \in \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$, suppose $I = \{i_1, i_2, \ldots, i_\alpha\}$ ($\alpha \leq s$) such that $i_1 \in I(x_{i_1}),\ i_2 \in I(x_{i_2}), \ldots, i_\alpha \in I(x_{i_\alpha})$ and $\{I(x_{i_1}), I(x_{i_2}), \ldots, I(x_{i_\alpha})\} \subseteq \{I(x_1), I(x_2), \ldots, I(x_s)\}$. According to the definition of $I(x)$, there exists $\{K_{i_1}, K_{i_2}, \ldots, K_{i_\alpha}\} \subseteq \mathcal{C}$ such that $i_1 \in K_{i_1},\ i_2 \in K_{i_2}, \ldots, i_\alpha \in K_{i_\alpha}$. Since $\{I(x_1), I(x_2), \ldots, I(x_s)\}$ forms a partition of E, thus $\{K_{i_1}, K_{i_2}, \ldots, K_{i_\alpha}\}$ are different blocks. According to the definition of transversal matroid, we have $I \in \mathcal{I}$. Hence, $\mathcal{I}_{\mathrm{SH}}(\mathcal{C}) \subseteq \mathcal{I}(\mathcal{C})$.

For all $X \in \mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$, $X = \mathrm{SH}(X) = \bigcup_{x \in X} I(x)$. Now we need to prove $X \in \mathcal{L}(M(\mathcal{C}))$, that is, $\mathrm{cl}_{M(\mathcal{C})}(X) = X = \{x \mid r_{M(\mathcal{C})}(X) = r_{M(\mathcal{C})}(X \cup \{x\})\}$. Since $X \subseteq \mathrm{cl}_{M(\mathcal{C})}(X)$, if $\mathrm{cl}_{M(\mathcal{C})}(X) \neq X$, then $\mathrm{cl}_{M(\mathcal{C})}(X) \nsubseteq X$, that is, there exists $y \notin X$ such that $r_{M(\mathcal{C})}(X) = r_{M(\mathcal{C})}(X \cup \{y\})$. Suppose $T = \{t_1, t_2, \ldots, t_t\}$ ($t \leq s$) is a maximal independent set included in X, then $\{t_1, t_2, \ldots, t_t\} \subseteq X = \bigcup_{x \in X} I(x)$ and there exist different $K_1, K_2, \ldots, K_t$ such that for all $i \in \{1, 2, \ldots, t\},\ t_i \in K_i$. Since $y \notin X$, $I(x) \cap I(y) = \emptyset$ for all $x \in X$. Based on $\{I(x_1), I(x_2), \ldots, I(x_s)\}$ forms a partition of E, there exists $K \subseteq I(y)$ such that $K_1, K_2, \ldots, K_t,\ K$ are different blocks and $y \in K$, thus $T \cup \{y\}$ is a maximal independent set included in $X \cup \{y\}$. Hence, we have $r_{M(\mathcal{C})}(X \cup \{y\}) = r_{M(\mathcal{C})}(X) + 1$ which contradicts $r_{M(\mathcal{C})}(X) = r_{M(\mathcal{C})}(X \cup \{y\})$. Thus we can obtain $\mathrm{cl}_{M(\mathcal{C})}(X) = X$. □

The following proposition illustrates that in what condition the indiscernible neighborhoods are included in the geometric lattice induced by $\mathcal{C}$.

Proposition 5.2. *Let $\mathcal{C}$ be a covering of E. If SH induced by $\mathcal{C}$ is a closure operator, then for all $x \in E,\ I(x) \in \mathcal{L}(M(\mathcal{C}))$.*

Proof. Since SH induced by $\mathcal{C}$ is a closure operator, $\{I(x) : x \in E\}$ forms a partition of E. Thus, for all $x \in E$, $\mathrm{SH}(I(x)) = \bigcup_{y \in I(x)} I(y) = I(x)$, that is, $I(x) \in \mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$. According to Proposition 5.1, $I(x) \in \mathcal{L}(M(\mathcal{C}))$. □

We give an example to help understand the relationship between $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$ and $\mathcal{L}(M(\mathcal{C}))$ better.

Example 5.3. Let $\mathcal{C} = \{\{1,2\}, \{1,3\}, \{2,3\}, \{4,5\}\}$. $I(1) = I(2) = I(3) = \{1,2,3\}$, $I(4) = I(5) = \{4,5\}$. Let $T = \{1,2\}$. $T \in \mathcal{I}(\mathcal{C})$ but $T \notin \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$ because $|T \cap I(1)| = 2$. Hence, $\mathcal{I}(\mathcal{C}) \nsubseteq \mathcal{I}_{\mathrm{SH}}(\mathcal{C})$. $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C})) = \{\emptyset, \{1,2,3\}, \{4,5\}, \{1,2,3,4,5\}\}$. $\mathcal{L}(M(\mathcal{C})) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4,5\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,3\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}$. It is obvious that $\mathcal{L}(M(\mathcal{C})) \nsubseteq \mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$. The structures of $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$ and $\mathcal{L}(M(\mathcal{C}))$ are showed in Figure 1.

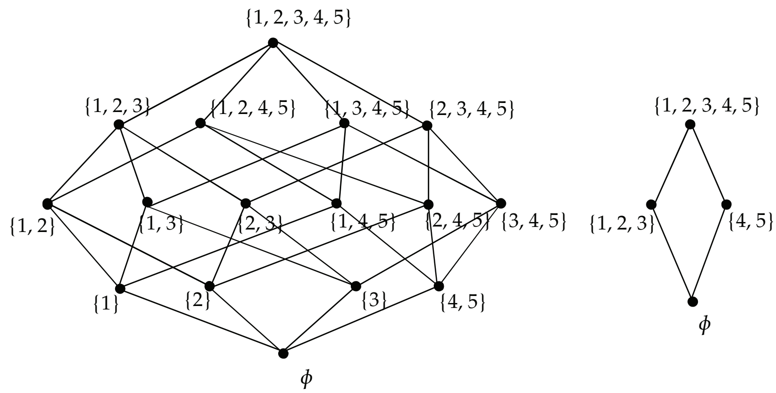

Figure 1: The lattice of $\mathcal{L}(M(\mathcal{C}))$ (resp., $\mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$) and $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$.

Remark 5.4. Let $\mathcal{C}$ be a covering of E. Although XH induced by $\mathcal{C}$ is a closure operator, it has no relationship between $\mathcal{I}_{\mathrm{XH}}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$, and has no relationship between $\mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$ and $\mathcal{L}(M(\mathcal{C}))$.

The following example illustrates the above statements.

Example 5.5. Let $\mathcal{C} = \{K_1, K_2, K_3, K_4\}$ be a covering of $E = \{a, b, c, d, e, f, g, h, i\}$, where $K_1 = \{a, b, i\}$, $K_2 = \{a, b, c, d, e, f\}$, $K_3 = \{f, g, h\}$, $K_4 = \{c, d, e, g, h, i\}$. Then $N(a) = N(b) = \{a, b\}, N(c) = N(d) = N(e) = \{c, d, e\}$, $N(f) = \{f\}$, $N(g) = N(h) = \{g, h\}$ and $N(i) = \{i\}$. Let $T = \{a, c, f, g, i\}$. It is clear that $T \in \mathcal{I}_{\mathrm{XH}}(\mathcal{C})$, but $T \notin \mathcal{I}(\mathcal{C})$ because $|T \cap K_2| = 2$, thus $\mathcal{I}_{\mathrm{XH}}(\mathcal{C}) \nsubseteq \mathcal{I}(\mathcal{C})$. Let $T' = \{a, c, d\}$. It is clear that $T' \in \mathcal{I}(\mathcal{C})$, but $T' \notin \mathcal{I}_{\mathrm{XH}}$ because $|T' \cap N(c)| = 2$, thus $\mathcal{I}(\mathcal{C}) \nsubseteq \mathcal{I}_{\mathrm{XH}}(\mathcal{C})$. Let $\mathrm{X} = \{a, b, i\}$. $\mathrm{X} \in \mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$ for $\mathrm{XH}(\mathrm{X}) = \mathrm{X}$. However, $\mathrm{X} \notin \mathcal{L}(M(\mathcal{C}))$ for $\mathrm{cl}_{M(\mathcal{C})}(\mathrm{X}) = \{a, b, c, d, e, i\}$. Let $\mathrm{X} = \{a\}$. $\mathrm{X} \in \mathcal{L}(M(\mathcal{C}))$ for $\mathrm{cl}_{M(\mathcal{C})}(\mathrm{X}) = \mathrm{X}$. However, $\mathrm{X} \notin \mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$ for $\mathrm{XH}(\mathrm{X}) = \{a, b\} \neq \mathrm{X}$.

The following proposition shows the relationship between $\mathcal{I}_{\mathrm{SH}}(\mathcal{C})$ and $\mathcal{I}_{\mathrm{XH}}(\mathcal{C})$, and the relationship between $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$ and $\mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$.

Proposition 5.6. *Let $\mathcal{C}$ be a covering of E. If XH and SH induced by $\mathcal{C}$ are closure operators, then $\mathcal{I}_{SH}(\mathcal{C}) \subseteq \mathcal{I}_{XH}(\mathcal{C})$ and $\mathcal{L}_{SH}(M(\mathcal{C})) \subseteq \mathcal{L}_{XH}(M(\mathcal{C}))$.*

Proof. If XH and SH induced by $\mathcal{C}$ are closure operators, then $\{N(x) : x \in E\}$ and $\{I(x) : x \in E\}$ form a partition of E, respectively. For all $x \in X$, $N(x) = \bigcap_{x \in K} K \subseteq \bigcup_{i \in K} K = I(x)$, thus $\{N(x) : x \in E\}$ is finer than $\{I(x) : x \in E\}$. Based on this, we can obtain $\mathcal{I}_{\mathrm{SH}}(\mathcal{C}) \subseteq \mathcal{I}_{\mathrm{XH}}(\mathcal{C})$.

For all $\mathrm{X} \in \mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$, $\mathrm{X} = \mathrm{SH}(\mathrm{X}) = \bigcup_{x \in X} I(x)$. Since $x \in N(x) \subseteq I(x)$, $\mathrm{X} = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} N(x) \subseteq \bigcup_{x \in X} I(x) = \mathrm{X}$, thus, $\mathrm{X} = \bigcup_{x \in X} N(x)$, that is, $\mathrm{X} \in \mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$. Hence, $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C})) \subseteq \mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$. $\square$

Example 5.7. From Example 5.3, we know $N(1) = \{1\}$, $N(2) = \{2\}$, $N(3) = \{3\}$, $N(4) = N(5) = \{4, 5\}$ and $I(1) = I(2) = I(3) = \{1, 2, 3\}$, $I(4) = I(5) = \{4, 5\}$. Let $I = \{1, 2\}$. It is clear that $I \in \mathcal{I}_{\mathrm{XH}}(M(\mathcal{C}))$ but $I \notin \mathcal{I}_{\mathrm{SH}}(M(\mathcal{C}))$ because $|I \cap I(1)| = 2$. Hence $\mathcal{I}_{\mathrm{XH}}(M(\mathcal{C})) \nsubseteq \mathcal{I}_{\mathrm{SH}}(M(\mathcal{C}))$. $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C})) = \{\emptyset, \{1, 2, 3\}, \{4, 5\}, \{1, 2, 3, 4, 5\}\}$ and $\mathcal{L}_{\mathrm{XH}}(M(\mathcal{C})) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4, 5\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3\},$

$\{1,2,4,5\}$, $\{1,3,4,5\}$, $\{2,3,4,5\}$, $\{1,2,3,4,5\}\}$. It is obvious that $\mathcal{L}_{\mathrm{XH}}(M(\mathcal{C})) \nsubseteq \mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$. The structures of them are showed in Figure 1.

When a covering degenerates into a partition, we can obtain the following result.

Theorem 5.8. *If $\mathcal{C}$ is a partition of E, then $\mathcal{I}_{SH}(\mathcal{C}) = \mathcal{I}_{XH}(\mathcal{C}) = \mathcal{I}(\mathcal{C})$ and $\mathcal{L}_{SH}(M(\mathcal{C})) = \mathcal{L}_{XH}(M(\mathcal{C})) = \mathcal{L}(M(\mathcal{C}))$.*

Proof. Since $\mathcal{C}$ is a partition of E, $I(x) = N(x) = K(x \in K)$ for all $x \in E$, and for all $X \subseteq E$, $\mathrm{SH}(X) = \mathrm{XH}(X) = R^*(X)$. Thus $\mathcal{I}_{\mathrm{SH}}(\mathcal{C}) = \mathcal{I}_{\mathrm{XH}}(\mathcal{C}) = \mathcal{I}(\mathcal{C})$ and $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C})) = \mathcal{L}_{\mathrm{XH}}(M(\mathcal{C})) = \mathcal{L}(M(\mathcal{C}))$. □

Next, we discuss the reducible element and immured element's influence on matroidal structures and geometric lattice structures. First, we study the reducible element and immured element's influence on $\mathcal{I}(\mathcal{C})$.

Theorem 5.9. *Let $\mathcal{F}$ be a family of subset of E and $K \in \mathcal{F}$. $\mathcal{I}(\mathcal{F} - \{K\}) \subseteq \mathcal{I}(\mathcal{F})$.*

Proof. For all $I \in \mathcal{I}(\mathcal{F} - \{K\})$, we may as well suppose $I = \{i_1, i_2, \ldots, i_t\}$ where $i_1, i_2, \ldots, i_t \in E$. According to the definition of transversal matroid, there exist different blocks $K_1, K_2, \ldots, K_t \in \mathcal{F}$ satisfy $K_i \neq K$ and $i_j \in K_j$ for all $1 \leq i, j \leq t$. Thus $I \in \mathcal{I}(\mathcal{F})$. □

The following example illustrates $\mathcal{I}(\mathcal{F}) \nsubseteq \mathcal{I}(\mathcal{F} - \{K\})$.

Example 5.10. Let $\mathcal{F} = \{K_1, K_2, K_3\}$ be a family of subset of $E = \{1,2,3,4\}$, where $K_1 = \{1,2\}$, $K_2 = \{1,3\}$, $K_3 = \{3\}$. $\mathcal{I}(\mathcal{F}) = 2^E$, $\mathcal{I}(\mathcal{F} - \{K_3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2\}\{2,3\}\}$. Hence, $\mathcal{I}(\mathcal{F}) \nsubseteq \mathcal{I}(\mathcal{F} - \{K\})$.

Let $\mathcal{C}$ be a covering of E and $\mathcal{F} = \mathcal{C}$. As we know, reducible elements and immured elements are members of $\mathcal{C}$. Based on Theorem 5.9, it is not difficult for us to obtain the following four corollaries.

Corollary 5.11. *Let $\mathcal{C}$ be a covering of E and $K \in \mathcal{C}$. If K is reducible, then $\mathcal{I}(\mathcal{C} - \{K\}) \subseteq \mathcal{I}(\mathcal{C})$.*

Corollary 5.12. *Let $\mathcal{C}$ be a covering of E. $\mathcal{I}(reduct(\mathcal{C})) \subseteq \mathcal{I}(\mathcal{C})$.*

Corollary 5.13. *Let $\mathcal{C}$ be a covering of E and $K \in \mathcal{C}$. If K is an immured element, then $\mathcal{I}(\mathcal{C} - \{K\}) \subseteq \mathcal{I}(\mathcal{C})$.*

Corollary 5.14. *Let $\mathcal{C}$ be a covering of E. $\mathcal{I}(exclusion(\mathcal{C})) \subseteq \mathcal{I}(\mathcal{C})$.*

The following theorem shows the reducible element and immured element's influence on geometric lattice structure $\mathcal{L}(M(\mathcal{C}))$.

Theorem 5.15. *Let $\mathcal{F}$ be a family of subset of E and for all $K \in \mathcal{F}$. $\mathcal{L}(M(\mathcal{F} - \{K\})) \subseteq \mathcal{L}(M(\mathcal{F}))$.*

Proof. First, we prove $\mathrm{cl}_{\mathcal{F}}(\{x\}) \subseteq \mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})$ for all $x \in E$. For all $y \notin \mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})$, $\{x, y\} \in \mathcal{I}_{\mathcal{F}-\{K\}} \subseteq \mathcal{I}_{\mathcal{F}}$. Thus $y \notin \mathrm{cl}_{\mathcal{F}}(\{x\})$ which implies $\mathrm{cl}_{\mathcal{F}}(\{x\}) \subseteq \mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})$.

Second, we need to prove that any atom of $\mathcal{L}(M(\mathcal{F} - \{K\}))$ is a closed set of $\mathcal{L}(M(\mathcal{F}))$, that is, $\mathrm{cl}_{\mathcal{F}}(\mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})) = \mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})$. Since $\mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\}) \subseteq \mathrm{cl}_{\mathcal{F}}(\mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})) \subseteq \mathrm{cl}_{\mathcal{F}-\{K\}}(\mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})) = \mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})$, $\mathrm{cl}_{\mathcal{F}}(\mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})) = \mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\})$.

Third, we need to prove $\mathcal{L}(M(\mathcal{F} - \{K\})) \subseteq \mathcal{L}(M(\mathcal{F}))$. For all $X \subseteq \mathcal{L}(M(\mathcal{F} - \{K\}))$, $X = \bigvee_{i=1}^{m} \mathrm{cl}_{\mathcal{F}-\{K\}}(\{x\}) = \bigvee_{i=1}^{m} \bigvee_{j=1}^{s_i} \mathrm{cl}_{\mathcal{F}}(\{x_j\})$ because $\mathcal{L}(M(\mathcal{F} - \{K\}))$ is a atomic lattice, thus $X \in \mathcal{L}(M(\mathcal{F}))$. □

The following example shows that $\mathcal{L}(M(\mathcal{F})) \nsubseteq \mathcal{L}(M(\mathcal{F} - \{K\}))$, where $K \in \mathcal{F}$.

Example 5.16. Based on Example 5.10, we have $\mathcal{L}(M(\mathcal{F})) = 2^E$ and $\mathcal{L}(M(\mathcal{F} - \{K_3\})) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}\}$. It is clear that $\mathcal{L}(M(\mathcal{F})) \nsubseteq \mathcal{L}(M(\mathcal{F} - \{K_3\}))$.

Similarly, when $\mathcal{F}$ is equal to $\mathcal{C}$, we can obtain the following four corollaries.

Corollary 5.17. *Let $\mathcal{C}$ be a covering of E and $K \in \mathcal{C}$. If K is reducible, then $\mathcal{L}(M(\mathcal{C} - \{K\})) \subseteq \mathcal{L}(M(\mathcal{C}))$.*

Corollary 5.18. *Let $\mathcal{C}$ be a covering of E. $\mathcal{L}(M(reduct(\mathcal{C}))) \subseteq \mathcal{L}(M(\mathcal{C}))$.*

Corollary 5.19. *Let $\mathcal{C}$ be a covering of E and $K \in \mathcal{C}$. If K is an immured element, then $\mathcal{L}(M(\mathcal{C} - \{K\})) \subseteq \mathcal{L}(M(\mathcal{C}))$.*

Corollary 5.20. *Let $\mathcal{C}$ be a covering of E. $\mathcal{L}(M(exclusion(\mathcal{C}))) \subseteq \mathcal{L}(M(\mathcal{C}))$.*

Let $\mathcal{C}$ be a covering of E and SH induced by $\mathcal{C}$ a closure operator. If a reducible element K of $\mathcal{C}$ is removed from the covering $\mathcal{C}$, then covering $\mathcal{C} - \{K\}$ may not still be a covering which makes SH be a closure operator. Hence, we omit the discussion of the relationship between $\mathcal{O}_{\mathrm{SH}}(\mathcal{C})$ and $\mathcal{O}_{\mathrm{SH}}(\mathcal{C} - \{K\})$.

Example 5.21. Let $E = \{1,2,3\}$ and $\mathcal{C} = \{K_1, K_2, K_3\}$ where $K_1 = \{1,2\}$, $K_2 = \{1,3\}$, $K_3 = \{1,2,3\}$. $I(1) = I(2) = I(3) = \{1,2,3\}$, thus $\{I(x) : x \in E\}$ forms a partition of E. Hence, SH is a closure operator induced by $\mathcal{C}$. It is clear that K_3 is a reducible element, $\mathcal{C} - \{K_3\} = \{K_1, K_2\}$. Then the indiscernible neighborhoods induced by $\mathcal{C} - \{K_3\}$ are $I(1) = \{1,2,3\}$, $I(2) = \{1,2\}$, $I(3) = \{1,3\}$. we find that $\{I(x) : x \in E\}$ cannot form a partition of E. Hence, SH is not a closure operator induced by $\mathcal{C} - \{K_3\}$.

The following theorem presents an immured element's influence on $\mathcal{O}_{\mathrm{SH}}(\mathcal{C})$ and $\mathcal{L}_{\mathrm{SH}}(M(\mathcal{C}))$.

Theorem 5.22. *Let $\mathcal{C}$ be a covering of E and K an immured element of $\mathcal{C}$. If SH induced by $\mathcal{C}$ is a closure operator, then SH induced by covering $\mathcal{C} - \{K\}$ is also a closure operator. Moreover, $\mathcal{O}_{SH}(\mathcal{C}) = \mathcal{O}_{SH}(\mathcal{C} - \{K\})$ and $\mathcal{L}_{SH}(M(\mathcal{C})) = \mathcal{L}_{SH}(M(\mathcal{C} - \{K\}))$.*

Proof. It comes from Lemma 4.8 and Theorem 4.3. □

As we know, if $\mathcal{C}$ is a covering of E, K_1 and K_2 are two elements of $\mathcal{C}$, and K_1 is an immured element of $\mathcal{C}$, then K_2 is an immured element of $\mathcal{C}$ if and only if K_2 is an immured element of the covering $\mathcal{C} - \{K_1\}$. Combining this with Theorem 5.22, we can obtain the following corollary.

Corollary 5.23. *Let $\mathcal{C}$ be a covering of E. If SH induced by $\mathcal{C}$ is a closure operator, then SH induced by $exclusion(\mathcal{C})$ is also a closure operator. Moreover, $\mathcal{O}_{SH}(\mathcal{C}) = \mathcal{O}_{SH}(exclusion(\mathcal{C}))$ and $\mathcal{L}_{SH}(M(\mathcal{C})) = \mathcal{L}_{SH}(M(exclusion(\mathcal{C})))$.*

Now we consider the reducible element's influence on $\mathcal{O}_{\mathrm{XH}}(\mathcal{C})$ and $\mathcal{L}_{\mathrm{XH}}(M(\mathcal{C}))$.

Theorem 5.24. *Let $\mathcal{C}$ be a covering of E and K be a reducible element. If XH induced by $\mathcal{C}$ is a closure operator, then XH induced by covering $\mathcal{C}-\{K\}$ is also a closure operator. Moreover, $\mathcal{I}_{XH}(\mathcal{C}) = \mathcal{I}_{XH}(\mathcal{C}-\{K\})$ and $\mathcal{L}_{XH}(M(\mathcal{C})) = \mathcal{L}_{XH}(M(\mathcal{C}-\{K\}))$.*

Proof. Since XH induced by $\mathcal{C}$ is a closure operator, $\{N(x) : x \in E\}$ induced by $\mathcal{C}$ forms a partition. Based on the definition of $\mathcal{I}_{\mathrm{XH}}(\mathcal{C})$ and Lemma 4.18, XH induced by $\mathcal{C}-\{K\}$ is also a closure operator and $\mathcal{I}_{\mathrm{XH}}(\mathcal{C}) = \mathcal{I}_{\mathrm{XH}}(\mathcal{C}-\{K\})$. □

As we know, if $\mathcal{C}$ is a covering of E, $K \in \mathcal{C}$, K is reducible in $\mathcal{C}$ and $K_1 \in \mathcal{C}-\{K\}$, then K_1 is reducible in covering $\mathcal{C}$ if and only if it is reducible in covering $\mathcal{C}-\{K\}$. Based on this and Theorem 5.24, we can obtain the following result.

Corollary 5.25. *Let $\mathcal{C}$ be a covering of E. If XH induced by $\mathcal{C}$ is a closure operator, then XH induced by $reduct(\mathcal{C})$ is a closure operator. Moreover, $\mathcal{I}_{XH}(\mathcal{C}) = \mathcal{I}_{XH}(reduct(\mathcal{C}))$ and $\mathcal{L}_{XH}(M(\mathcal{C})) = \mathcal{L}_{XH}(M(reduct(\mathcal{C})))$.*

Let $\mathcal{C}$ be a covering of E and XH induced by $\mathcal{C}$ a closure operator. If an immured element K is removed from the covering $\mathcal{C}$, then $\mathcal{C}-\{K\}$ may not still be a covering which makes XH be a closure operator. So we omit the discussion of the relationship between $\mathcal{I}_{\mathrm{XH}}(\mathcal{C})$ and $\mathcal{I}_{\mathrm{XH}}(\mathcal{C}-\{K\})$.

Example 5.26. Suppose $K_1 = \{1\}$, $K_2 = \{1,2\}$, $K_3 = \{2,3\}$, $K_4 = \{3\}$, $K_5 = \{1,2,3\}$ and $\mathcal{C}_1 = \{K_1, K_2, K_3, K_4\}$. Then $N(1) = \{1\}$, $N(2) = \{2\}$ and $N(3) = \{3\}$, thus $\{N(x) : x \in E\}$ forms a partition of E. Hence, XH is a closure operator. It is clear that K_1 is an immured element, and the neighborhoods induced by $\mathcal{C}-\{K_1\}$ are $N(1) = \{1,2\}$, $N(2) = \{2\}$ and $N(3) = \{3\}$, thus $\{N(x) : x \in E\}$ cannot form a partition of E. Hence, $\mathcal{C}-\{K_1\}$ is not a covering which makes XH be a closure operator.

6. Conclusions

This paper has studied the geometric lattice structures of covering based-rough sets through matroids. The important contribution of this paper is that we have established a geometric lattice structure of covering-based rough sets through the transversal matroid induced by a covering and have presented two geometric lattice structures of covering-based rough sets through two types of covering upper approximation operators. Moreover, we have discussed the relationship among the three geometric lattice structures. To study other properties of the geometric lattice structure induced by a covering and to study other geometric lattices from the viewpoint of other upper approximation operators are our future work.

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