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Research Article

Comparison Theorems for the Multidimensional BDSDEs and Applications

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A class of backward doubly stochastic differential equations (BDSDEs) are studied. We obtain a comparison theorem of these multidimensional BDSDEs. As its applications, we derive the existence of solutions for this multidimensional BDSDEs with continuous coefficients. We can also prove that this solution is the minimal solution of the BDSDE.

1. Introduction

Backward stochastic differential equations (BSDEs in short) were first introduced in [1] in order to give a probabilistic interpretation (Feynman-Kac formula) for the solutions of semilinear parabolic PDEs, one can see [2, 3]. Moreover, BSDEs have been considered with great interests because for their connections with mathematical finance [4] as well as stochastic optimal control and stochastic games [5, 6]. As we know, the comparison theorem is a very useful result in the theory of BSDEs, for example, it can be used to study viscosity solutions of partial differential equations. In 1992, Peng [2] gave the comparison of the one-dimensional BSDE. In 1994, Christel Geiß and Ralf Manthey [7] proved the comparison theorems for stochastic differential equations in finite and infinite dimensions.

After they introduced the theory of BSDEs, Pardoux and Peng [8] in 1994 brought forward a new kind of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals, that is, the equations involve both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t. They have proved the existence and uniqueness of solutions to BDSDEs under uniformly Lipschitz conditions on coefficients. That is, for a given terminal time $T > 0$, under the uniformly Lipschitz assumptions on coefficients f, g, for any square

integrable terminal value ξ, the following BDSDE has a unique solution pair (y_t, z_t) in the interval $[0, T]$:

$$y_t = \xi + \int_t^T f(s, y_s, z_s)ds + \int_t^T g(s, y_s, z_s)dB_s - \int_t^T z_s dW_s. \tag{1.1}$$

Pardoux and Peng [8] showed that BDSDEs can produce a probabilistic representation for certain quasilinear stochastic partial differential equations (SPDEs). Recently, [9] Zhang and Zhao, [10] Zhu and Han studied the infinite horizon BDSDEs. Many researchers do their work in this area (cf. [11–18] and the references therein). As we know, the comparison theorem is a very useful result in the theory of BDSDEs, for example, it can be used to study viscosity solutions of SPDEs [19]. Shi et al. proved the comparison theorem of these BDSDEs, but they studied the one-dimensional BDSDEs.

In this paper, we will prove the comparison theorem of the multidimensional BDSDEs. Then we study the multidimensional BDSDEs with continuous coefficients as an application of the comparison theorem.

The paper is organized as follows: in Section 2 we introduce some preliminaries and notations; in Section 3 we prove the comparison theorem of multidimensional BDSDEs; at the end, we give the multi-dimension BDSDEs with continuous coefficients in Section 4.

2. Preliminaries: The Existence and Uniqueness of BDSDEs and an Extension of Itô Formula

Notation 1. The Euclidean norm of a vector $x \in R^k$ will be denoted by $|x|$, and for a $d \times k$ matrix A, we define $\|A\| = \sqrt{TrAA^*}$, where A^* is the transpose of A. We also define (α, β) as the inner product of α and β.

For $a \in R^k$, let a_i be the ith component of a; $c \in R^{k\times d}$, let c_i be the ith row of c, let $c_{i,j}$ be an element for the ith row and the jth column of c. For $a^1, a^2 \in R^k$, we define

$$a^1 \geq a^2 \Longleftrightarrow a_j^1 \geq a_j^2, \quad j = 1, 2, \ldots, k. \tag{2.1}$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let T be an arbitrarily fixed positive constant throughout this paper. Let $\{W_t;\ 0 \leq t \leq T\}$ and $\{B_t;\ 0 \leq t \leq T\}$ be two mutually independent standard Brownian motions with values in R^d and R^l, respectively, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \tag{2.2}$$

where for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$.

Note that the collection $\{\mathcal{F}_t; t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a filtration.

For any $n \in N$, let $M^2(0, T; R^n)$ denote the set of (classes of $dP \otimes dt$ a.e. equal) n-dimensional jointly measurable stochastic processes $\{\varphi_t; t \in [0, T]\}$ which satisfy

(i) $\|\varphi\|_{M^2}^2 := E\int_0^T |\varphi_t|^2 dt < \infty$;

(ii) φ_t is $\mathcal{F}_t$-measurable, for a.e. $t \in [0, T]$.

Similarly, we denote by $S^2([0,T];R^n)$ the set of n-dimensional continuous stochastic processes $\{\varphi_t; t \in [0,T]\}$ which satisfy

(iii) $\|\varphi\|^2_{S^2} := E(\sup_{0\le t\le T}|\varphi_t|^2) < \infty$;

(iv) φ_t is $\mathcal{F}_t$-measurable, for any $t \in [0,T]$.

Obviously, $M^2(0,T;R^n)$ and $S^2([0,T];R^n)$ are Hilbert spaces.

Let

$$f : \Omega \times [0,T] \times R^k \times R^{k\times d} \longrightarrow R^k, \qquad g : \Omega \times [0,T] \times R^k \times R^{k\times d} \longrightarrow R^{k\times l} \tag{2.3}$$

be jointly measurable and such that for any $(y,z) \in R^k \times R^{k\times d}$.

(H1) $f(\cdot,y,z) \in M^2(0,T;R^k)$, $g(\cdot,y,z) \in M^2(0,T;R^{k\times l})$.

(H2) There exist constants $C > 0$ and $0 < \alpha < 1$ such that for any $(\omega,t) \in \Omega \times [0,T]$, (y_1,z_1), $(y_2,z_2) \in R^k \times R^{k\times d}$,

$$\begin{aligned} \left|f(t,y_1,z_1) - f(t,y_2,z_2)\right|^2 &\le C\left(\left|y_1-y_2\right|^2 + \|z_1-z_2\|^2\right) \\ \left\|g(t,y_1,z_1) - g(t,y_2,z_2)\right\|^2 &\le C\left|y_1-y_2\right|^2 + \alpha\|z_1-z_2\|^2. \end{aligned} \tag{2.4}$$

Given $\xi \in L^2(\Omega,\mathcal{F}_T,P;R^k)$, we consider the following backward doubly stochastic differential equation:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)dB_s - \int_t^T Z_s dW_s, \quad 0 \le t \le T. \tag{2.5}$$

Note that the integral with respect to $\{B_t\}$ is a "backward Itô integral" and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. The forward integrals were defined in [20] Da Prato and Zabczyk and [21] D. Nualart, and E. Pardoux. To see the backward one, for any $s \ge 0$ let $\{h(s)\}_{s\ge 0}$ be a stochastic process such that $\{h(s)\}$ is $\mathcal{F}_s$ measurable, and locally square integrable, that is, for any $0 \le a \le b < \infty$, $\int_a^b \|h(s)\|^2 ds < \infty$ almost surely. Since $\mathcal{F}_s$ is a backward filtration with respect to B, so from the one-dimensional backward Itô's integral and relation with forward integral, for $0 \le T \le T'$, we have $\int_t^T h(s)dBs = -\int_{T'-T}^{T'-t} h(T'-s)d\widehat{B}_s$ a.s, where $\widehat{B}_s = B_{T'-s} - B_{T'}$.

Definition 2.1. A pair of processes $(y,z) : \Omega\times[0,T] \to R^k \times R^{k\times d}$ is called a solution of BDSDE (2.5), if $(y,z) \in S^2([0,T];R^k) \times M^2(0,T;R^{k\times d})$ and satisfies BDSDE (2.5).

Proposition 2.2. *Under conditions (H1) and (H2), BDSDE* (2.5) *has a unique solution* $(Y,Z) \in S^2([0,T];R^k) \times M^2(0,T;R^{k\times d})$.

Proposition 2.3. *Let* $\alpha \in S^2([0,T];R^k)$, $\beta \in M^2(0,T;R^k)$, $\gamma \in M^2(0,T;R^{k\times l})$, $\delta \in M^2(0,T;R^{k\times d})$ *satisfy*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \delta_s dW_s, \quad 0 \le t \le T. \tag{2.6}$$

Then

$$
\begin{aligned}
|\alpha_t|^2 &= |\alpha_0|^2 + 2\int_0^t (\alpha_s, \beta_s)ds + 2\int_0^t (\alpha_s, \gamma_s dB_s) + 2\int_0^t (\alpha_s, \delta_s dW_s) \\
&\quad - \int_0^t \|\gamma_s\|^2 ds + \int_0^t \|\delta_s\|^2 ds, \\
E|\alpha_t|^2 &= E|\alpha_0|^2 + 2E\int_0^t (\alpha_s, \beta_s)ds - E\int_0^t \|\gamma_s\|^2 ds + E\int_0^t \|\delta_s\|^2 ds.
\end{aligned} \tag{2.7}
$$

This two propositions were derived in [8].

3. Comparison Theorem of Multidimensional BDSDEs

In this section, we will prove the comparison theorem of the multidimensional BDSDEs. Firstly, we give the definition of the indicator function for the positive and negative parts.

Definition 3.1. If $f : \Omega \mapsto \overline{R}$, then the positive part of f is defined by the formula $f^+ \doteq \max\{f, 0\}$. Similarly, the negative part of f is defined as $f^- \doteq \max\{-f, 0\} = -\min\{f, 0\}$

Note that both f^+ and f^- are nonnegative functions.

We consider the following k-dimensional BDSDEs: $(0 \le t \le T)$

$$
\begin{aligned}
y_t^1 &= \xi^1 + \int_t^T f^1\left(s, y_s^1, z_s^1\right)ds + \int_t^T g\left(s, y_s^1, z_s^1\right)dB_s - \int_t^T z_s^1 dW_s, \\
y_t^2 &= \xi^2 + \int_t^T f^2\left(s, y_s^2, z_s^2\right)ds + \int_t^T g\left(s, y_s^2, z_s^2\right)dB_s - \int_t^T z_s^2 dW_s,
\end{aligned} \tag{3.1}
$$

where $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, P; R^k)$, $f^1(\omega, t, y, z)$, $f^2(\omega, t, y, z) : \Omega \times [0, T] \times R^k \times R^{k\times d} \to R^k$, $g(\omega, t, y, z) : \Omega \times [0, T] \times R^k \times R^{k\times d} \to R^{k\times l}$. We assume ξ^1, ξ^2 and f^1, f^2, g satisfy the following conditions:

(i) $\xi^1 \ge \xi^2$ P-a.s.;

(ii) For all $j = 1, 2, \ldots, k$, for all $(\omega, t) \in \Omega \times [0, T]$, y^1, $y^2 \in R^k$, z^1, $z^2 \in R^{k\times d}$, $y_j^1 = y_j^2$, $z_j^1 = z_j^2$, $y_l^1 \ge y_l^2$, $l \ne j$

$$
f_j^1\left(\omega, t, y^1, z^1\right) \ge f_j^2\left(\omega, t, y^2, z^2\right); \tag{3.2}
$$

(iii) for all $(\omega, t) \in \Omega \times [0, T]$, y^1, $y^2 \in R^k$, z^1, $z^2 \in R^{k\times d}$

$$
\left|f_j^i(t, y^1, z^1) - f_j^i(t, y^2, z^2)\right|^2 \le C\left(\left|y^1 - y^2\right|^2 + \left\|z^1 - z^2\right\|^2\right), \quad i = 1, 2; \tag{3.3}
$$

(iv) for all $(\omega,t,y,z)\in\Omega\times[0,T]\times R^k\times R^{k\times d}$, $\exists h(\omega,t,\mu,\nu):\Omega\times[0,T]\times R\times R^d\to R^l$ such that

$$\begin{aligned} g_j(\omega,t,y,z) &= h_j(\omega,t,y_j,z_j),\\ \left|h_j\left(t,y_j^1,z_j^1\right)-h_j\left(t,y_j^2,z_j^2\right)\right|^2 &\le C\left|y_j^1-y_j^2\right|^2+\alpha\left|z_j^1-z_j^2\right|^2, \end{aligned} \tag{3.4}$$

where $C>0$, $0<\alpha<1$ are two constants.

From Proposition 2.2, under conditions (iii) and (iv), there exist two pairs of measurable processes $(y^1,z^1)\in S^2([0,T];R^k)\times M^2(0,T;R^{k\times d})$ and $(y^2,z^2)\in S^2([0,T];R^k)\times M^2(0,T;R^{k\times d})$ satisfying BDSDEs (3.1), respectively.

Theorem 3.2. *Assume the conditions (i)–(iv) hold, let (y^1,z^1) and (y^2,z^2) be solutions of BDSDEs (3.1), respectively. Then $y_t^1\ge y_t^2$, a.s. for all $t\in[0,T]$.*

Proof. for all $\varepsilon>0$, we define a function $\phi_\varepsilon(y):R\to R$

$$\phi_\varepsilon(y)=\begin{cases} y^2, & y\le 0,\\ y^2-\dfrac{1}{6\varepsilon}y^3, & 0\le y\le 2\varepsilon,\\ 2\varepsilon y-\dfrac{4}{3}\varepsilon^2, & y\ge 2\varepsilon. \end{cases} \tag{3.5}$$

Obviously, $\phi_\varepsilon(y)\in C^2(R)$ and $\phi_\varepsilon''(y)$ is bounded, for all $y\in R$,

$$\phi_\varepsilon(y)\longrightarrow\left|y^-\right|^2,\qquad \phi_\varepsilon''(y)\longrightarrow -2y^-,\qquad \phi_\varepsilon''(y)\longrightarrow 2\mathbf{I}_{\{y\le 0\}},\quad \text{when } \varepsilon\to 0. \tag{3.6}$$

We set

$$\overline{y}_t=y_t^1-y_t^2,\qquad \overline{z}_t=z_t^1-z_t^2,\qquad \overline{\xi}=\xi^1-\xi^2. \tag{3.7}$$

Then $(\overline{y}_t,\overline{z}_t)$ satisfy the following BDSDE:

$$\overline{y}_t=\overline{\xi}+\int_t^T\left[f^1\left(s,y_s^1,z_s^1\right)-f^2\left(s,y_s^2,z_s^2\right)\right]ds+\int_t^T\left[g\left(s,y_s^1,z_s^1\right)-g\left(s,y_s^2,z_s^2\right)\right]dB_s-\int_t^T\overline{z}_s dW_s. \tag{3.8}$$

Applying Itô's formula to $\phi_\varepsilon(\overline{y}_j(t))$, we get

$$\begin{aligned} \phi_\varepsilon\left(\overline{y}_j(t)\right)=\phi_\varepsilon\left(\overline{\xi}_j\right)&+\int_t^T\phi_\varepsilon'\left(\overline{y}_j(s)\right)\left[f_j^1\left(s,y_s^1,z_s^1\right)-f_j^2\left(s,y_s^2,z_s^2\right)\right]ds\\ &+\int_t^T\phi_\varepsilon'\left(\overline{y}_j(s)\right)\left[g_j\left(s,y_s^1,z_s^1\right)-g_j\left(s,y_s^2,z_s^2\right)\right]dB_s-\int_t^T\phi_\varepsilon'\left(\overline{y}_j(s)\right)\overline{z}_j(s)dW_s\\ &+\frac{1}{2}\int_t^T\phi'_\varepsilon\left(\overline{y}_j(s)\right)\left|g_j\left(s,y_s^1,z_s^1\right)-g_j\left(s,y_s^2,z_s^2\right)\right|^2ds-\frac{1}{2}\int_t^T\phi_\varepsilon''\left(\overline{y}_j(s)\right)\left|\overline{z}_j(s)\right|^2ds. \end{aligned} \tag{3.9}$$

Let $\varepsilon \to 0$, we get

$$\begin{aligned}\left|\overline{y}_j(t)^-\right|^2 &= \left|\overline{\xi_j^-}\right|^2 - \int_t^T 2\overline{y}_j(s)^-\left[f_j^1\left(s,y_s^1,z_s^1\right) - f_j^2\left(s,y_s^2,z_s^2\right)\right]ds \\ &\quad - \int_t^T 2\overline{y}_j(s)^-\left[g_j\left(s,y_s^1,z_s^1\right) - g_j\left(s,y_s^2,z_s^2\right)\right]dB_s + \int_t^T 2\overline{y}_j(s)^-\overline{z}_j(s)dW_s \\ &\quad + \int_t^T \mathbf{I}_{(\overline{y}_j(s)\le 0)}\left|g_j\left(s,y_s^1,z_s^1\right) - g_j\left(s,y_s^2,z_s^2\right)\right|^2 ds - \int_t^T \mathbf{I}_{(\overline{y}_j(s)\le 0)}\left|\overline{z}_j(s)\right|^2 ds.\end{aligned} \tag{3.10}$$

From (i), we have $\xi^1 - \xi^2 \ge 0$, a.s., so

$$E\left|\left(\xi^1 - \xi^2\right)^-\right|^2 = 0. \tag{3.11}$$

Since (y^1, z^1) and (y^2, z^2) are in $S^2([0,T];R^k) \times M^2(0,T;R^{k\times d})$, it easily follows that

$$E\int_t^T \overline{y}_j(s)^-\left[g_j\left(s,y_s^1,z_s^1\right) - g_j\left(s,y_s^2,z_s^2\right)\right]dB_s = 0, \tag{3.12}$$

$$E\int_t^T \overline{y}_j(s)^-\overline{z}_j(s)dW_s = 0. \tag{3.13}$$

Let

$$\begin{aligned}\bigtriangleup^1 &= -2\int_t^T \overline{y}_j(s)^-\left[f_j^1\left(s,y_s^1,z_s^1\right) - f_j^2\left(s,y_s^2,z_s^2\right)\right]ds\Bigg] \\ &= -2\int_t^T \overline{y}_j(s)^-\Big[f_j^1\left(y_1^1,\ldots,y_j^1,\ldots,y_k^1,z_1^1,\ldots,z_j^1,\ldots,z_k^1\right) \\ &\qquad -f_j^1\left(y_1^1+\overline{y_1^-},\ldots,y_j^2,\ldots,y_k^1+\overline{y_k^-},z_1^1,\ldots,z_j^2,\ldots,z_k^1\right)\Big]ds \\ &\quad -2\int_t^T \overline{y}_j(s)^-\Big[f_j^1\left(y_1^1+\overline{y_1^-},\ldots,y_j^2,\ldots,y_k^1+\overline{y_k^-},z_1^1,\ldots,z_j^2,\ldots,z_k^1\right) \\ &\qquad -f_j^2\left(y_1^2,\ldots,y_j^2,\ldots,y_k^2,z_1^2,\ldots,z_j^2,\ldots,z_k^2\right)\Big]ds \\ &= \bigtriangleup_1^1 + \bigtriangleup_2^1,\end{aligned} \tag{3.14}$$

where $\bigtriangleup_1^1$ is the first integral, and $\bigtriangleup_2^1$ is the second integral. Since

$$y_l^1 + \overline{y_l^-} \ge y_l^2, \quad l \ne j \tag{3.15}$$

and (ii) we have

$$f_j^1\left(y_1^1+\overline{y}_1^-,\ldots,y_j^2,\ldots,y_k^1+\overline{y}_k^-,z_1^1,\ldots,z_j^2,\ldots,z_k^1\right)\geq f_j^2\left(y_1^2,\ldots,y_j^2,\ldots,y_k^2,z_1^2,\ldots,z_j^2,\ldots,z_k^2\right). \tag{3.16}$$

So $\bigtriangleup_2^1\leq 0$.

From (iii) and Young's inequality, it follows that

$$\begin{aligned}
\bigtriangleup_1^1 &\leq \int_t^T 2\overline{y}_j^- L\left(\left|\overline{y}_1^-\right|+\cdots+\left|\overline{y}_{j-1}^-\right|+\left|\overline{y}_j\right|+\left|\overline{y}_{j+1}^-\right|+\cdots+\left|\overline{y}_k^-\right|+\left|\overline{z}_j\right|\right)ds\\
&= \int_t^T L\left(2\overline{y}_j^-\left|\overline{y}_1^-\right|+\cdots+2\overline{y}_j^-\left|\overline{y}_{j-1}^-\right|+2\overline{y}_j^-\left|\overline{y}_j\right|+2\overline{y}_j^-\left|\overline{y}_{j+1}^-\right|+\cdots+2\overline{y}_j^-\left|\overline{y}_k^-\right|+2\overline{y}_j^-\left|\overline{z}_j\right|\right)ds\\
&= \int_t^T L\left(2\overline{y}_j^-\left|\overline{y}_1^-\right|+\cdots+2\overline{y}_j^-\left|\overline{y}_{j-1}^-\right|+2\left|\overline{y}_j\right|^2+2\overline{y}_j^-\left|\overline{y}_{j+1}^-\right|+\cdots+2\overline{y}_j^-\left|\overline{y}_k^-\right|+2\overline{y}_j^-\left|\overline{z}_j\right|\right)ds\\
&\leq \int_t^T L\left(\left|\overline{y}_j^-\right|^2+\left|\overline{y}_1^-\right|^2+\cdots+\left|\overline{y}_j^-\right|^2+\left|\overline{y}_{j-1}^-\right|^2+2\left|\overline{y}_j^-\right|^2+\left|\overline{y}_j^-\right|^2+\left|\overline{y}_{j+1}^-\right|^2\right.\\
&\qquad\left.+\cdots+\left|\overline{y}_j^-\right|^2+\left|\overline{y}_k^-\right|^2+\frac{L}{1-\alpha}\left|\overline{y}_j^-\right|^2+\frac{1-\alpha}{L}\mathbf{I}_{(\overline{y}_j\leq 0)}\left|\overline{z}_j\right|^2\right)ds\\
&= \left(Lk+\frac{L^2}{1-\alpha}\right)\int_t^T\left|\overline{y}_j^-\right|^2ds+L\sum_{l=1}^k\int_t^T\left|\overline{y}_l^-\right|^2ds+(1-\alpha)\int_t^T\mathbf{I}_{(\overline{y}_j\leq 0)}\left|\overline{z}_j\right|^2\Bigg)ds,
\end{aligned} \tag{3.17}$$

where $L>0$ only depends on the Lipschitz constant C in (iii). From (iv) we deduce

$$\begin{aligned}
\bigtriangleup^2 &= \int_t^T\mathbf{I}_{(\overline{y}_j(s)\leq 0)}\left|g_j\left(s,y_s^1,z_s^1\right)-g_j\left(s,y_s^2,z_s^2\right)\right|^2ds\\
&= \int_t^T\mathbf{I}_{(\overline{y}_j(s)\leq 0)}\left|h_j\left(s,y_j^1(s),z_j^1(s)\right)-h_j\left(s,y_j^2(s),z_j^2(s)\right)\right|^2ds\\
&\leq \int_t^T\mathbf{I}_{(\overline{y}_j(s)\leq 0)}\left[C\left|y_j^1(s)-y_j^2(s)\right|^2+\alpha\left|z_j^1(s)-z_j^2(s)\right|^2\right]ds\\
&= C\int_t^T\left|\overline{y}_j^-\right|^2ds+\alpha\int_t^T\mathbf{I}_{(\overline{y}_j(s)\leq 0)}\left|\overline{z}_j\right|^2ds.
\end{aligned} \tag{3.18}$$

So there $\exists M>0$ only depends on C,α,k,d, such that

$$\bigtriangleup^1+\bigtriangleup^2\leq M\sum_{l=1}^k\int_t^T\left|\overline{y}_l^-\right|^2ds+\int_t^T\mathbf{I}_{(\overline{y}_j(s)\leq 0)}\left|\overline{z}_j\right|^2ds. \tag{3.19}$$

Taking expectation on both sides of (3.10) and noting (3.11)–(3.19), we get

$$E\left|\overline{y}_j(t)^-\right|^2 \le ME\sum_{l=1}^{k}\int_t^T \left|\overline{y_l}^-\right|^2 ds. \tag{3.20}$$

We sum from 1 to k and get $\sum_{j=1}^{k} E|\overline{y}_j(t)^-|^2 \le MkE\sum_{l=1}^{k}\int_t^T |\overline{y_l}^-|^2 ds$. That is,

$$E\sum_{j=1}^{k}\left|\overline{y}_j(t)^-\right|^2 \le Mk\int_t^T E\sum_{j=1}^{k}\left|\overline{y_j}^-\right|^2 ds. \tag{3.21}$$

By Gronwall's inequality, it follows that

$$E\sum_{j=1}^{k}\left|\overline{y}_j(t)^-\right|^2 = 0, \quad \forall t \in [0,T].\ dt - \text{a.s.} \tag{3.22}$$

So $\overline{y}_j(t) \ge 0\ dt \otimes dP - \text{a.s.}\ j = 1,2,\ldots,k$. □

Example 3.3. If conditions (ii) and (iv) of the theorem are replaced by the following: (ii′) for all $j = 1,2,\ldots,k$, $\forall(\omega,t) \in \Omega \times [0,T]$, $y \in R^k$, $z^1, z^2 \in R^{k\times d}$, $z_j^1 = z_j^2$,

$$f_j^1\left(\omega,t,y,z^1\right) \ge f_j^2\left(\omega,t,y,z^2\right), \tag{3.23}$$

and (iv′) for all $(\omega,t,y,z) \in \Omega \times [0,T] \times R^k \times R^{k\times d}$,

$$\left\|g(t,y_1,z_1) - g(t,y_2,z_2)\right\|^2 \le C\left|y_1 - y_2\right|^2 + \alpha\|z_1 - z_2\|^2, \tag{3.24}$$

where $C > 0, 0 < \alpha < 1$ are two constants. By the following example we can see, the comparison theorem is untenable at the above conditions.

If $m = 2$, $f^1(y,z) = f^2(y,z) = f(y,z) = Ay$, $g(y,z) = 0$, where $A_{2\times 2}$ is a constant coefficient matrix, obviously f, g satisfy (ii′) and (iv′). We consider the following k-dimensional BDSDEs: $(0 \le t \le T)$

$$y_t^1 = \xi^1 + \int_t^T Ay_s^1 ds - \int_t^T z_s^1 dW_s, \qquad y_t^2 = \xi^2 + \int_t^T Ay_s^2 ds - \int_t^T z_s^2 dW_s, \tag{3.25}$$

then

$$y_t^1 = E\left[\frac{e^{A(T-t)}\xi^1}{\mathcal{F}_t^w}\right], \qquad y_t^2 = E\left[\frac{e^{A(T-t)}\xi^2}{\mathcal{F}_t^w}\right]. \tag{3.26}$$

Already know $\xi^1 \ge \xi^2$, P a.s., to be sure $y_t^1 \ge y_t^2$, $dt \otimes dP$ a.s., this requires

$$e^{A(T-t)}\xi^1 \ge e^{A(T-t)}\xi^2, \quad dt \otimes dP \text{ a.s.}, \tag{3.27}$$

then

$$e^{A(T-t)}\left(\xi^1-\xi^2\right)\geq 0,\quad dt\otimes dP \text{ a.s.} \tag{3.28}$$

Let

$$A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tag{3.29}$$

then

$$e^{A(T-t)}=\begin{bmatrix}\cos(T-t) & \sin(T-t)\\ -\sin(T-t) & \cos(T-t)\end{bmatrix}. \tag{3.30}$$

Here f, g satisfy (ii′) and (iv′) does not satisfy (ii) and (iv). Let $T=2\pi, t\in[0,\pi/2]$

$$e^{A(T-t)}=\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}. \tag{3.31}$$

Obviously we cannot guarantee, for all $\xi^1\geq\xi^2$, there is $y_t^1\geq y_t^2$, $dt\otimes dP$ a.s.

4. The Multidimensional BDSDEs with Continuous Coefficients

In this section, we do not assume a Lipschitz property. Our main interest in this section is to study the multidimensional BDSDEs with continuous coefficients as an application of the comparison theorem obtained in Section 3. First, we give the definition of quasimonotonously increasing mapping [22].

Definition 4.1. A mapping $f:R^k\to R^k$ is said to be quasi-monotonously increasing mapping, if for each $j=1,2,\ldots,k$, $f_j(x)\leq f_j(y)$ provided $x_j=y_j$, and $x_l\leq y_l$, $l\neq j$.

We give our main result.

Theorem 4.2. *Assume $f:\Omega\times[0,T]\times R^k\times R^{k\times d}\to R^k$ and $g:\Omega\times[0,T]\times R^k\times R^{k\times d}\to R^{k\times l}$ are jointly measurable functions and satisfy.*

(1) There exists $r(\omega,t,y,\gamma):\Omega\times[0,T]\times R^k\times R^d\to R^k$, such that for all $(\omega,t,y,z)\in\Omega\times[0,T]\times R^k\times R^{k\times d}$,

$$f_j(\omega,t,y,z)=r_j(\omega,t,y,z_j). \tag{4.1}$$

(2) Linear growth: for all $(\omega,t,y,z)\in\Omega\times[0,T]\times R^k\times R^{k\times d}$, $\exists K>0$, such that

$$\left|f(\omega,t,y,z)\right|\leq K(1+|y|+\|z\|). \tag{4.2}$$

(3) Continuous: for fixed ω and t, $f(\omega,t,\cdot,\cdot)$ is continuity.

(4) Quasimonotonously increasing: for fixed ω, t and z, $f(\omega,t,\cdot,z)$ is quasi-monotonously increase, that is for all $j = 1,2,\ldots,k$, such that

$$f_j\left(\omega,t,y^1,z\right) \geq f_j\left(\omega,t,y^2,z\right), \quad \forall y^1,y^2 \in R^k, \qquad y_j^1 = y_j^2, \qquad y_l^1 \geq y_l^2, \quad l \neq j; \tag{4.3}$$

(5) Function g satisfies the (iv) of Theorem 3.2, that is for all $(\omega,t,y,z) \in \Omega \times [0,T] \times R^k \times R^{k\times d}$, $\exists h(\omega,t,\mu,\nu) : \Omega \times [0,T] \times R \times R^d \to R$ such that $g_j(\omega,t,y,z) = h_j(\omega,t,y_j,z_j)$ and

$$\left|h_j\left(t,y_j^1,z_j^1\right) - h_j\left(t,y_j^2,z_j^2\right)\right|^2 \leq C\left|y_j^1 - y_j^2\right|^2 + \alpha\left|z_j^1 - z_j^2\right|^2, \tag{4.4}$$

where $C > 0$, $0 < \alpha < 1$ are two constants. Then, if $\xi \in L^2(\Omega,\mathcal{F}_T,P;R^k)$, BDSDE (2.5) has a solution $(y,z) \in S^2([0,T];R^k) \times M^2(0,T;R^{k\times d})$.

We give a lemma, this method was first introduced by [23] Barlow and Perkins. For fixed $(\omega,t) \in \Omega \times [0,T]$, we define the sequence $f^n(\omega,t,y,z) : \Omega \times [0,T] \times R^k \times R^{k\times d} \to R^k$ as the following: $j = 1,2,\ldots,k$.

$$f_j^n(\omega,t,y,z) \doteq \inf_{x\in R^k,\gamma\in R^{k\times d}} \left\{ f_j(\omega,t,x,\gamma) + \sum_{l=1,l\neq j}^{k} n(y_l - x_l)^+ + n\left|y_j - x_j\right| + n\left\|z-\gamma\right\| \right\}. \tag{4.5}$$

Lemma 4.3.

Let $n \geq K$ and f^n be define as in (4.5), then f^n satisfies the following properties.

(1) For all $(\omega,t,y,z) \in \Omega \times [0,T] \times R^k \times R^{k\times d}$,

$$f_j^n(\omega,t,y,z) \leq f_j(\omega,t,y,z). \tag{4.6}$$

(2) Uniformly linear growth: for all $(\omega,t,y,z) \in \Omega \times [0,T] \times R^k \times R^{k\times d}$,

$$\left|f_j^n(\omega,t,y,z)\right| \leq K(1 + |y| + |z_j|). \tag{4.7}$$

(3) Quasimonotonous increase with respect to y: for all $(\omega,t) \in \Omega \times [0,T]$, $z^1,z^2 \in R^{k\times d}$

$$f_j^n\left(\omega,t,y^1,z^1\right) \geq f_j^n\left(\omega,t,y^2,z^2\right), \quad \forall y_j^1 = y_j^2,\ z_j^1 = z_j^2,\ y_l^1 \geq y_l^2,\ l \neq j. \tag{4.8}$$

(4) Monotonous increase with respect to n: for all $(\omega,t,y,z) \in \Omega \times [0,T] \times R^k \times R^{k\times d}$,

$$f_j^n(\omega,t,y,z) \leq f_j^{n+1}(\omega,t,y,z). \tag{4.9}$$

(5) Lipschitz condition: for all $(\omega,t) \in \Omega \times [0,T]$, (y^1,z^1), $(y^2,z^2) \in R^k \times R^{k\times d}$,

$$\left|f_j^n\left(\omega,t,y^1,z^1\right) - f_j^n\left(\omega,t,y^2,z^2\right)\right| \leq n\left(\left|y^1 - y^2\right| + \left\|z^1 - z^2\right\|\right), \tag{4.10}$$

(6) Strong convergence: for all $(\omega,t)\in\Omega\times[0,T]$, (y,z), $(y^n,z^n)\in R^k\times R^{k\times d}$, $n=1,2,\ldots$, *if* $y^n\to y$, $z^n\to z$, *when* $n\to\infty$. *Then*

$$f^n(\omega,t,y^n,z^n)\longrightarrow f(\omega,t,y,z). \tag{4.11}$$

We consider the following k-dimensional BDSDEs

$$y_t^n=\xi+\int_t^T f^n(s,y_s^n,z_s^n)ds+\int_t^T g(s,y_s^n,z_s^n)dB_s-\int_t^T z_s^n dW_s, \tag{4.12}$$

$$U_t=\xi+\int_t^T F(s,U_s,V_s)ds+\int_t^T g(s,U_s,V_s)dB_s-\int_t^T V_s dW_s, \tag{4.13}$$

where $F:\Omega\times[0,T]\times R^k\times R^{k\times d}\to R^k$, and

$$F_j(\omega,t,y,z)=K\left(1+|y|+|z_j|\right),\quad j=1,2,\ldots,k. \tag{4.14}$$

From Lemma 4.3, for all $(\omega,t)\in\Omega\times[0,T]$, (y^1,z^1), $(y^2,z^2)\in R^k\times R^{k\times d}$, $y_j^1=y_j^2$, $z_j^1=z_j^2$, $y_l^1\geq y_l^2$, $l\neq j$, we have

$$\begin{aligned}
f_j^n\left(t,y^2,z^2\right)&\leq f_j^{n+1}\left(t,y^2,z^2\right)\\
&\leq f_j^{n+1}\left(t,y^1,z^1\right)\\
&\leq K\left(1+\left|y^1\right|+\left|z_j^1\right|\right)\\
&=F_j\left(t,y^1,z^1\right),\quad n=K,K+1,\ldots.
\end{aligned} \tag{4.15}$$

From Pardoux and Peng [8], BDSDE (4.12) and (4.13) has a unique solution

$$(y^n,z^n)\in S^2\left([0,T];R^k\right)\times M^2\left(0,T;R^{k\times d}\right),\qquad (U,V)\in S^2\left([0,T];R^k\right)\times M^2\left(0,T;R^{k\times d}\right). \tag{4.16}$$

From the comparison Theorem 3.2, for all $n\geq m\geq K$,

$$y_j^m(t)\leq y_j^n(t)\leq U_j(t),\quad dt\otimes dP-\text{a.s.} \tag{4.17}$$

Lemma 4.4. *There exists constant* $A>0$ *depending only on* K,C,α,T *and* ξ, *such that*

$$\forall n\geq K,\quad \left\|y^n\right\|_{S^2}\leq A,\qquad \left\|z^n\right\|_{M^2}\leq A,\qquad \|U\|_{S^2}\leq A,\qquad \|V\|_{M^2}\leq A. \tag{4.18}$$

Lemma 4.5. $\{(y^n, z^n)\}_{n=1}^{\infty}$ *converges in* $S^2([0,T];R^k) \times M^2(0,T;R^{k\times d})$.

The proof of the Lemmas 4.4 and 4.5 is similar to Lemma 4.2 and Lemma 4.3 in [19], so we omit it.

We give the proof of Theorem 4.2.

Proof. For all $n \geq n_0 \geq K$, we have $y^{n_0} \leq y^n \leq U$, and y^n converges in $S^2([0,T];R^k)$, $dt \otimes dP$ – a.s. to $y \in S^2([0,T];R^k)$.

On the other hand, since z^n converges in $M^2(0,T;R^{k\times d})$ to z, we can assume, choosing a subsequence if needed, that $z^n \to z$ $dt \otimes dP$ – a.s. and $\overline{G}_t = \sup_n \|z_t^n\|$ is $dt \otimes dP$ integrable. Therefore, from (3.11) and (2.5) of the Lemma 4.3, we get for almost all ω,

$$\begin{aligned} f^n(t, y_t^n, z_t^n) &\longrightarrow f(t, y_t, z_t), \quad (n \to \infty)\ dt - \text{a.e.} \\ \left|f^n(t, y_t^n, z_t^n)\right| &\leq K\left(1 + \sup_n \left|y_t^n\right| + \sup_n \left\|z_t^n\right\|\right) \\ &= K\left(1 + \sup_n \left|y_t^n\right| + \overline{G}_t\right) \in L^1([0,T];dt). \end{aligned} \tag{4.19}$$

Thus, for almost ω and uniformly in t, it holds that

$$\int_t^T f^n(s, y_s^n, z_s^n)ds \longrightarrow \int_t^T f(s, y_s, z_s)ds, \quad (n \to \infty). \tag{4.20}$$

From the continuity properties of the stochastic integral, it follows that

$$\begin{aligned} &\sup_{0\leq t\leq T}\left|\int_t^T z_s^n dW_s - \int_t^T z_s dW_s\right| \longrightarrow 0 \quad \text{in probability,} \\ &\sup_{0\leq t\leq T}\left|\int_t^T g(s, y_s^n, z_s^n)dB_s - g(s, y_s, z_s)dB_s\right| \longrightarrow 0 \quad \text{in probability.} \end{aligned} \tag{4.21}$$

Choosing, again, a subsequence, we can assume that the above convergence is P-a.s. Finally,

$$\begin{aligned} \left|y_t^n - y_t^m\right| \leq &\int_t^T \left|f^n(s, y_s^n, z_s^n) - f^m(s, y_s^m, z_s^m)\right|ds \\ &+ \left|\int_t^T \left(g(s, y_s^n, z_s^n) - g(s, y_s^m, z_s^m)\right)dB_s\right| + \left|\int_t^T (z_s^n - z_s^m)dW_s\right| \end{aligned} \tag{4.22}$$

and taking limits in m and supremum over t, we get

$$\begin{aligned}\sup_{0\le t\le T}\left|y_t^n - y_t\right| \le{}& \int_0^T \left|f^n(s,y_s^n,z_s^n) - f(s,y_s,z_s)\right|ds \\ &+ \sup_{0\le t\le T}\left|\int_t^T \left(g(s,y_s^n,z_s^n) - g(s,y_s,z_s)\right)dB_s\right| \\ &+ \sup_{0\le t\le T}\left|\int_t^T (z_s^n - z_s)dW_s\right| \quad P-a.s.\end{aligned} \tag{4.23}$$

From which it follows that y^n converges uniformly in t to y (in particular, y is a continuous process). Note that y^n is monotone, therefore, we actually have the uniform convergence for the entire sequence and not just for a subsequence. Taking limits in (4.12), we deduce that (y,z) is the solution of (2.5). □

From the comparison theorem, we can get a result.

Lemma 4.6. *y is the minimal solution of BDSDE* (2.5).

Proof. If let $(\widehat{y},\widehat{z}) \in S^2([0,T];R^k) \times M^2(0,T;R^{k\times d})$ be any solution of (2.5), from the comparison Theorem 3.2, we get that $y^n \le \widehat{y}$, for all $n \in N$, and therefore $y \le \widehat{y}$. That is, y is the minimal solution of BDSDE (2.5). □

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