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Research Article

Interval Oscillation Criteria for Super-Half-Linear Impulsive Differential Equations with Delay

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We study the following second-order super-half-linear impulsive differential equations with delay $[r(t)\varphi_\gamma(x'(t))]' + p(t)\varphi_\gamma(x(t-\sigma)) + q(t)f(x(t-\sigma)) = e(t)$, $t \neq \tau_k$, $x(t^+) = a_k x(t)$, $x'(t^+) = b_k x'(t)$, $t = \tau_k$, where $t \geq t_0 \in \mathbb{R}$, $\varphi_*(u) = |u|^{*-1}u$, σ is a nonnegative constant, $\{\tau_k\}$ denotes the impulsive moments sequence with $\tau_1 < \tau_2 < \cdots < \tau_k < \cdots$, $\lim_{k\to\infty}\tau_k = \infty$, and $\tau_{k+1} - \tau_k > \sigma$. By some classical inequalities, Riccati transformation, and two classes of functions, we give several interval oscillation criteria which generalize and improve some known results. Moreover, we also give two examples to illustrate the effectiveness and nonemptiness of our results.

1. Introduction

We consider the following second-order super-half-linear impulsive differential equations with delay

$$\begin{gathered}
\left[r(t)\varphi_\gamma\left(x'(t)\right)\right]' + p(t)\varphi_\gamma(x(t-\sigma)) + q(t)f(x(t-\sigma)) = e(t), \quad t \neq \tau_k, \\
x(t^+) = a_k x(t), \quad x'(t^+) = b_k x'(t), \quad t = \tau_k, \ k = 1, 2, \ldots,
\end{gathered} \tag{1.1}$$

where $t \geq t_0 \in \mathbb{R}$, $\varphi_*(u) = |u|^{*-1}u$, σ is a nonnegative constant, $\{\tau_k\}$ denotes the impulsive moments sequence with $\tau_1 < \tau_2 < \cdots < \tau_k < \cdots$, $\lim_{k\to\infty}\tau_k = \infty$, and $\tau_{k+1} - \tau_k > \sigma$.

Let $J \subset \mathbb{R}$ be an interval, we define

$$PLC(J,\mathbb{R}) := \left\{ y : J \to \mathbb{R} \,\middle|\, \begin{array}{c} y \text{ is continuous everywhere except each } \tau_k \text{ at which} \\ y(\tau_k^+) \text{ and } y(\tau_k^-) \text{ exist and } y(\tau_k^-) = y(\tau_k),\ k \in \mathbb{N} \end{array} \right\}. \tag{1.2}$$

For given t_0 and $\phi \in PLC([t_0 - \sigma, t_0], \mathbb{R})$, we say $x \in PLC([t_0 - \sigma, \infty), \mathbb{R})$ is a solution of (1.1) with initial value ϕ if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x(t) = \phi(t)$ for $t \in [t_0 - \sigma, t_0]$.

A solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

Impulsive differential equation is an adequate mathematical apparatus for the simulation of processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, and so forth. Because it has more richer theory than its corresponding nonimpulsive differential equation, much research has been done on the qualitative behavior of certain impulsive differential equations (see [1, 2]).

In the last decades, there has been an increasing interest in obtaining sufficient conditions for oscillation and/or nonoscillation of different classes impulsive differential equations with delay (constant or variable), see, for example, [1–9] and the references cited therein.

In recent years, interval oscillation of impulsive differential equations was also arousing the interest of many researchers, see [10–15].

However, for the impulsive equations almost all of interval oscillation results in the existing literature were established only for the case of "without delay," in other words, for the case of "with delay" the study on the interval oscillation is very scarce. To the best of our knowledge, Huang and Feng [16] gave the first research in this direction recently. They considered the second-order delay differential equations with impulses

$$\begin{gathered} x''(t) + p(t) f(x(t-\tau)) = e(t), \quad t \geq t_0,\ t \neq t_k,\ k = 1, 2, \ldots, \\ x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k), \quad k = 1, 2, \ldots, \end{gathered} \tag{1.3}$$

and established some interval oscillation criteria which developed some known results for the equation without delay or impulses [13, 17, 18].

Motivated mainly by [16], in this paper, we study the interval oscillation of the delay impulsive equation (1.1). By using classical inequalities, Riccati transformation, and two classes of functions (introduced first by Philos [19]), we establish some interval oscillation criteria which generalize and improve some known results of [13, 16–18]. Moreover, we also give two examples to illustrate the effectiveness and nonemptiness of our results.

2. Main Results

Throughout the paper, we always assume that the following conditions hold:

(A_1) $r(t) \in C([t_0, \infty), (0, \infty))$ and is nondecreasing, $p(t), q(t), e(t) \in PLC([t_0, \infty), \mathbb{R})$;

(A_2) γ is a quotient of odd positive integers a_k, b_k are real constants satisfying $b_k \geq a_k > 0$, $k = 1, 2, \ldots$;

(A_3) $f \in C(\mathbb{R}, \mathbb{R})$, $x f(x) > 0$ and there exist some positive constants η and α such that $f(x)/\varphi_\alpha(x) > \eta$ for all $x \neq 0$ with $\alpha \geq \gamma$.

We introduce the following notations at intervals $[c_1, d_1]$ and $[c_2, d_2]$

$$\begin{gathered} k(s) := \max\{i : t_0 < \tau_i < s\}, \qquad M_j := \max\{r(t) : t \in [c_j, d_j]\}, \quad j = 1,2, \\ \Omega_j(c_j, d_j) := \left\{w \in C^1[c_j, d_j] \mid w(t) \not\equiv 0, w(c_j) = w(d_j) = 0\right\}, \quad j = 1,2. \end{gathered} \tag{2.1}$$

For two constants $c, d \notin \{\tau_k\}$ with $c < d$, $k(c) < k(d)$ and a function $\varphi \in C([c,d],\mathbb{R})$, we define an operator $Q : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$Q_c^d[\varphi] = \varphi(\tau_{k(c)+1}) \frac{b_{k(c)+1}^{\gamma} - a_{k(c)+1}^{\gamma}}{a_{k(c)+1}^{\gamma}(\tau_{k(c)+1} - c)^{\gamma}} + \sum_{i=k(c)+2}^{k(d)} \varphi(\tau_i) \frac{b_i^{\gamma} - a_i^{\gamma}}{a_i^{\gamma}(\tau_i - \tau_{i-1})^{\gamma}}, \tag{2.2}$$

where $\sum_s^t = 0$ if $s > t$.

In the discussion of the impulse moments of $x(t)$ and $x(t-\sigma)$, we need to consider the following cases for $k(c_j) < k(d_j)$,

(S1) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma > d_j$; (S2) $\tau_{k(c_j)} + \sigma < c_j$ and $\tau_{k(d_j)} + \sigma < d_j$;

(S3) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma > d_j$; (S4) $\tau_{k(c_j)} + \sigma > c_j$ and $\tau_{k(d_j)} + \sigma < d_j$, and the cases for $k(c_j) = k(d_j)$;

($\overline{\text{S}}$1) $\tau_{k(c_j)} + \sigma < c_j$; ($\overline{\text{S}}$2) $c_j < \tau_{k(c_j)} + \sigma < d_j$; ($\overline{\text{S}}$3) $\tau_{k(c_j)} + \sigma > d_j$.

Combining (S$*$) with ($\overline{\text{S}}*$), we can get 12 cases. In order to save space, throughout the paper, we study (1.1) under the case of combination of (S1) with ($\overline{\text{S}}$1) only. The discussions for other cases are similar and omitted.

The following preparatory lemmas will be useful to prove our theorems. The first is derived from [20] and the second from [21].

Lemma 2.1. *Let λ and δ be positive real numbers with $\lambda > \delta$. Then*

$$AX^{\lambda-\delta} + BX^{-\delta} \geq \lambda\delta^{-\delta/\lambda}(\lambda-\delta)^{(\delta/\lambda)-1} A^{\delta/\lambda} B^{1-\delta/\lambda}, \tag{2.3}$$

for all $A, B \geq 0$ and $X > 0$.

Lemma 2.2. *Suppose X and Y are nonnegative, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda-1)Y^{\lambda}, \quad \lambda > 1, \tag{2.4}$$

where equality holds if and if $X = Y$.

Let $\gamma > 0$, $B \geq 0$, $A > 0$ and $y \geq 0$. Put

$$\lambda = 1 + \frac{1}{\gamma}, \quad X = A^{\gamma/(\gamma+1)} y, \quad Y = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma} B^{\gamma} A^{-\gamma^2/(\gamma+1)}. \tag{2.5}$$

It follows from Lemma 2.2 that

$$By - Ay^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}. \tag{2.6}$$

Lemma 2.3. *Assume that for any $T \geq t_0$, there exists $c_j, d_j \notin \{\tau_k\}$, $j = 1, 2$, such that $T < c_1 < d_1 \leq c_2 < d_2$ and*

$$\begin{aligned} p(t), q(t) \geq 0, &\quad t \in [c_1 - \sigma, d_1] \cup [c_2 - \sigma, d_2] \setminus \{\tau_k\}, \\ e(t) \leq 0, &\quad t \in [c_1 - \sigma, d_1] \setminus \{\tau_k\}, \\ e(t) \geq 0, &\quad t \in [c_2 - \sigma, d_2] \setminus \{\tau_k\}. \end{aligned} \tag{2.7}$$

If $x(t)$ is a nonoscillatory solution of (1.1), *then there exist the following estimations of $x(t-\sigma)/x(t)$:*

$$\begin{aligned} &(a) \text{ for } t \in (\tau_i + \sigma, \tau_{i+1}], \quad \frac{x(t-\sigma)}{x(t)} > \frac{t - \tau_i - \sigma}{t - \tau_i}, \\ &(b) \text{ for } t \in (\tau_i, \tau_i + \sigma), \quad \frac{x(t-\sigma)}{x(t)} > \frac{t - \tau_i}{b_i(t + \sigma - \tau_i)}, \\ &(c) \text{ for } t \in \left[c_j, \tau_{k(c_j)+1}\right], \quad \frac{x(t-\sigma)}{x(t)} > \frac{t - \tau_{k(c_j)} - \sigma}{t - \tau_{k(c_j)}}, \\ &(d) \text{ for } t \in \left(\tau_{k(d_j)}, d_j\right], \quad \frac{x(t-\sigma)}{x(t)} > \frac{t - \tau_{k(d_j)}}{b_{k(d_j)}\left(t + \sigma - \tau_{k(d_j)}\right)}, \end{aligned} \tag{2.8}$$

where $i = k(c_j), \ldots, k(d_j) - 1$, $j = 1, 2$.

Proof. Without loss of generality, we assume that $x(t) > 0$ and $x(t - \sigma) > 0$ for $t \geq t_0$. In this case the selected interval of t is $[c_1, d_1]$. From (1.1) and (2.7), we obtain

$$\left[r(t)\varphi_{\gamma}(x'(t))\right]' = e(t) - p(t)\varphi_{\gamma}(x(t-\sigma)) - q(t)f(x(t-\sigma)) \leq 0. \tag{2.9}$$

Hence $r(t)\varphi_{\gamma}(x'(t))$ is nonincreasing on the interval $[c_1, d_1] \setminus \{\tau_k\}$.

Case (a) (if $\tau_i + \sigma < t \leq \tau_{i+1}$, then $(t - \sigma, t) \subset (\tau_i, \tau_{i+1}]$). Thus there is no impulsive moment in $(t - \sigma, t)$. For any $s \in (t - \sigma, t)$, we have

$$x(s) - x\left(\tau_i^+\right) = x'(\xi_1)(s - \tau_i), \quad \xi_1 \in (\tau_i, s). \tag{2.10}$$

Since $x(\tau_i^+) > 0$, the function $\varphi_\gamma(\cdot)$ is an increasing function and $r(s)\varphi_\gamma(x'(s))$ is non-increasing on (τ_i, τ_{i+1}), we have

$$\begin{aligned} \varphi_\gamma(x(s)) > \varphi_\gamma\left(x'(\xi_1)(s-\tau_i)\right) &= \frac{r(\xi_1)\varphi_\gamma(x'(\xi_1))}{r(\xi_1)}(s-\tau_i)^\gamma \\ &\geq \frac{r(s)\varphi_\gamma(x'(s))}{r(\xi_1)}(s-\tau_i)^\gamma, \quad \xi_1 \in (\tau_i, s). \end{aligned} \tag{2.11}$$

From (2.11) and the conditions $r(s) > 0$ and $r(s)$ is nondecreasing, we have

$$\varphi_\gamma(x(s)) > \varphi_\gamma\left(x'(s)\right)(s-\tau_i)^\gamma = \varphi_\gamma\left(x'(s)(s-\tau_i)\right), \quad \xi_1 \in (\tau_i, s). \tag{2.12}$$

Thus

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_i}. \tag{2.13}$$

Integrating both sides of above inequality from $t-\sigma$ to t, we obtain

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-\tau_i-\sigma}{t-\tau_i}, \quad t \in (\tau_i+\sigma, \tau_{i+1}]. \tag{2.14}$$

Case (b) (if $\tau_i < t < \tau_i + \sigma$, then $\tau_i - \sigma < t - \sigma < \tau_i < t < \tau_i + \sigma$). There is an impulsive moment τ_i in $(t-\sigma, t)$. For any $t \in (\tau_i, \tau_i + \sigma)$, we have

$$x(t) - x\left(\tau_i^+\right) = x'(\xi_2)(t-\tau_i), \quad \xi_2 \in (\tau_i, t). \tag{2.15}$$

Using the impulsive condition of (1.1) and the monotone properties of $r(t)$, $\varphi_\gamma(\cdot)$ and $r(t)\varphi_\gamma(x'(t))$, we get

$$\begin{aligned} \varphi_\gamma(x(t) - a_i x(\tau_i)) &= \frac{r(\xi_2)\varphi_\gamma(x'(\xi_2))}{r(\xi_2)}(t-\tau_i)^\gamma \\ &\leq \frac{r\left(\tau_i^+\right)\varphi_\gamma\left(x'\left(\tau_i^+\right)\right)}{r(\xi_2)}(t-\tau_i)^\gamma \\ &= \frac{r(\tau_i)\varphi_\gamma(b_i x'(\tau_i)(t-\tau_i))}{r(\xi_2)}. \end{aligned} \tag{2.16}$$

Since $x(\tau_i) > 0$, we have

$$\varphi_\gamma\left(\frac{x(t)}{x(\tau_i)} - a_i\right) \leq \frac{r(\tau_i)}{r(\xi_2)}\varphi_\gamma\left(b_i \frac{x'(\tau_i)}{x(\tau_i)}(t-\tau_i)\right). \tag{2.17}$$

In addition,

$$x(\tau_i) > x(\tau_i) - x(\tau_i - \sigma) = x'(\xi_3)\sigma, \quad \xi_3 \in (\tau_i - \sigma, \tau_i). \tag{2.18}$$

Similar to the analysis of (2.11)–(2.14), we have

$$\frac{x'(\tau_i)}{x(\tau_i)} < \frac{1}{\sigma}. \tag{2.19}$$

From (2.17) and (2.19) and note that the monotone properties of $\varphi_\gamma(\cdot)$ and $r(t)$, we get

$$\frac{x(t)}{x(\tau_i)} < a_i + \frac{b_i}{\sigma}(t - \tau_i). \tag{2.20}$$

In view of (A_3), we have

$$\frac{x(\tau_i)}{x(t)} > \frac{\sigma}{\sigma a_i + b_i(t - \tau_i)} \geq \frac{\sigma}{b_i(t + \sigma - \tau_i)} > 0. \tag{2.21}$$

On the other hand, using similar analysis of (2.11)–(2.19), we get

$$\frac{x'(s)}{x(s)} < \frac{1}{s - \tau_i + \sigma}, \quad s \in (\tau_i - \sigma, \tau_i). \tag{2.22}$$

Integrating (2.22) from $t - \sigma$ to τ_i, where $t \in (\tau_i, \tau_i + \sigma)$, we have

$$\frac{x(t - \sigma)}{x(\tau_i)} > \frac{t - \tau_i}{\sigma} \geq 0. \tag{2.23}$$

From (2.21) and (2.23), we obtain

$$\frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_i}{b_i(t + \sigma - \tau_i)}, \quad t \in (\tau_i, \tau_i + \sigma). \tag{2.24}$$

Case (c) ($t \in [c_1, \tau_{k(c_1)+1}]$). Since $\tau_{k(c_1)} + \sigma < c_1$, then $t - \sigma \in [c_1 - \sigma, \tau_{k(c_1)+1} - \sigma] \subset (\tau_{k(c_1)}, \tau_{k(c_1)+1} - \sigma]$. So, there is no impulsive moment in $(t - \sigma, t)$. Similar to (2.14) of Case (a), we have

$$\frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_{k(c_1)} - \sigma}{t - \tau_{k(c_1)}}, \quad t \in \left[c_1, \tau_{k(c_1)+1}\right]. \tag{2.25}$$

Case (d) ($t \in (\tau_{k(d_1)}, d_1]$). Since $\tau_{k(d_1)} + \sigma > d_1$, then $t - \sigma \in (\tau_{k(d_1)} - \sigma, d_1 - \sigma] \subset (\tau_{k(d_1)} - \sigma, \tau_{k(d_1)})$. Hence, there is an impulsive moment $\tau_{k(d_1)}$ in $(t - \sigma, t)$. Making a similar analysis of Case (b), we obtain

$$\frac{x(t - \sigma)}{x(t)} > \frac{t - \tau_{k(d_1)}}{b_{k(d_1)}\left(t + \sigma - \tau_{k(d_1)}\right)} \geq 0, \quad t \in \left(\tau_{k(d_1)}, d_1\right]. \tag{2.26}$$

When $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study (1.1). The proof is similar and will be omitted. Therefore we complete the proof. □

Theorem 2.4. *Assume that for any $T \geq t_0$, there exists $c_j, d_j \notin \{\tau_k\}$, $j = 1,2$, such that $T < c_1 < d_1 \leq c_2 < d_2$ and (2.7) holds. If there exists $w_j(t) \in \Omega_j(c_j, d_j)$ ($j = 1,2$) such that, for $k(c_j) < k(d_j)$,*

$$\begin{aligned}
&\int_{c_j}^{\tau_{k(c_j)+1}} W_j(t) \frac{\left(t - \tau_{k(c_j)} - \sigma\right)^{\gamma}}{\left(t - \tau_{k(c_j)}\right)^{\gamma}} dt \\
&\quad + \sum_{i=k(c_j)+1}^{k(d_j)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} W_j(t) \frac{(t-\tau_i)^{\gamma}}{b_i^{\gamma}(t+\sigma-\tau_i)^{\gamma}} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} W_j(t) \frac{(t-\tau_i-\sigma)^{\gamma}}{(t-\tau_i)^{\gamma}} dt\right] \\
&\quad + \int_{\tau_{k(d_j)}}^{d_j} W_j(t) \frac{\left(t - \tau_{k(d_j)}\right)^{\gamma}}{b_{k(d_j)}^{\gamma}\left(t + \sigma - \tau_{k(d_j)}\right)^{\gamma}} dt - \int_{c_j}^{d_j} \left(r(t)\left|w_j'(t)\right|^{\gamma+1}\right) dt \\
&\geq M_j Q_{c_j}^{d_j}\left[w_j^{\gamma+1}\right],
\end{aligned} \tag{2.27}$$

and for $k(c_j) = k(d_j)$,

$$\int_{c_j}^{d_j} \left(W_j(t) \frac{(t-c_i)^{\gamma}}{(t-c_i+\sigma)^{\gamma}} - r(t)\left|w_j'(t)\right|^{\gamma+1}\right) dt \geq 0, \tag{2.28}$$

where $W_j(t) = (p(t) + \alpha\gamma^{-\gamma/\alpha}(\alpha-\gamma)^{\gamma/\alpha-1}\eta^{\gamma/\alpha}q^{\gamma/\alpha}(t)|e(t)|^{1-\gamma/\alpha})w_j^{\gamma+1}(t)$, then (1.1) is oscillatory.

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$.

We define

$$u(t) = r(t) \frac{\varphi_{\gamma}(x'(t))}{x^{\gamma}(t)}, \quad \text{for } t \in [c_1, d_1]. \tag{2.29}$$

Differentiating $u(t)$ and in view of (1.1) we obtain, for $t \neq t_k$,

$$\begin{aligned}
u'(t) &= -p(t)\frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)} - q(t)\frac{f(x(t-\sigma))}{x^{\gamma}(t)} - \frac{|e(t)|}{x^{\gamma}(t)} - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma} \\
&\leq -\left\{p(t) + \eta q(t)x^{\alpha-\gamma}(t-\sigma) + |e(t)|x^{-\gamma}(t-\sigma)\right\}\frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)} - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}.
\end{aligned} \tag{2.30}$$

Putting $A = \eta q(t)$, $B = |e(t)|$, $X = x(t-\sigma)$, $\lambda = \alpha$, and $\delta = \gamma$, by Lemma 2.1, we see that

$$\begin{aligned} u'(t) &\le -\left(p(t) + \alpha\gamma^{-\gamma/\alpha}(\alpha-\gamma)^{(\gamma/\alpha)-1}(\eta q(t))^{\gamma/\alpha}|e(t)|^{1-(\gamma/\alpha)}\right)\frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)} - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma} \\ &= -\frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)}R(t) - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}, \end{aligned} \tag{2.31}$$

where $R(t) = p(t) + \alpha\gamma^{-\gamma/\alpha}(\alpha-\gamma)^{\gamma/\alpha-1}(\eta q(t))^{\gamma/\alpha}(t)|e(t)|^{1-\gamma/\alpha}$.

First, we consider the case $k(c_1) < k(d_1)$.

In this case, we assume impulsive moments in $[c_1, d_1]$ are $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \dots, \tau_{k(d_1)}$. Choosing a $w_1(t) \in \Omega_1(c_1, d_1)$, multiplying both sides of (2.31) by $w_1^{\gamma+1}(t)$ and then integrating it from c_1 to d_1, we obtain

$$\begin{aligned} &\int_{c_1}^{\tau_{k(c_1)+1}} u'(t)w_1^{\gamma+1}(t)dt + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} u'(t)w_1^{\gamma+1}(t)dt + \cdots + \int_{\tau_{k(d_1)}}^{d_1} u'(t)w_1^{\gamma+1}(t)dt \\ &\quad\le -\int_{c_1}^{\tau_{k(c_1)+1}} \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}w_1^{\gamma+1}(t)dt - \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}w_1^{\gamma+1}(t)dt \\ &\qquad - \cdots - \int_{\tau_{k(d_1)}}^{d_1} \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}w_1^{\gamma+1}(t)dt - \int_{c_1}^{\tau_{k(c_1)+1}} \frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)}W_1(t)dt \\ &\qquad - \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+1}+\sigma} \frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)}W_1(t)dt - \int_{\tau_{k(c_1)+1}+\sigma}^{\tau_{k(c_1)+2}} \frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)}W_1(t)dt \\ &\qquad - \cdots - \int_{\tau_{k(d_1)-1}+\sigma}^{\tau_{k(d_1)}} \frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)}W_1(t)dt - \int_{\tau_{k(d_1)}}^{d_1} \frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)}W_1(t)dt, \end{aligned} \tag{2.32}$$

where $W_1(t) = R(t)w_1^{\gamma+1}(t)$. Using the integration by parts on the left side of above inequality and noting the condition $w_1(c_1) = w_1(d_1) = 0$, we obtain

$$\begin{aligned} &\sum_{i=k(c_1)+1}^{k(d_1)} w_1^{\gamma+1}(\tau_i)\left[u(\tau_i) - u(\tau_i^+)\right] \\ &\quad\le \int_{c_1}^{\tau_{k(c_1)+1}} \left[(\gamma+1)\left|w_1^{\gamma}(t)w_1'(t)\right||u(t)| - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}w_1^{\gamma+1}(t)\right]dt \\ &\qquad + \sum_{i=k(c_1)+1}^{k(d_1)-1}\int_{\tau_i}^{\tau_{i+1}} \left[(\gamma+1)\left|w_1^{\gamma}(t)w_1'(t)\right||u(t)| - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}w_1^{\gamma+1}(t)\right]dt \\ &\qquad + \int_{\tau_{k(d_1)}}^{d_1} \left[(\gamma+1)\left|w_1^{\gamma}(t)w_1'(t)\right||u(t)| - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}w_1^{\gamma+1}(t)\right]dt \end{aligned}$$

$$
\begin{aligned}
&-\int_{c_1}^{\tau_{k(c_1)+1}} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt \\
&-\sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt\right] \\
&-\int_{\tau_{k(d_1)}}^{d_1} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt.
\end{aligned}
\tag{2.33}
$$

Letting $y = |w_1^\gamma(t)|\,|u(t)|$, $B = (\gamma+1)|w_1'(t)|$, $A = \gamma/r^{1/\gamma}(t)$ and using (2.6), for the integrand function in above inequality we have that

$$
(\gamma+1)\left|w_1^\gamma(t)w_1'(t)\right||u(t)| - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma} w_1^{\gamma+1}(t) \le r(t)\left|w_1'(t)\right|^{\gamma+1}. \tag{2.34}
$$

Meanwhile, for $t = \tau_k$, $k = 1, 2, \ldots$, we have

$$
u(\tau_k^+) = \left(\frac{b_k}{a_k}\right)^\gamma u(\tau_k). \tag{2.35}
$$

Hence

$$
\sum_{i=k(c_1)+1}^{k(d_1)} w_1^{\gamma+1}(\tau_i)\left[u(\tau_i) - u(\tau_i^+)\right] = \sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^\gamma - b_i^\gamma}{a_i^\gamma} w_1^{\gamma+1}(\tau_i)u(\tau_i). \tag{2.36}
$$

Therefore, we get

$$
\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i^\gamma - b_i^\gamma}{a_i^\gamma} w_1^{\gamma+1}(\tau_i)u(\tau_i) \\
&\quad\le \int_{c_1}^{d_1} r(t)\left|w_1'(t)\right|^{\gamma+1}dt - \int_{c_1}^{\tau_{k(c_1)+1}} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt \\
&\quad\quad-\sum_{i=k(c_1)+1}^{k(d_1)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt\right] \\
&\quad\quad-\int_{\tau_{k(d_1)}}^{d_1} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t)dt.
\end{aligned}
\tag{2.37}
$$

On the other hand, for $t \in (\tau_{i-1}, \tau_i] \subset [c_1, d_1]$, $i = k(c_1)+2, \ldots, k(d_1)$, we have

$$
x(t) - x(\tau_{i-1}) = x'(\xi)(t - \tau_{i-1}), \quad \xi \in (\tau_{i-1}, t). \tag{2.38}
$$

In view of $x(\tau_{i-1}) > 0$, and note that the monotone properties of $\varphi_\gamma(\cdot)$, $r(t)\varphi_\gamma(x'(t))$, and $r(t)$, we obtain

$$\varphi_\gamma(x(t)) > \varphi_\gamma(x'(\xi))\varphi_\gamma(t-\tau_{i-1}) \geq \frac{r(t)}{r(\xi)}\varphi_\gamma(x'(t))\varphi_\gamma(t-\tau_{i-1}). \tag{2.39}$$

This is

$$\frac{r(t)\varphi_\gamma(x'(t))}{\varphi_\gamma(x(t))} < \frac{r(\xi)}{(t-\tau_{i-1})^\gamma}. \tag{2.40}$$

Let $t \to \tau_i^-$, it follows

$$u(\tau_i) = \frac{r(\tau_i)\varphi_\gamma(x'(\tau_i))}{\varphi_\gamma(x(\tau_i))} < \frac{M_1}{(\tau_i-\tau_{i-1})^\gamma}, \quad i = k(c_1)+2,\dots,k(d_1). \tag{2.41}$$

Similar analysis on $(c_1, \tau_{k(c_1)+1}]$, we can get

$$u(\tau_{k(c_1)+1}) < \frac{M_1}{(\tau_{k(c_1)+1}-c_1)^\gamma}. \tag{2.42}$$

Then from (2.41), (2.42), and (A_2), we have

$$\begin{aligned}\sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i^\gamma - a_i^\gamma}{a_i^\gamma} w_1^{\gamma+1}(\tau_i)u(\tau_i) &< M_1\left[w_1^{\gamma+1}(\tau_{k(c_1)+1})\theta(c_1) + \sum_{i=k(c_1)+2}^{k(d_1)} w_1^{\gamma+1}(\tau_i)\zeta(\tau_i)\right] \\ &= M_1 Q_{c_1}^{d_1}\left[w_1^{\gamma+1}\right],\end{aligned} \tag{2.43}$$

where $\theta(c_1) = (b_{k(c_1)+1}^\gamma - a_{k(c_1)+1}^\gamma)/(a_{k(c_1)+1}^\gamma(\tau_{k(c_1)+1}-c_1)^\gamma)$ and $\zeta(\tau_i) = (b_i^\gamma - a_i^\gamma)/(a_i^\gamma(\tau_i-\tau_{i-1})^\gamma)$. From (2.37) and (2.43) and applying Lemma 2.3, we obtain

$$\begin{aligned}&\int_{c_1}^{\tau_{k(c_1)+1}} W_1(t)\frac{(t-\tau_{k(c_1)}-\sigma)^\gamma}{(t-\tau_{k(c_1)})^\gamma}dt \\ &\quad + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} W_1(t)\frac{(t-\tau_i)^\gamma}{b_i^\gamma(t+\sigma-\tau_i)^\gamma}dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} W_1(t)\frac{(t-\tau_i-\sigma)^\gamma}{(t-\tau_i)^\gamma}dt\right] \\ &\quad + \int_{\tau_{k(d_1)}}^{d_1} W_1(t)\frac{(t-\tau_{k(d_1)})^\gamma}{b_{k(d_1)}^\gamma(t+\sigma-\tau_{k(d_1)})^\gamma}dt \\ &\quad - \int_{c_1}^{d_1} r(t)\left|w_1'(t)\right|^{\gamma+1}dt \\ &< M_1 Q_{c_1}^{d_1}\left[w_1^{\gamma+1}\right].\end{aligned} \tag{2.44}$$

This contradicts (2.27).

Next we consider the case $k(c_1) = k(d_1)$. By the condition $(\bar{S}1)$ we know there is no impulsive moment in $[c_1, d_1]$. Multiplying both sides of (2.31) by $w_1^{\gamma+1}(t)$ and integrating it from c_1 to d_1, we obtain

$$\int_{c_1}^{d_1} u'(t) w_1^{\gamma+1}(t) dt \le -\int_{c_1}^{d_1} \frac{\gamma}{r^{1/\gamma}(t)} |u(t)|^{(\gamma+1)/\gamma} w_1^{\gamma+1}(t) dt - \int_{c_1}^{d_1} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t) dt. \tag{2.45}$$

Using the integration by parts on the left-hand side and noting the condition $w_1(c_1) = w_1(d_1) = 0$, we obtain

$$\int_{c_1}^{d_1} \left[(\gamma+1) w_1^\gamma(t) w_1'(t) u(t) - \frac{\gamma}{r^{1/\gamma}(t)} w_1^{\gamma+1}(t) |u(t)|^{(\gamma+1)/\gamma} \right] dt - \int_{c_1}^{d_1} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t) dt \ge 0. \tag{2.46}$$

It follows that

$$\int_{c_1}^{d_1} \left[(\gamma+1) \left| w_1^\gamma(t) w_1'(t) \right| |u(t)| - \frac{\gamma}{r^{1/\gamma}(t)} w_1^{\gamma+1}(t) |u(t)|^{(\gamma+1)/\gamma} \right] dt - \int_{c_1}^{d_1} \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t) dt \ge 0. \tag{2.47}$$

Letting $A = (\gamma / r^{1/\gamma}(t)) w_1^{\gamma+1}(t)$, $B = (\gamma+1)|w_1^\gamma(t) w_1'(t)|$, and $y = |u(t)|$ and applying the inequality (2.6), we get

$$\int_{c_1}^{d_1} \left[r(t) \left| w_1'(t) \right|^{\gamma+1} - \frac{x^\gamma(t-\sigma)}{x^\gamma(t)} W_1(t) \right] dt \ge 0. \tag{2.48}$$

Using same way as Case (a) we get

$$\frac{x(t-\sigma)}{x(t)} > \frac{t-c_1}{t-c_1+\sigma}, \quad t \in [c_1, d_1]. \tag{2.49}$$

From (2.48) and (2.49) we obtain

$$\int_{c_1}^{d_1} \left(W_1(t) \frac{(t-c_1)^\gamma}{(t-c_1+\sigma)^\gamma} - r(t) \left| w_1'(t) \right|^{\gamma+1} \right) dt < 0. \tag{2.50}$$

This contradicts our assumption (2.28).

When $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study (1.1). The proof is similar and will be omitted. Therefore we complete the proof. □

Remark 2.5. When $\gamma = 1$, $p(t) = 0$, $f(x) = |x|^{\alpha-1}x$ and the delay term $\sigma = 0$, (1.1) reduces to that studied by Liu and Xu [13]. Therefore our Theorem 2.4 generalizes Theorem 2.1 of [13].

Remark 2.6. When $\gamma = \alpha = 1$, $r(t) = 1$ and $p(t) = 0$, (1.1) reduces to the (1.3) studied by Huang and Feng [16]. Therefore our Theorem 2.4 extends Theorem 2.1 of [16].

Remark 2.7. When $a_k = b_k = 1$, for all $k = 1, 2, \ldots$, the impulses in (1.1) disappear, Theorem 2.4 reduces to the main results of [17, 18].

In the following we will establish a Kemenev type interval oscillation criteria for (1.1) by the ideas of Philos [19] and Kong [22].

Let $D = \{(t,s) : t_0 \le s \le t\}$, $H_1, H_2 \in C^1(D, \mathbb{R})$, then a pair function H_1, H_2 is said to belong to a function set $\mathscr{H}$, defined by $(H_1, H_2) \in \mathscr{H}$, if there exists $h_1, h_2 \in L_{\text{loc}}(D, \mathbb{R})$ satisfying the following conditions:

(A_4) $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0$, $H_2(t,s) > 0$ for $t > s$;

(A_5) $(\partial/\partial t)H_1(t,s) = h_1(t,s)H_1(t,s)$, $(\partial/\partial s)H_2(t,s) = h_2(t,s)H_2(t,s)$.

We assume there exist $c_j, d_j, \delta_j \notin \{\tau_k, k = 1, 2, \ldots\}$ $(j = 1, 2)$ which satisfy $T < c_1 < \delta_1 < d_1 \le c_2 < \delta_2 < d_2$ for any $T \ge t_0$. Noticing whether or not there are impulsive moments of $x(t)$ in $[c_j, \delta_j]$ and $[\delta_j, d_j]$, we should consider the following four cases, namely,

(S1) $k(c_j) < k(\delta_j) < k(d_j)$; (S2) $k(c_j) = k(\delta_j) < k(d_j)$;

(S3) $k(c_j) < k(\delta_j) = k(d_j)$; (S4) $k(c_j) = k(\delta_j) = k(d_j)$.

Moreover, in the discussion of the impulse moments of $x(t - \sigma)$, it is necessary to consider the following two cases:

$(\overline{\text{S}}1)$ $\tau_{k(\delta_j)} + \sigma > \delta_j$; $(\overline{\text{S}}2)$ $\tau_{k(\delta_j)} + \sigma \le \delta_j$.

In the following theorem, we only consider the case of combination of (S1) with $(\overline{\text{S}}1)$. For the other cases, similar conclusions can be given and their proofs will be omitted here.

For convenience in the expressing blow, we define

$$\begin{aligned}
\Pi_{1,j} =: \frac{1}{H_1(\delta_j, c_j)} \Bigg\{ & \int_{c_j}^{\tau_{k(c_j)+1}} \widetilde{H}_1(t, c_j) \frac{\left(t - \tau_{k(c_j)} - \sigma\right)^{\gamma}}{\left(t - \tau_{k(c_j)}\right)^{\gamma}} dt \\
& + \sum_{i=k(c_j)+1}^{k(\delta_j)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} \widetilde{H}_1(t, c_j) \frac{(t-\tau_i)^{\gamma}}{b_i^{\gamma}(t+\sigma-\tau_i)^{\gamma}} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} \widetilde{H}_1(t, c_j) \frac{(t-\tau_i-\sigma)^{\gamma}}{(t-\tau_i)^{\gamma}} dt \right] \\
& + \int_{\tau_{k(\delta_j)}}^{\delta_j} \widetilde{H}_1(t, c_j) \frac{\left(t - \tau_{k(\delta_j)}\right)^{\gamma}}{b_{k(\delta_j)}^{\gamma} \left(t + \sigma - \tau_{k(\delta_j)}\right)^{\gamma}} dt \\
& - \frac{1}{(\gamma+1)^{\gamma+1}} \int_{c_j}^{\delta_j} r(t) H_j(t, c_j) \left| h_j(t, c_j) \right|^{\gamma+1} dt \Bigg\},
\end{aligned}$$

$$\Pi_{2,j} =: \frac{1}{H_2(d_j,\delta_j)}\Bigg\{\int_{\delta_j}^{\tau_{k(\delta_j)}+\sigma}\widetilde{H_2}(d_j,t)\frac{\left(t-\tau_{k(\delta_j)}\right)^{\gamma}}{b_{k(\delta_j)}^{\gamma}\left(t+\sigma-\tau_{k(\delta_j)}\right)^{\gamma}}dt$$
$$+\int_{\tau_{k(\delta_j)}+\sigma}^{\tau_{k(\delta_j)+1}}\widetilde{H_2}(d_j,t)\frac{\left(t-\tau_{k(\delta_j)}-\sigma\right)^{\gamma}}{\left(t-\tau_{k(\delta_j)}\right)^{\gamma}}dt$$
$$+\sum_{i=k(\delta_j)+1}^{k(d_j)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H_2}(d_j,t)\frac{(t-\tau_i)^{\gamma}}{b_i^{\gamma}(t+\sigma-\tau_i)^{\gamma}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H_2}(d_j,t)\frac{(t-\tau_i-\sigma)^{\gamma}}{(t-\tau_i)^{\gamma}}dt\right]$$
$$+\int_{\tau_{k(d_j)}}^{d_j}\widetilde{H_2}(d_j,t)\frac{\left(t-\tau_{k(d_j)}\right)^{\gamma}}{b_{k(d_j)}^{\gamma}\left(t+\sigma-\tau_{k(d_j)}\right)^{\gamma}}dt$$
$$-\frac{1}{(\gamma+1)^{\gamma+1}}\int_{\delta_j}^{d_j}r(t)H_2(d_j,t)\left|h_2(d_j,t)\right|^{\gamma+1}dt\Bigg\}, \tag{2.51}$$

where $\widetilde{H_1}(t,c_j) = H_1(t,c_j)R(t)$, $\widetilde{H_2}(d_j,t) = H_2(d_j,t)R(t)$ $(j = 1,2)$ and $R(t) = p(t)+\alpha\gamma^{-\gamma/\alpha}(\alpha-\gamma)^{\gamma/\alpha-1}\eta^{\gamma/\alpha}q^{\gamma/\alpha}(t)|e(t)|^{1-\gamma/\alpha}$.

Theorem 2.8. *Assume that for any $T \geq t_0$, there exists $c_j, d_j \notin \{\tau_k\}$, $j = 1,2$, such that $T < c_1 < d_1 \leq c_2 < d_2$ and if there exists a pair of $(H_1,H_2) \in \mathscr{H}$ such that*

$$\Pi_{1,j}+\Pi_{2,j} > \frac{M_j}{H_1(\delta_j,c_j)}Q_{c_j}^{\delta_j}[H_1(\cdot,c_j)]+\frac{M_j}{H_2(d_j,\delta_j)}Q_{\delta_j}^{d_j}[H_2(d_j,\cdot)], \quad j = 1,2, \tag{2.52}$$

then (1.1) *is oscillatory.*

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1,d_1]$. Using the same proof as in Theorem 2.4, we can get (2.31). Multiplying both sides of (2.31) by $H_1(t,c_1)$ and integrating it from c_1 to δ_1, we have

$$\int_{c_1}^{\delta_1}H_1(t,c_1)u'(t)dt \leq -\int_{c_1}^{\delta_1}H_1(t,c_1)\frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(1+\gamma)/\gamma}dt$$
$$-\int_{c_1}^{\delta_1}\widetilde{H_1}(t,c_1)\frac{x^{\gamma}(t-\sigma)}{x^{\gamma}(t)}dt, \tag{2.53}$$

where $\widetilde{H}_1(t,c_1) = H_1(t,c_1)(p(t) + \alpha\gamma^{-\gamma/\alpha}(\alpha-\gamma)^{\gamma/\alpha-1}\eta^{\gamma/\alpha}q^{\gamma/\alpha}(t)|e(t)|^{1-\gamma/\alpha})$. Noticing impulsive moments $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(\delta_1)}$ are in $[c_1,\delta_1]$ and using the integration by parts on the left-hand side of above inequality, we obtain

$$\begin{aligned}\int_{c_1}^{\delta_1} H_1(t,c_1)u'(t)dt &= \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)du(t) \\ &= \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{a_i^\gamma - b_i^\gamma}{a_i^\gamma} H_1(\tau_i,c_1)u(\tau_i) + H_1(\delta_1,c_1)u(\delta_1) \\ &\quad - \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)h_1(t,c_1)u(t)dt.\end{aligned} \tag{2.54}$$

Substituting (2.54) into (2.53), we have

$$\begin{aligned}\int_{c_1}^{\delta_1} \widetilde{H}_1(t,c_1)\frac{x^\gamma(t-\sigma)}{x^\gamma(t)}dt &\le \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\gamma - a_i^\gamma}{a_i^\gamma} H_1(\tau_i,c_1)u(\tau_i) - H_1(\delta_1,c_1)u(\delta_1) \\ &+ \left(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \cdots + \int_{\tau_{k(\delta_1)}}^{\delta_1}\right) H_1(t,c_1)\left[|h_1(t,c_1)||u(t)| - \frac{\gamma}{r^{1/\gamma}(t)}|u(t)|^{(\gamma+1)/\gamma}\right]dt.\end{aligned} \tag{2.55}$$

Letting $A = \gamma/r^{1/\gamma}(t)$, $B = |h_1(t,c_1)|$, $y = |u(t)|$ and using (2.6) to the right side of above inequality, we have

$$\begin{aligned}\int_{c_1}^{\delta_1} \widetilde{H}_1(t,c_1)\frac{x^\gamma(t-\sigma)}{x^\gamma(t)}dt &\le \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\gamma - a_i^\gamma}{a_i^\gamma} H_1(\tau_i,c_1)u(\tau_i) - H_1(\delta_1,c_1)u(\delta_1) \\ &\quad + \frac{1}{(\gamma+1)^{\gamma+1}} \int_{c_1}^{\delta_1} r(t)H_1(t,c_1)|h_1(t,c_1)|^{\gamma+1}dt.\end{aligned} \tag{2.56}$$

Because there are different integration intervals in (2.56), we need to divide the integration interval $[c_1,\delta_1]$ into several subintervals for estimating the function $x(t-\sigma)/x(t)$. Using Lemma 2.2, (2.6), we get estimation for the left-hand side of above inequality as follows,

$$\begin{aligned}&\int_{c_1}^{\delta_1} \widetilde{H}_1(t,c_1)\frac{x^\gamma(t-\sigma)}{x^\gamma(t)}dt \\ &\quad > \int_{c_1}^{\tau_{k(c_1)+1}} \widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^\gamma}{\left(t-\tau_{k(c_1)}\right)^\gamma}dt \\ &\quad + \sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} \widetilde{H}_1(t,c_1)\frac{(t-\tau_i)^\gamma}{b_i^\gamma(t+\sigma-\tau_i)^\gamma}dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} \widetilde{H}_1(t,c_1)\frac{(t-\tau_i-\sigma)^\gamma}{(t-\tau_i)^\gamma}dt\right] \\ &\quad + \int_{\tau_{k(\delta_1)}}^{\delta_1} \widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(\delta_1)}\right)^\gamma}{b_{k(\delta_1)}^\gamma\left(t+\sigma-\tau_{k(\delta_1)}\right)^\gamma}dt.\end{aligned} \tag{2.57}$$

From (2.56) and (2.57), we have

$$\begin{aligned}
&\int_{c_1}^{\tau_{k(c_1)+1}} \widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(c_1)}-\sigma\right)^{\gamma}}{\left(t-\tau_{k(c_1)}\right)^{\gamma}}dt\\
&\quad+\sum_{i=k(c_1)+1}^{k(\delta_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i)^{\gamma}}{b_i^{\gamma}(t+\sigma-\tau_i)^{\gamma}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_1(t,c_1)\frac{(t-\tau_i-\sigma)^{\gamma}}{(t-\tau_i)^{\gamma}}dt\right]\\
&\quad+\int_{\tau_{k(\delta_1)}}^{\delta_1}\widetilde{H}_1(t,c_1)\frac{\left(t-\tau_{k(\delta_1)}\right)^{\gamma}}{b_{k(\delta_1)}^{\gamma}\left(t+\sigma-\tau_{k(\delta_1)}\right)^{\gamma}}dt-\frac{1}{(\gamma+1)^{\gamma+1}}\int_{c_1}^{\delta_1}r(t)H_1(t,c_1)|h_1(t,c_1)|^{\gamma+1}dt\\
&\quad<\sum_{i=k(c_1)+1}^{k(\delta_1)}\frac{b_i^{\gamma}-a_i^{\gamma}}{a_i^{\gamma}}H_1(\tau_i,c_1)u(\tau_i)-H_1(\delta_1,c_1)u(\delta_1).
\end{aligned}\tag{2.58}$$

Multiplying both sides of (2.31) by $H_2(d_1,t)$ and using similar analysis to the above, we can obtain

$$\begin{aligned}
&\int_{\delta_1}^{\tau_{k(\delta_1)}+\sigma}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}\right)^{\gamma}}{b_{k(\delta_1)}^{\gamma}\left(t+\sigma-\tau_{k(\delta_1)}\right)^{\gamma}}+\int_{\tau_{k(\delta_1)}+\sigma}^{\tau_{k(\delta_1)+1}}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}-\sigma\right)^{\gamma}}{\left(t-\tau_{k(\delta_1)}\right)^{\gamma}}dt\\
&\quad+\sum_{i=k(\delta_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_2(d_1,t)\frac{(t-\tau_i)^{\gamma}}{b_i^{\gamma}(t+\sigma-\tau_i)^{\gamma}}dt+\int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_2(d_1,t)\frac{(t-\tau_i-\sigma)^{\gamma}}{(t-\tau_i)^{\gamma}}dt\right]\\
&\quad+\int_{\tau_{k(d_1)}}^{d_1}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(d_1)}\right)^{\gamma}}{b_{k(d_1)}^{\gamma}\left(t+\sigma-\tau_{k(d_1)}\right)^{\gamma}}dt-\frac{1}{(\gamma+1)^{\gamma+1}}\int_{\delta_1}^{d_1}r(t)H_2(d_1,t)|h_2(d_1,t)|^{\gamma+1}dt\\
&\quad<\sum_{i=k(\delta_1)+1}^{k(d_1)}\frac{b_i^{\gamma-}a_i^{\gamma}}{a_i^{\gamma}}H_2(d_1,\tau_i)u(\tau_i)+H_2(d_1,\delta_1)u(\delta_1).
\end{aligned}\tag{2.59}$$

Dividing (2.58) and (2.59) by $H_1(\delta_1,c_1)$ and $H_2(d_1,\delta_1)$, respectively, and adding them, we get

$$\begin{aligned}
\Pi_{1,1}+\Pi_{2,1}<{}&\frac{1}{H_1(\delta_1,c_1)}\sum_{i=k(c_1)+1}^{k(\delta_1)}\frac{b_i^{\gamma}-a_i^{\gamma}}{a_i^{\gamma}}H_1(\tau_i,c_1)u(\tau_i)\\
&+\frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)}\frac{b_i^{\gamma}-a_i^{\gamma}}{a_i^{\gamma}}H_2(d_1,\tau_i)u(\tau_i).
\end{aligned}\tag{2.60}$$

On the other hand, similar to (2.43), we have

$$\sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^{\gamma}-a_i^{\gamma}}{a_i^{\gamma}} H_1(\tau_i,c_1)u(\tau_i) \le M_1 Q_{c_1}^{\delta_1}[H_1(\cdot,c_1)],$$
$$\sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i^{\gamma}-a_i^{\gamma}}{a_i^{\gamma}} H_2(d_1,\tau_i)u(\tau_i) \le M_1 Q_{\delta_1}^{d_1}[H_2(d_1,\cdot)]. \tag{2.61}$$

From (2.60) and (2.61), we can obtain a contradiction to the condition (2.52).

When $x(t) < 0$, we can choose interval $[c_2, d_2]$ to study (1.1). The proof is similar and will be omitted. Therefore we complete the proof. □

Remark 2.9. Let $H_1(t,s) = H_2(t,s) = H(t,s)$, $h_1(t,s) = 2h_1^*(t,s)/\sqrt{H(t,s)}$ and $h_2(t,s) = -2h_2^*(t,s)/\sqrt{H(t,s)}$, the conditions (A_4), (A_5) can be changed into

(A_6) $H(t,t) = 0$, $H(t,s) > 0$, for $t > s$;

(A_7) $(\partial/\partial t)H(t,s) = 2h_1^*(t,s)\sqrt{H(t,s)}$, $(\partial/\partial s)H(t,s) = -2h_2^*(t,s)\sqrt{H(t,s)}$.

We know that (A_6) and (A_7) are the main assumptions used in [13, 16] to obtain Kemenev type oscillation criteria. Therefore, Theorem 2.8 is a generalization of Theorem 2.3 in [13] and Theorem 2.5 in [16].

3. Examples

In this section, we give two examples to illustrate the effectiveness and nonemptiness of our results.

Example 3.1. Consider the following equation

$$[\varphi_\gamma(x'(t))]' + \nu_1 \sin t\varphi_\gamma\left(x\left(t-\frac{\pi}{12}\right)\right) + \nu_2^2 \cos t\varphi_\alpha\left(x\left(t-\frac{\pi}{12}\right)\right) = -\cos 2t, \quad t \ne \tau_{k,i},$$
$$x(t^+) = a_k x(t),\ x'(t^+) = b_k x'(t), \quad t = \tau_{k,i}, \tag{3.1}$$

where $\tau_{k,i} = 2k\pi + (i-1)(\pi/2) + (-1)^{(i-1)}(\pi/8)$, $i = 1,2$, $k = 1,2,\ldots$, $t \ge 0$, $\gamma = 3$, $\alpha = 6$, ν_1 and ν_2 are positive constants.

For any $T > 0$, we can choose large $n_0 \in \mathbb{N}$ such that

$$T < c_1 = 2n\pi + \frac{\pi}{12}, \quad d_1 = 2n\pi + \frac{\pi}{4}, \quad c_2 = 2n\pi + \frac{\pi}{3}, \quad d_2 = 2n\pi + \frac{\pi}{2}, \quad n = n_0, n_0+1, \ldots. \tag{3.2}$$

There are impulsive moments $\tau_{n,1} = 2n\pi + \pi/8$ in $[c_1, d_1]$ and $\tau_{n,2} = 2n\pi + 3\pi/8$ in $[c_2, d_2]$. From $\tau_{n,2} - \tau_{n,1} = \pi/4 > \pi/12$ and $\tau_{n+1,1} - \tau_{n,2} = 13\pi/8 > \pi/12$ for all $n > n_0$, we know that condition $\tau_{k+1} - \tau_k > \sigma$ is satisfied. Moreover, we also see the conditions (S1) and (2.7) are satisfied.

Let $w(t) = \sin 12t$. It is easy to get that $W_1(t) = (\nu_1 \sin t + 2\nu_2\sqrt{\cos t \cos 2t})\sin^4 12t$. In view of $\sum_{i=k(c_1)+1}^{k(d_1)-1} = 0$ as $k(c_1)+1 > k(d_1)-1$, by a simple calculation, the left side of (2.27) is the following

$$\begin{aligned}
&\int_{c_1}^{\tau_{k(c_1)+1}} W_1(t)\frac{(t-\tau_{k(c_1)}-\sigma)^\gamma}{(t-\tau_{k(c_1)})^\gamma}dt \\
&\quad + \sum_{i=k(c_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma} W_1(t)\frac{(t-\tau_i)^\gamma}{b_i^\gamma(t+\sigma-\tau_i)^\gamma}dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} W_1(t)\frac{(t-\tau_i-\sigma)^\gamma}{(t-\tau_i)^\gamma}dt\right] \\
&\quad + \int_{\tau_{k(d_1)}}^{d_1} W_1(t)\frac{(t-\tau_{k(d_1)})^\gamma}{b_{k(d_1)}^\gamma(t+\sigma-\tau_{k(d_1)})^\gamma}dt - \int_{c_1}^{d_1} r(t)|w_{1'}(t)|^{\gamma+1}dt \\
&= \int_{2n\pi+\pi/12}^{2n\pi+\pi/8} W_1(t)\left(\frac{t-2(n-1)\pi-3\pi/8-\pi/12}{t-2(n-1)\pi-3\pi/8}\right)^3 dt \\
&\quad + \int_{2n\pi+\pi/8}^{2n\pi+\pi/4} W_1(t)\left(\frac{t-2n\pi-\pi/8}{b_{n,1}(t+\pi/12-2n\pi-\pi/8)}\right)^3 dt - \int_{2n\pi+\pi/12}^{2n\pi+\pi/4} 12^4\cos^4 12t\,dt \\
&= \int_{\pi/12}^{\pi/8} W_1(t)\left(\frac{t+37\pi/24}{t+13\pi/8}\right)^3 dt + \int_{\pi/8}^{\pi/4} W_1(t)\left(\frac{t-\pi/8}{b_{n,1}(t-\pi/24)}\right)^3 dt - 12^4\int_{\pi/12}^{\pi/4}\cos^4 12t\,dt \\
&\approx 0.671\nu_1 + 1.326\nu_2 + b_{n,1}^{-3}(0.008\nu_1 + 0.002\nu_2) - 6^4\pi.
\end{aligned} \tag{3.3}$$

On the other hand, we have

$$Q_{c_1}^{d_1}\left[w^4\right] = \left(\frac{24}{\pi}\right)^3 \frac{b_{n,1}^3 - a_{n,1}^3}{a_{n,1}^3}. \tag{3.4}$$

Thus condition (2.27) is satisfied for $t \in [c_1, d_1]$ if

$$0.671\nu_1 + 1.326\nu_2 + b_{n,1}^{-3}(0.008\nu_1 + 0.002\nu_2) - 6^4\pi \geq \left(\frac{24}{\pi}\right)^3 \frac{b_{n,1}^3 - a_{n,1}^3}{a_{n,1}^3}. \tag{3.5}$$

Similarly, for $t \in [c_2, d_2]$ we can get the following condition

$$0.012\nu_1 + 0.003\nu_2 + b_{n,2}^{-3}(0.012\nu_1 + 0.001\nu_2) - 6^4\pi \geq \left(\frac{24}{\pi}\right)^3 \frac{b_{n,2}^3 - a_{n,2}^3}{a_{n,2}^3} \tag{3.6}$$

which ensures (2.27). Hence, by Theorem 2.4, if (3.5) and (3.6) hold, (3.1) is oscillatory. It is easy to see that (3.5) and (3.6) may be satisfied when ν_1 or ν_2 is large enough. Particularly, let $a_{n,i} = b_{n,i}$, for $i = 1,2$ and $n = 1,2,\ldots$, condition (3.5) and (3.6) become a simple form

$$\begin{aligned}
&0.671\nu_1 + 1.326\nu_2 + b_{n,1}^{-3}(0.008\nu_1 + 0.002\nu_2) \geq 6^4\pi, \\
&0.012\nu_1 + 0.003\nu_2 + b_{n,2}^{-3}(0.012\nu_1 + 0.001\nu_2) \geq 6^4\pi.
\end{aligned} \tag{3.7}$$

Example 3.2. Consider the following equation

$$\begin{gathered}
x''(t) + \mu_1 p(t) x\left(t - \frac{2}{3}\right) + \mu_2 q(t) x^2\left(t - \frac{2}{3}\right) = e(t), \quad t \neq \tau_k, \\
x(t^+) = a_k x(t), \quad x'(t^+) = b_k x'(t), \quad t = \tau_k,
\end{gathered} \tag{3.8}$$

where τ_k: $\tau_{n,1} = 8n + 1/2$, $\tau_{n,2} = 8n + 3/2$, $\tau_{n,3} = 8n + 9/2$, $\tau_{n,4} = 8n + (11/2)$ $(n = 0,1,2,\ldots)$, $\gamma = 1$, $\alpha = 2$, $\sigma = 2/3$, $b_k \geq a_k > 0$, μ_1, μ_2 (>0) is a constant. Clearly, $\tau_{k+1} - \tau_k > \sigma$. In addition, let

$$\begin{gathered}
p(t) = q(t) = \begin{cases} (t - 8n)^3, & t \in [8n, 8n+2], \\ (8n + 4 - t)^3, & t \in [8n+2, 8n+4], \\ (t - 8n - 4)^3, & t \in [8n+4, 8n+6], \\ (8n + 8 - t)^3, & t \in [8n+6, 8n+8], \end{cases} \\
e(t) = \begin{cases} (t - 8n - 2)^3, & t \in [8n, 8n+4], \\ (8n + 6 - t)^3, & t \in [8n+4, 8n+8]. \end{cases}
\end{gathered} \tag{3.9}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 8n$ and let $[c_1, d_1] = [8n, 8n+2]$, $[c_2, d_2] = [8n+4, 8n+6]$, $\delta_1 = 8n+1$ and $\delta_2 = 8n+5$. Then $p(t)$, $q(t)$, and $e(t)$ on $[c_1, d_1]$ and $[c_2, d_2]$ satisfy (2.7). Let $H_1(t,s) = H_2(t,s) = (t-s)^3$, then $h_1(t,s) = -h_2(t,s) = 3/(t-s)$. By simple calculation, we get

$$\begin{aligned}
\Pi_{1,1} =: \frac{1}{H_1(\delta_1, c_1)} \Bigg\{ &\int_{c_1}^{\tau_{k(c_1)+1}} \widetilde{H}_1(t, c_1) \frac{(t - \tau_{k(c_1)} - \sigma)^\gamma}{(t - \tau_{k(c_1)})^\gamma} dt \\
&+ \sum_{i=k(c_1)+1}^{k(\delta_1)-1} \left[\int_{\tau_i}^{\tau_i+\sigma} \widetilde{H}_1(t, c_1) \frac{(t - \tau_i)^\gamma}{b_i^\gamma (t + \sigma - \tau_i)^\gamma} dt + \int_{\tau_i+\sigma}^{\tau_{i+1}} \widetilde{H}_1(t, c_1) \frac{(t - \tau_i - \sigma)^\gamma}{(t - \tau_i)^\gamma} dt \right] \\
&- \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \int_{c_1}^{\delta_1} r(t) H_1(t, c_1) |h_1(t, c_1)|^{\gamma+1} dt \Bigg\} \\
+ \int_{\tau_{k(\delta_1)}}^{\delta_1} \widetilde{H}_1(t, c_1) &\frac{(t - \tau_{k(\delta_1)})^\gamma}{b_{k(\delta_1)}^\gamma (t + \sigma - \tau_{k(\delta_1)})^\gamma} dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{8n}^{8n+1/2} (t-8n)^3\left[\mu_1(t-8n)^3 + 2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2}\right]\frac{t-8n+11/6}{t-8n+5/2}dt \\
&\quad + \int_{8n+1/2}^{8n+1} (t-8n)^3\left[\mu_1(t-8n)^3+2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2}\right]\frac{(t-8n-1/2)}{b_{n,1}(t-8n+1/6)}dt-\frac{9}{8} \\
&= \mu_1\int_0^{1/2}\frac{u^6(u+11/6)}{u+5/2}du + 2\mu_2^{1/2}\int_0^{1/2}\frac{u^{9/2}(2-u)^{1/2}(u+11/6)}{u+5/2}du \\
&\quad + \frac{\mu_1}{b_{n,1}}\int_{1/2}^{1}\frac{u^6(u-1/2)}{u+1/6}du + \frac{2\mu_2^{1/2}}{b_{n,1}}\int_{1/2}^{1}\frac{u^{9/2}(2-u)^{1/2}(u-1/2)}{u+1/6}du - \frac{9}{8} \\
&\approx \left(0.0008+\frac{0.0005}{b_{n,1}}\right)\mu_1 + \left(0.0081+\frac{0.1268}{b_{n,1}}\right)\mu_2^{1/2} - \frac{9}{8},
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{2,1} =:\ & \frac{1}{H_2(d_1,\delta_1)}\Bigg\{\int_{\delta_1}^{\tau_{k(\delta_1)}+\sigma}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}\right)^\gamma}{b_{k(\delta_1)}^\gamma\left(t+\sigma-\tau_{k(\delta_1)}\right)^\gamma} + \int_{\tau_{k(\delta_1)}+\sigma}^{\tau_{k(\delta_1)+1}}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(\delta_1)}-\sigma\right)^\gamma}{\left(t-\tau_{k(\delta_1)}\right)^\gamma}dt \\
&\qquad + \sum_{i=k(\delta_1)+1}^{k(d_1)-1}\left[\int_{\tau_i}^{\tau_i+\sigma}\widetilde{H}_2(d_1,t)\frac{(t-\tau_i)^\gamma}{b_i^\gamma(t+\sigma-\tau_i)^\gamma}dt + \int_{\tau_i+\sigma}^{\tau_{i+1}}\widetilde{H}_2(d_1,t)\frac{(t-\tau_i-\sigma)^\gamma}{(t-\tau_i)^\gamma}dt\right] \\
&\qquad - \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\int_{\delta_1}^{d_1} r(t)H_2(d_1,t)|h_2(d_1,t)|^{\gamma+1}dt\Bigg\} \\
&\quad + \int_{\tau_{k(d_1)}}^{d_1}\widetilde{H}_2(d_1,t)\frac{\left(t-\tau_{k(d_1)}\right)^\gamma}{b_{k(d_1)}^\gamma\left(t+\sigma-\tau_{k(d_1)}\right)^\gamma}dt \\
&= \int_{8n+1}^{8n+7/6}(8n+2-t)^3\left[\mu_1(t-8n)^3+2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2}\right]\frac{t-8n-1/2}{b_{n,1}(t-8n+1/6)}dt \\
&\quad + \int_{8n+7/6}^{8n+3/2}(8n+2-t)^3\left[\mu_1(t-8n)^3+2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2}\right]\frac{t-8n-7/6}{t-8n-1/2}dt \\
&\quad + \int_{8n+3/2}^{8n+2}(8n+2-t)^3\left[\mu_1(t-8n)^3+2\mu_2^{1/2}(t-8n)^{3/2}(8n+2-t)^{1/2}\right]\frac{t-8n-3/2}{b_{n,2}(t-8n-5/6)}dt-\frac{9}{8} \\
&= \frac{\mu_1}{b_{n,1}}\int_1^{7/6}\frac{u^3(2-u)^3(u-1/2)}{u+1/6}du + \frac{2\mu_2^{1/2}}{b_{n,1}}\int_1^{7/6}\frac{u^{3/2}(2-u)^{7/2}(u-1/2)}{u+1/6}du \\
&\quad + \mu_1\int_{7/6}^{3/2}\frac{u^3(2-u)^3(u-7/6)}{u-1/2}du + 2\mu_2^{1/2}\int_{7/6}^{3/2}\frac{u^{3/2}(2-u)^{7/2}(u-7/6)}{u-1/2}du \\
&\quad + \frac{\mu_1}{b_{n,2}}\int_{3/2}^{2}\frac{u^3(2-u)^3(u-3/2)}{u-5/6}du + \frac{2\mu_2^{1/2}}{b_{n,2}}\int_{3/2}^{2}\frac{u^{3/2}(2-u)^{7/2}(u-3/2)}{u-5/6}du - \frac{9}{8} \\
&\approx \left(0.0389+\frac{0.0754}{b_{n,1}}+\frac{0.0087}{b_{n,2}}\right)\mu_1 + \left(0.0394+\frac{0.1281}{b_{n,1}}+\frac{0.0046}{b_{n,2}}\right)\mu_2^{1/2} - \frac{9}{8}.
\end{aligned}
\tag{3.10}
$$

Then the left side of the inequality (2.52) is

$$\Pi_{1,1}+\Pi_{2,1}\approx\left(0.0497+\frac{0.0759}{b_{n,1}}+\frac{0.0087}{b_{n,2}}\right)\mu_1+\left(0.0475+\frac{0.2541}{b_{n,1}}+\frac{0.0046}{b_{n,2}}\right)\mu_2^{1/2}-\frac{9}{4}. \tag{3.11}$$

Because $M_1=M_2=1$, $\tau_{k(c_1)+1}=\tau_{k(\delta_1)}=\tau_{n,1}=8n+1/2\in(c_1,\delta_1)$ and $\tau_{k(\delta_1)+1}=\tau_{k(d_1)}=\tau_{n,2}=8n+3/2\in(\delta_1,d_1)$, it is easy to get that the right side of the inequality (2.52) for $j=1$ is

$$\frac{M_1}{H_1(\delta_1,c_1)}Q_{c_1}^{\delta_1}[H_1(\cdot,c_1)]+\frac{M_1}{H_2(d_1,\delta_1)}Q_{\delta_1}^{d_1}[H_2(d_1,\cdot)]=\frac{b_{n,1}-a_{n,1}}{4a_{n,1}^2}+\frac{29(b_{n,2}-a_{n,2})}{8a_{n,2}}. \tag{3.12}$$

Thus (2.52) is satisfied for $j=1$ if

$$\begin{aligned}&\left(0.0497+\frac{0.0759}{b_{n,1}}+\frac{0.0087}{b_{n,2}}\right)\mu_1+\left(0.0475+\frac{0.2541}{b_{n,1}}+\frac{0.0046}{b_{n,2}}\right)\mu_2^{1/2}\\&\quad>\frac{9}{4}+\frac{b_{n,1}-a_{n,1}}{4a_{n,1}^2}+\frac{29(b_{n,2}-a_{n,2})}{8a_{n,2}}.\end{aligned} \tag{3.13}$$

When $j=2$, with the similar argument above we get that the left side of inequality (2.52) is

$$\begin{aligned}\Pi_{1,2}+\Pi_{2,2}&=\mu_1\int_0^{1/2}\frac{u^6(u+11/6)}{u+5/2}du+2\mu_2^{1/2}\int_0^{1/2}\frac{u^{9/2}(2-u)^{1/2}(u+11/6)}{u+5/2}du\\&\quad+\frac{\mu_1}{b_{n,3}}\int_{1/2}^{1}\frac{u^6(u-1/2)}{u+1/6}du+\frac{2\mu_2^{1/2}}{b_{n,3}}\int_{1/2}^{1}\frac{u^{9/2}(2-u)^{1/2}(u-1/2)}{u+1/6}du\\&\quad+\frac{\mu_1}{b_{n,4}}\int_{1}^{7/6}\frac{u^3(2-u)^3(u-1/2)}{u+1/6}du+\frac{2\mu_2^{1/2}}{b_{n,4}}\int_{1}^{7/6}\frac{u^{3/2}(2-u)^{7/2}(u-1/2)}{u+1/6}du\\&\quad+\mu_1\int_{7/6}^{3/2}\frac{u^3(2-u)^3(u-7/6)}{u-1/2}du+2\mu_2^{1/2}\int_{7/6}^{3/2}\frac{u^{3/2}(2-u)^{7/2}(u-7/6)}{u-1/2}du\\&\quad+\frac{\mu_1}{b_{n,4}}\int_{3/2}^{2}\frac{u^3(2-u)^3(u-3/2)}{u-5/6}du+\frac{2\mu_2^{1/2}}{b_{n,4}}\int_{3/2}^{2}\frac{u^{3/2}(2-u)^{7/2}(u-3/2)}{u-5/6}du-\frac{9}{4}\\&\approx\left(0.0497+\frac{0.0759}{b_{n,3}}+\frac{0.0087}{b_{n,4}}\right)\mu_1+\left(0.0475+\frac{0.2541}{b_{n,3}}+\frac{0.0046}{b_{n,4}}\right)\mu_2^{1/2}-\frac{9}{4}\end{aligned} \tag{3.14}$$

and the right side of the inequality (2.52) is

$$\frac{M_2}{H_2(\delta_2,c_2)}Q_{c_2}^{\delta_2}[H_1(\cdot,c_2)]+\frac{M_2}{H_2(d_2,\delta_2)}Q_{\delta_2}^{d_2}[H_2(d_2,\cdot)]=\frac{b_{n,3}-a_{n,3}}{4a_{n,3}^2}+\frac{29(b_{n,4}-a_{n,4})}{8a_{n,4}}. \tag{3.15}$$

Therefore (2.52) is satisfied for $j = 2$ if

$$\begin{aligned}&\left(0.0497+\frac{0.0759}{b_{n,3}}+\frac{0.0087}{b_{n,4}}\right)\mu_1+\left(0.0475+\frac{0.2541}{b_{n,4}}+\frac{0.0046}{b_{n,4}}\right)\mu_2^{1/2}\\&\quad>\frac{9}{4}+\frac{b_{n,3}-a_{n,3}}{4a_{n,3}^2}+\frac{29(b_{n,4}-a_{n,4})}{8a_{n,4}}.\end{aligned}\tag{3.16}$$

Hence, by Theorem 2.8, (3.8) is oscillatory if (3.13) and (3.16) hold. Particularly, let $a_{n,i} = b_{n,i}$, for $i = 1, 2, 3, 4$, condition (3.13) and (3.16) become a simple form

$$\begin{aligned}&\left(0.0497+\frac{0.0759}{b_{n,1}}+\frac{0.0087}{b_{n,2}}\right)\mu_1+\left(0.0475+\frac{0.2541}{b_{n,1}}+\frac{0.0046}{b_{n,2}}\right)\mu_2^{1/2}>\frac{9}{4},\\&\left(0.0497+\frac{0.0759}{b_{n,3}}+\frac{0.0087}{b_{n,4}}\right)\mu_1+\left(0.0475+\frac{0.2541}{b_{n,3}}+\frac{0.0046}{b_{n,4}}\right)\mu_2^{1/2}>\frac{9}{4}.\end{aligned}\tag{3.17}$$

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