Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 343794, 13 pages
doi:10.1155/2012/343794

Research Article

On a Dual Model with Barrier Strategy

Yuzhen Wen and Chuancun Yin

School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China

Correspondence should be addressed to Yuzhen Wen, wenyzhen@163.com

Received 5 February 2012; Revised 16 April 2012; Accepted 20 April 2012

Academic Editor: Laurent Gosse

We consider the dual of the generalized Erlang(n) risk model with a barrier dividend strategy. We derive integro-differential equations with boundary conditions satisfied by the expectation of the sum of discounted dividends until ruin and the moment-generating function of the discounted dividend payments until ruin, respectively. The results are illustrated by several examples.

1. Introduction

Many interesting results have been obtained on a model that is dual to the classical insurance risk model in recent years. See, for example, Albrecher et al. [1], Ng [2], Avanzi et al. [3], and Avanzi and Gerber [4]. In the classical dual model (see Grandell [5]), the surplus at time t is

$$U(t) = u - ct + \sum_{k=1}^{N(t)} Z_k \equiv u - ct + S(t), \quad t \geq 0, \tag{1.1}$$

where u and c are constants, $u \geq 0$ is the initial surplus and $c > 0$ is the rate of expenses, $S(t) = \sum_{k=1}^{N(t)} Z_k$ is the aggregate positive gains process, $\{Z_k\}_{k\geq 1}$ is a sequence of independent and identically distributed nonnegative random variables with a common probability distribution function $P(x)$, and $\{N(\mathrm{t})\}$ is a Poisson process with rate λ. Moreover, it is assumed that $\{N(t)\}$ and $\{Z_k\}_{k\geq 1}$ are independent. In (1.1), the expected increment of the surplus per unit time is

$$\mu = E[S(1)] - c = \lambda \int_0^{+\infty} xp(x)dx - c. \tag{1.2}$$

It is assumed that $\mu > 0$.

In the model (1.1), the premium rate is negative, causing the surplus to decrease. Claims, on the other hand, cause the surplus to jump up. Thus the premium rate should be viewed as an expense rate and claims should be viewed as profits or gains. While not very popular in insurance mathematics, this model has appeared in various literature (see Cramér [6], Seal [7], Tákacs [8], and the references cited therein). In Avanzi et al. [3], the authors studied the expected total discounted dividends until ruin for the dual model under the barrier strategy by means of integro-differential equations.

Recently, the research to models with two-sided jumps has been attracting a lot of attention in applied probability. For example, Perry et al. [9] studied the one- and two-sided first exit problems for a compound Poisson process with negative and positive jumps and linear deterministic decrease between jumps and assumed that the jumps have hyperexponential distributions. Kou and Wang [10] used a double exponential jump diffusion process to model the asset return. Asmussen et al. [11] considered the stock price models as an exponential Lévy process with phase-type jumps in two directions. For some related work see, among others, Jacobsen [12], Dong and Wang [13], Dong and Wang [14], Cai et al. [15], Zhang et al. [16], Chi and Lin [17], Cai and Kou [18], and the references therein.

Motivated by some related work mentioned above, we consider a more general risk process. We will assume that the number of gains up to time t is an ordinary renewal process:

$$N(t) = \max\{k \geq 1 : W_1 + W_2 + \cdots + W_k \leq t\}, \tag{1.3}$$

where the random variables $\{W_i\}_{i\geq 1}$ are independent and identically generalized Erlang(n)-distributed, that is, the W_i's are distributed as the sum of n independent and exponentially distributed random variables:

$$W_i = \xi_1 + \xi_2 + \cdots + \xi_n, \quad i = 1, 2, \ldots, \tag{1.4}$$

where ξ_j $(j = 1, 2, \ldots, n)$ may have different exponential parameters $\lambda_j > 0$. We also assumed that the jumps are two-sided. The upward jumps can be interpreted as the random gains of the company, while the downward jumps are interpreted as the random loss of the company. The common density of the jumps is given by

$$p(x) = pp_1(x)I_{\{x\geq 0\}} + qp_2(-x)I_{\{x<0\}}, \tag{1.5}$$

where $p_1(x)$ and $p_2(x)$ are two arbitrary probability density functions on $[0, \infty)$ and $p, q \geq 0$ are two constants such that $p + q = 1$. Denoted are the probability distribution functions of $p_1(x)$ and $p_2(x)$, respectively, by $P_1(x)$ and $P_2(x)$.

We then consider the modification of the surplus process by a barrier strategy with a barrier b. Whenever the surplus exceeds the barrier, the excess is paid out immediately as a dividend. But when $U(t)$ is below b, no dividends are paid. The modified surplus at time t is given by

$$X(t) = U(t) - D(t), \quad t \geq 0, \tag{1.6}$$

where $D(t)$ denote the aggregate dividends paid between time 0 and time t, that is,

$$D(t) = \left(\max_{0\le s\le t} U(s) - b\right)^{+}. \tag{1.7}$$

Let

$$T = \inf\{t \ge 0 : X(t) < 0\} \tag{1.8}$$

be the time of ruin for the modified surplus X, and let

$$D = \int_0^T e^{-\delta t} dD(t) \tag{1.9}$$

be the sum of the discounted dividend payments, where $\delta > 0$ is the force of interest for valuation.

In this paper, we consider the expectation and the moment-generating function of the sum of the discounted dividends until ruin. In Section 2, we derive an integro-differential equation with boundary conditions for the expectation of the discounted dividends until ruin. In Section 3, we obtain an integro-differential equation with boundary conditions for the moment-generating function of the discounted dividend payments until ruin.

2. Expectation of the Discounted Dividends

Denote by $V(u;b)$ the expectation of the discounted dividends until ruin if the barrier strategy with parameter b is applied:

$$V(u;b) = E[D \mid X(0) = u], \quad 0 \le u \le b. \tag{2.1}$$

Note that

$$V(u;b) = u - b + V(b;b), \quad u > b. \tag{2.2}$$

Let $\partial/\partial u$ denote the differentiation operator with respect to u. And we define $\prod_{j=2}^{1} \cdot = 1$.

Theorem 2.1. *The function $V(u;b)$ satisfies the following integro-differential equation:*

$$\begin{aligned}\left[\prod_{j=1}^{n}\left(\frac{c}{\lambda_j}\frac{\partial}{\partial u} + 1 + \frac{\delta}{\lambda_j}\right)\right] V(u;b) &= p\int_0^{b-u} V(u+x;b)p_1(x)dx \\ &\quad + p\int_{b-u}^{\infty}[1 - P_1(x)]dx + pV(b;b)[1 - P_1(b-u)] \\ &\quad + q\int_0^{u} V(u-x;b)p_2(x)dx, \quad 0 < u < b\end{aligned} \tag{2.3}$$

with boundary conditions

$$V(0;b) = 0, \tag{2.4}$$

$$\left[\prod_{j=2}^{k}\left(\frac{c}{\lambda_{j-1}}\frac{\partial}{\partial u} + 1 + \frac{\delta}{\lambda_{j-1}}\right)\right]\frac{\partial V(u;b)}{\partial u}\bigg|_{u=b} = 1, \quad k = 1,2,\ldots,n. \tag{2.5}$$

Proof. We let $V_j(u;b)$ denote the expectation of the discounted dividends if the risk process is in state j $(j = 1,\ldots,n)$. Eventually, we are interested in $V(u;b) = V_1(u;b)$. Conditioning on the occurrence of a (sub-) claim within an infinitesimal time interval, we obtain for $0 \le u < b$ and $j = 1,\ldots,n-1$,

$$V_j(u;b) = e^{-\delta dt}\left\{P\left(\xi_j > dt\right)V_j(u - cdt;b) + P\left(\xi_j \le dt\right)V_{j+1}(\mathrm{u} - cdt;b)\right\}. \tag{2.6}$$

Note that we have

$$\begin{aligned} e^{-\delta dt} &= 1 - \delta dt + o(dt), \\ P\left(\xi_j > dt\right) &= 1 - \lambda_j dt + o(dt), \\ P\left(\xi_j \le dt\right) &= \lambda_j dt + o(dt), \\ V_j(u - cdt;b) &= V_j(u;b) - c\frac{\partial V_j(u;b)}{\partial u}dt + o(dt). \end{aligned} \tag{2.7}$$

Substituting these formulas into (2.6), after some careful calculations, we have for $j = 1,\ldots,n-1$

$$\lambda_j V_{j+1}(u;b) = \left[\left(\lambda_j + \delta\right) + c\frac{\partial}{\partial u}\right]V_j(u;b). \tag{2.8}$$

For $j = n$, we have

$$\begin{aligned} V_n(u;b) &= e^{-\delta dt}\left[P(\xi_n > dt)V_n(u - cdt;b) + P(\xi_n \le dt)\int_{-\infty}^{\infty} V(u + x;b)p(x)dx\right] + o(dt) \\ &= V_n(u;b) - (\delta + \lambda_n)V_n(u;b)dt - c\frac{\partial V_n(u;b)}{\partial u}dt + p\lambda_n dt \\ &\quad \times \left[\int_0^{b-u} V(u + x;b)p_1(x)dx + \int_{b-u}^{\infty}(u - b + x)p_1(x)dx + V(b;b)(1 - P_1(b - u))\right] \\ &\quad + q\lambda_n dt\int_{-u}^{0} V(u + x;b)p_2(-x)dx + o(dt) \end{aligned}$$

$$
\begin{aligned}
&= V_n(u;b) - (\delta + \lambda_n) V_n(u;b)dt - c\frac{\partial V_n(u;b)}{\partial u}dt \\
&\quad + p\lambda_n dt\left[\int_0^{b-u} V(u+x;b)p_1(x)dx + \int_{b-u}^{\infty}[1-P_1(x)]dx + V(b;b)(1-P_1(b-u))\right] \\
&\quad + q\lambda_n dt\int_0^u V(u-x;b)p_2(x)dx + o(dt),
\end{aligned}
\tag{2.9}
$$

which leads to

$$
\begin{aligned}
\left[(\lambda_n+\delta) + c\frac{\partial}{\partial u}\right]V_n(u;b) &= \lambda_n p\int_0^{b-u} V(u+x;b)p_1(x)dx + \lambda_n p\int_{b-u}^{\infty}[1-P_1(x)]dx \\
&\quad + \lambda_n pV(b;b)[1-P_1(b-u)] + \lambda_n q\int_0^u V(u-x;b)p_2(x)dx.
\end{aligned}
\tag{2.10}
$$

It follows from (2.8) that

$$
V_{j+1}(u;b) = \left[\frac{(\lambda_j+\delta) + c(\partial/\partial u)}{\lambda_j}\right]V_j(u;b), \quad (j = 1,2,\dots,n-1) \tag{2.11}
$$

and subsequently

$$
V_n(u;b) = \left[\prod_{j=1}^{n-1}\frac{(\lambda_j+\delta) + c(\partial/\partial u)}{\lambda_j}\right]V_1(u;b), \tag{2.12}
$$

which together with (2.10) yields (2.3).

Since the ruin is immediate if $u = 0$, we have the boundary condition (2.4).

For $u = b$, we obtain analogously for $j = 1,2,\dots,n-1$ that

$$
V_j(b;b) = e^{-\delta dt}\left\{P(\xi_j > dt)V_j(b;b) - cdt + P(\xi_j \le dt)V_{j+1}(b;b)\right\}, \tag{2.13}
$$

which by Taylor expansion leads to

$$
\lambda_j V_{j+1}(b;b) = (\lambda_j+\delta)V_j(b;b) + c. \tag{2.14}
$$

Comparing these equations with the corresponding ones in (2.8), the continuity of $V_j(u;b)$ at $u = b$ then implies that

$$
\left.\frac{\partial V_j(u;b)}{\partial u}\right|_{u=b} = 1, \quad j = 1,2,\dots,n-1. \tag{2.15}
$$

Similarly, one can verify that (2.15) also holds for $j = n$. For $j = 1$, (2.15) is equivalent to (2.5) with $k = 1$. It follows from (2.11) that

$$V_k(u;b) = \left[\prod_{j=2}^{k} \frac{(\lambda_{j-1} + \delta) + c(\partial/\partial u)}{\lambda_{j-1}}\right] V(u;b), \quad k = 2,3,\dots,n. \tag{2.16}$$

Applying the operator $\partial/\partial u$ to both sides of the above equation, we get

$$\left[\prod_{j=2}^{k} \frac{(\lambda_{j-1} + \delta) + c(\partial/\partial u)}{\lambda_{j-1}}\right] \frac{\partial V(u;b)}{\partial u}\bigg|_{u=b} = \frac{\partial V_k(u;b)}{\partial u}\bigg|_{u=b} = 1, \quad k = 2,3,\dots,n, \tag{2.17}$$

which is the boundary condition (2.5). □

Remark 2.2. When the gains waiting time W_i have an exponential distribution with $k(y) = \lambda e^{-\lambda y}$ for $\lambda > 0$, $y \geq 0$, $q = 0$, $p = 1$, and $p(x) = p_1(x)$, we can get the integro-differential equation of $V(u;b)$:

$$\begin{aligned} cV'(u;b) + (\lambda + \delta)V(u;b) &= \lambda \int_0^{b-u} V(u+x;b)p(x)dx \\ &\quad + \lambda \int_{b-u}^{\infty} [1 - P(x)]dx + \lambda V(b;b)[1 - P(b-u)]. \end{aligned} \tag{2.18}$$

This is the result (2.3) in Avanzi et al. [3].

Example 2.3. For $n = 1$, $\lambda_1 := \lambda$, let $p_1(x) = \beta_1 e^{-\beta_1 x}$, $p_2(x) = \beta_2 e^{-\beta_2 x}$, $x > 0$, where β_1 and β_2 are two positive constants. Then

$$p(x) = p\beta_1 e^{-\beta_1 x} I_{\{x \geq 0\}} + q\beta_2 e^{\beta_2 x} I_{\{x<0\}}. \tag{2.19}$$

From (2.3), we have

$$\begin{aligned} \left[\frac{c}{\lambda}\frac{\partial}{\partial u} + \left(1 + \frac{\delta}{\lambda}\right)\right] V(u;b) &= p\beta_1 e^{\beta_1 u} \int_u^b V(x;b)e^{-\beta_1 x}dx + \frac{p}{\beta_1}e^{-\beta_1(b-u)} \\ &\quad + pV(b;b)e^{-\beta_1(b-u)} + q\beta_2 e^{-\beta_2 u}\int_0^u V(x;b)e^{\beta_2 x}dx. \end{aligned} \tag{2.20}$$

Applying the operator $((\partial/\partial u) - \beta_1)((\partial/\partial u) + \beta_2)$ to both sides of (2.20), we get

$$\left(\frac{\partial}{\partial u} - \beta_1\right)\left(\frac{\partial}{\partial u} + \beta_2\right)\left[\frac{c}{\lambda}\frac{\partial}{\partial u} + \left(1 + \frac{\delta}{\lambda}\right)\right]V(u;b) = (q\beta_2 - p\beta_1)\frac{\partial V(u;b)}{\partial u} - (p+q)\beta_1\beta_2 V(u;b). \tag{2.21}$$

The characteristic equation of (2.21) is

$$(r-\beta_1)(r+\beta_2)\left[\frac{c}{\lambda}r+\left(1+\frac{\delta}{\lambda}\right)\right]=(q\beta_2-p\beta_1)r-(p+q)\beta_1\beta_2. \tag{2.22}$$

That is,

$$\frac{c}{\lambda}r+\left(1+\frac{\delta}{\lambda}\right)=\frac{p\beta_1}{\beta_1-r}+\frac{q\beta_2}{\beta_2+r}. \tag{2.23}$$

The expression on the left-hand side is a linear function of r, while the expression on the right-hand side is a rational function with poles at $r = \beta_1, -\beta_2$. By a graphical arguments, it can be verified that the characteristic equation above has exactly three real roots r_1, r_2, r_3 satisfying

$$-\infty < r_1 < -\beta_2 < r_2 < 0 < r_3 < \beta_1 < \infty. \tag{2.24}$$

Hence, we set

$$V(u;b) = c_1e^{r_1u}+c_2e^{r_2u}+c_3e^{r_3u}, \tag{2.25}$$

where $c_1 = c_1(b)$, $c_2 = c_2(b)$, and $c_3 = c_3(b)$ are constants need to be determined. It follows from (2.4) and (2.5) that we have

$$\begin{gathered} c_1+c_2+c_3=0,\\ c_1r_1e^{r_1b}+c_2r_2e^{r_2b}+c_3r_3e^{r_3b}=1. \end{gathered} \tag{2.26}$$

Substituting (2.25) into (2.20), and since this equation must be satisfied for $u = b$, we have

$$\begin{aligned} &c_1\left[\left(\frac{c}{\lambda}r_1+1+\frac{\delta}{\lambda}-p\right)e^{r_1b}-q\beta_2\frac{e^{r_1b}-e^{-\beta_2b}}{r_1+\beta_2}\right]+c_2\left[\left(\frac{c}{\lambda}r_2+1+\frac{\delta}{\lambda}-p\right)e^{r_2b}-q\beta_2\frac{e^{r_2b}-e^{-\beta_2b}}{r_2+\beta_2}\right]\\ &+c_3\left[\left(\frac{c}{\lambda}r_3+1+\frac{\delta}{\lambda}-p\right)e^{r_3b}-q\beta_2\frac{e^{r_3b}-e^{-\beta_2b}}{r_3+\beta_2}\right]-\frac{p}{\beta_1}=0, \end{aligned} \tag{2.27}$$

which can be rewritten as

$$\Delta_1c_1+\Delta_2c_2+\Delta_3c_3=\frac{p}{\beta_1}, \tag{2.28}$$

where

$$\Delta_1 = \frac{pr_1}{\beta_1 - r_1} e^{r_1 b} + \frac{q\beta_2}{\beta_2 + r_1} e^{-\beta_2 b},$$
$$\Delta_2 = \frac{pr_2}{\beta_1 - r_2} e^{r_2 b} + \frac{q\beta_2}{\beta_2 + r_2} e^{-\beta_2 b}, \tag{2.29}$$
$$\Delta_3 = \frac{pr_3}{\beta_1 - r_3} e^{r_3 b} + \frac{q\beta_2}{\beta_2 + r_3} e^{-\beta_2 b}.$$

Solving system (2.26) and (2.28) gives $c_1 = A_1/A$, $c_2 = A_2/A$, and $c_3 = A_3/A$, where

$$A = \frac{pr_1 r_2 (r_2 - r_1)}{(\beta_1 - r_1)(\beta_1 - r_2)} e^{(r_1+r_2)b} + \frac{pr_1 r_3 (r_1 - r_3)}{(\beta_1 - r_1)(\beta_1 - r_3)} e^{(r_1+r_3)b} + \frac{pr_2 r_3 (r_3 - r_2)}{(\beta_1 - r_2)(\beta_1 - r_3)} e^{(r_2+r_3)b}$$
$$+ q\beta_2 \left(\frac{r_3 e^{r_3 b} - r_2 e^{r_2 b}}{r_1 + \beta_2} + \frac{r_1 e^{r_1 b} - r_3 e^{r_3 b}}{r_2 + \beta_2} + \frac{r_2 e^{r_2 b} - r_1 e^{r_1 b}}{r_3 + \beta_2} \right) e^{-\beta_2 b},$$
$$A_1 = \frac{pr_2^2}{\beta_1(\beta_1 - r_2)} e^{r_2 b} - \frac{pr_3^2}{\beta_1(\beta_1 - r_3)} e^{r_3 b} + q\beta_2 \left(\frac{1}{r_2 + \beta_2} - \frac{1}{r_3 + \beta_2} \right) e^{-\beta_2 b}, \tag{2.30}$$
$$A_2 = -\frac{pr_1^2}{\beta_1(\beta_1 - r_1)} e^{r_1 b} + \frac{pr_3^2}{\beta_1(\beta_1 - r_3)} e^{r_3 b} + q\beta_2 \left(\frac{1}{r_3 + \beta_2} - \frac{1}{r_1 + \beta_2} \right) e^{-\beta_2 b},$$
$$A_3 = \frac{pr_1^2}{\beta_1(\beta_1 - r_1)} e^{r_1 b} - \frac{pr_2^2}{\beta_1(\beta_1 - r_2)} e^{r_2 b} + q\beta_2 \left(\frac{1}{r_1 + \beta_2} - \frac{1}{r_2 + \beta_2} \right) e^{-\beta_2 b}.$$

Example 2.4. For $n = 2$, $q = 0$, $p = 1$, and $p(x) = p_1(x) = \beta e^{-\beta x}$, $\beta > 0$, $x \geq 0$, we have

$$\left[\prod_{j=1}^{2} \left(\frac{c}{\lambda_j} \frac{\partial}{\partial u} + \left(1 + \frac{\delta}{\lambda_j} \right) \right) \right] V(u;b) = \beta e^{\beta u} \int_u^b V(x;b) e^{-\beta x} dx$$
$$+ \frac{1}{\beta} e^{-\beta(b-u)} + V(b;b) e^{-\beta(b-u)}. \tag{2.31}$$

Applying the operator $(\partial/\partial u) - \beta$ to both sides of (2.31), we get

$$\left(\frac{\partial}{\partial u} - \beta \right) \prod_{j=1}^{2} \left(\frac{c}{\lambda_j} \frac{\partial}{\partial u} + \left(1 + \frac{\delta}{\lambda_j} \right) \right) V(u;b) = -\beta V(u;b), \tag{2.32}$$

from which we get the characteristic equation

$$\frac{(cr + \lambda_1 + \delta)(cr + \lambda_2 + \delta)}{\lambda_1 \lambda_2} = \frac{\beta}{\beta - r}. \tag{2.33}$$

By a graphical arguments, it can be verified that the characteristic equation above has exactly three real roots r_1, r_2, r_3 satisfying

$$r_1 < r_2 < 0 < r_3 < \beta. \tag{2.34}$$

Hence we get

$$V(u,b) = c_1 e^{r_1 u} + c_2 e^{r_2 u} + c_3 e^{r_3 u}, \tag{2.35}$$

where $c_1 = c_1(b),\ c_2 = c_2(b)$, and $c_3 = c_3(b)$ are constants. It follows from (2.4) and (2.5) that

$$c_1 + c_2 + c_3 = 0, \tag{2.36}$$

$$c_1 r_1 e^{r_1 b} + c_2 r_2 e^{r_2 b} + c_3 r_3 e^{r_3 b} = 1. \tag{2.37}$$

Substituting (2.35) into (2.31), and because this equation must be satisfied for all $0 \le u \le b$, the sum of the coefficients of $e^{-\beta(u-b)}$ must be zero. Therefore,

$$c_1 \frac{r_1}{r_1 - \beta} e^{r_1 b} + c_2 \frac{r_2}{r_2 - \beta} e^{r_2 b} + c_3 \frac{r_3}{r_3 - \beta} e^{r_3 b} + \frac{1}{\beta} = 0. \tag{2.38}$$

It follows from (2.36)–(2.38) that

$$c_1 = \frac{B_1}{B}, \qquad c_2 = \frac{B_2}{B}, \qquad c_3 = \frac{B_3}{B}, \tag{2.39}$$

where

$$\begin{aligned} B &= \beta(r_3 - \beta)(r_1 - r_2) r_1 r_2 e^{(r_1 + r_2)b} + \beta(r_2 - \beta)(r_3 - r_1) r_1 r_3 e^{(r_1 + r_3)b} \\ &\quad + \beta(r_1 - \beta)(r_2 - r_3) r_2 r_3 e^{(r_2 + r_3)b}, \\ B_1 &= (r_1 - \beta)(r_3 - \beta) r_2^2 e^{r_2 b} - (r_1 - \beta)(r_2 - \beta) r_3^2 e^{r_3 b}, \\ B_2 &= (r_1 - \beta)(r_2 - \beta) r_3^2 e^{r_3 b} - (r_2 - \beta)(r_3 - \beta) r_1^2 e^{r_1 b}, \\ B_3 &= (r_2 - \beta)(r_3 - \beta) r_1^2 e^{r_1 b} - (r_1 - \beta)(r_3 - \beta) r_2^2 e^{r_2 b}. \end{aligned} \tag{2.40}$$

3. Moment-Generating Function of the Discounted Dividends

We denote moment-generating function of D by

$$M(u, y, b) = E\left[e^{yD} \mid X(0) = u\right]. \tag{3.1}$$

Let $\partial/\partial y$ denote the differentiation operator with respect to y and correspondingly $\partial/\partial u$ the differentiation operator with respect to u.

Theorem 3.1. *The moment-generating function $M(u,y,b)$ $(0 < u < b)$ satisfies the following integro-differential equation:*

$$\left[\prod_{j=1}^{n}\left(\frac{\delta y}{\lambda_j}\frac{\partial}{\partial y}+\frac{c}{\lambda_j}\frac{\partial}{\partial u}+1\right)\right]M(u,y,b) = p\int_0^{\infty}M(u+x,y,b)p_1(x)dx+q\int_0^{u}M(u-x,y,b)p_2(x)dx+q[1-P_2(u)], \tag{3.2}$$

with boundary conditions

$$\left[\prod_{j=2}^{k}\left(\delta y\frac{\partial}{\partial y}+c\frac{\partial}{\partial u}+\lambda_{j-1}\right)\right]\frac{\partial M(u,y,b)}{\partial u}\bigg|_{u=b} = y\left[\prod_{j=2}^{k}\left(\delta y\frac{\partial}{\partial y}+c\frac{\partial}{\partial u}+\lambda_{j-1}\right)\right]M(u,y,b)\bigg|_{u=b}, \quad k=1,2,\ldots,n. \tag{3.3}$$

Proof. As in Albrecher et al. [19], let $M_j(u,y,b)$ denote the moment-generating function of D if the risk process is in state j $(j=1,\ldots,n-1)$. Eventually, we are interested in $M(u,y,b) := M_1(u,y,b)$. Conditioning on the occurrence of a (sub-)claim within an infinitesimal time interval, we obtain for $0 \le u < b$ and $j=1,\ldots,n-1$,

$$M_j(u,y,b) = P(\xi_j > dt)M_j\left(u-cdt, ye^{-\delta dt}, b\right)+P(\xi_j \le dt)M_{j+1}\left(u-cdt, ye^{-\delta dt}, b\right). \tag{3.4}$$

It follows from (3.4) that

$$\lambda_j M_{j+1}(u,y,b) = \left[\lambda_j+\delta y\frac{\partial}{\partial y}+c\frac{\partial}{\partial u}\right]M_j(u,y,b). \tag{3.5}$$

For $j=n$, we have

$$\left[\delta y\frac{\partial}{\partial y}+c\frac{\partial}{\partial u}+\lambda_n\right]M_n(u,y,b) = p\lambda_n\int_0^{\infty}M_1(u+x,y,b)p_1(x)dx + q\lambda_n\int_{-u}^{0}M_1(u+x,y,b)p_2(-x)dx+q\lambda_n[1-P_2(u)]. \tag{3.6}$$

It follows from (3.5) that we have

$$\left(\prod_{j=1}^{n-1}\lambda_j\right)M_n(u,y,b) = \left[\prod_{j=1}^{n-1}\left(\delta y\frac{\partial}{\partial y}+c\frac{\partial}{\partial u}+\lambda_j\right)\right]M_1(u,y,b), \tag{3.7}$$

which together with (3.6) yields (3.2).

For $u = b$, we obtain analogously for $j = 1, 2, \ldots, n-1$

$$M_j(b,y,b) = (1-\lambda_j dt)e^{-ycdt}M_j\left(b, ye^{-\delta dt}, b\right) + \lambda_j dte^{-ycdt}M_{j+1}\left(b, ye^{-\delta dt}, b\right) + o(dt) \quad (3.8)$$

which leads to

$$\lambda_j M_{j+1}(b,y,b) = (\lambda_j + yc)M_j(b,y,b) + y\delta \frac{\partial M_j}{\partial y}(b,y,b). \quad (3.9)$$

Comparing these equations with the corresponding equations in (3.5), the continuity of $M_j(u,y,b)$ at $u = b$ implies

$$\left.\frac{\partial M_j(u,y,b)}{\partial u}\right|_{u=b} = yM_j(b,y,b), \quad j = 1, 2, \ldots, n-1. \quad (3.10)$$

Similarly, we can show that (3.10) holds true for $j = n$. For $j = 1$, (3.10) is equivalent to (3.3) for $k = 1$. Now it just remains to express equations (3.10) for $j = 2, \ldots, n$ in terms of $M_1 = M$, which is done by virtue of (3.9). □

For $m \in \mathbb{N}$, we denote the mth moment of D by

$$W_m(u;b) = E[D^m \mid X(0) = u]. \quad (3.11)$$

Theorem 3.2. *The* m*th moment* $W_m(u;b)$ $(0 < u < b)$ *satisfies the following integro-differential equation*

$$\begin{aligned}\left[\prod_{j=1}^{n}\left(\frac{\delta m}{\lambda_j} + \frac{c}{\lambda_j}\frac{\partial}{\partial u} + 1\right)\right]W_m(u;b) &= p\int_0^{b-u} W_m(u+x;b)p_1(x)dx \\ &\quad + p\sum_{i=0}^{m}\binom{m}{i}W_i(b;b)\int_{b-u}^{\infty}(u-b+x)^{m-i}p_1(x)dx \\ &\quad + q\int_0^{u} W_m(u-x;b)p_2(x)dx\end{aligned} \quad (3.12)$$

with boundary conditions

$$\begin{aligned}&\left.\left[\prod_{j=2}^{k}\left(\delta m + c\frac{\partial}{\partial u} + \lambda_{j-1}\right)\right]\frac{\partial W_m(u;b)}{\partial u}\right|_{u=b} \\ &= m\left.\left[\prod_{j=2}^{k}\left(\delta(m-1) + c\frac{\partial}{\partial u} + \lambda_{j-1}\right)\right]W_{m-1}(u;b)\right|_{u=b}, \quad k = 1, 2, \ldots, n.\end{aligned} \quad (3.13)$$

Proof. Since

$$M(u, y, b) = 1 + \sum_{m=1}^{\infty} \frac{y^m}{m!} W_m(u, b), \tag{3.14}$$

the result follows if we equate the coefficients of y^m $(m = 0, 1, 2, \ldots)$ in (3.2) and (3.3). □

Remark 3.3. We remark that when $m = 1$, $W_0(u; b) = 1$, and $W_1(u; b) = V(u; b)$, we reobtained the result of Theorem 2.1; when $n = 1$, $p = 0$, and $q = 1$, (3.12) reduces to (2.3) of Cheung and Drekic [20]; when $p = 0$, $q = 1$, (3.2) and (3.3) reduce to (2) and (3) of Albrecher et al. [19], and (3.12) and (3.13) reduce to (9) and (10) of Albrecher et al. [19].

Acknowledgments

The authors would like to thank the anonymous referees for their constructive and insightful suggestions and comments on the previous version of this paper. This paper is supported by the National Natural Science Foundation of China (no. 11171179).

References

[1] H. Albrecher, A. L. Badescu, and D. Landriault, “On the dual risk model with tax payments,” *Insurance: Mathematics and Economics*, vol. 42, no. 3, pp. 1086–1094, 2008.
[2] A. C. Y. Ng, “On a dual model with a dividend threshold,” *Insurance: Mathematics and Economics*, vol. 44, no. 2, pp. 315–324, 2009.
[3] B. Avanzi, H. U. Gerber, and E. S. W. Shiu, “Optimal dividends in the dual model,” *Insurance: Mathematics and Economics*, vol. 41, no. 1, pp. 111–123, 2007.
[4] B. Avanzi and H. U. Gerber, “Optimal dividends in the dual model with diffusion,” *Astin Bulletin*, vol. 38, no. 2, pp. 653–667, 2008.
[5] J. Grandell, *Aspects of Risk Theory*, Springer, New York, NY, USA, 1991.
[6] H. Cramér, *Collective Risk Theory: A Survey of the Theory from the Point of View of the Theory of Stochastic Processes*, A. B. Nordiska Bokhandeln, Stockholm, Sweden, 1955.
[7] H. L. Seal, *Stochastic Theory of a Risk Business*, Wiley, New York, NY, USA, 1969.
[8] L. Takács, *Combinatorial Methods in the Theory of Stochastic Processes*, Wiley, New York, NY, USA, 1967.
[9] D. Perry, W. Stadje, and S. Zacks, “First-exit times for compound Poisson processes for some types of positive and negative jumps,” *Stochastic Models*, vol. 18, no. 1, pp. 139–157, 2002.
[10] S. G. Kou and H. Wang, “First passage times of a jump diffusion process,” *Advances in Applied Probability*, vol. 35, no. 2, pp. 504–531, 2003.
[11] S. Asmussen, F. Avram, and M. R. Pistorius, “Russian and American put options under exponential phase-type Lévy models,” *Stochastic Processes and Their Applications*, vol. 109, no. 1, pp. 79–111, 2004.
[12] M. Jacobsen, “The time to ruin for a class of Markov additive risk process with two-sided jumps,” *Advances in Applied Probability*, vol. 37, no. 4, pp. 963–992, 2005.
[13] Y. H. Dong and G. J. Wang, “Ruin probability for renewal risk model with negative risk sums,” *Journal of Industrial and Management Optimization*, vol. 2, no. 2, pp. 229–236, 2006.
[14] Y. H. Dong and G. J. Wang, “On a compounding assets model with positive jumps,” *Applied Stochastic Models in Business and Industry*, vol. 24, no. 1, pp. 21–30, 2008.
[15] N. Cai, N. Chen, and X. Wan, “Pricing double-barrier options under a flexible jump diffusion model,” *Operations Research Letters*, vol. 37, no. 3, pp. 163–167, 2009.
[16] Z. Zhang, H. Yang, and S. Li, “The perturbed compound Poisson risk model with two-sided jumps,” *Journal of Computational and Applied Mathematics*, vol. 233, no. 8, pp. 1773–1784, 2010.
[17] Y. Chi and X. S. Lin, “On the threshold dividend strategy for a generalized jump-diffusion risk model,” *Insurance: Mathematics and Economics*, vol. 48, no. 3, pp. 326–337, 2011.

[18] N. Cai and S. G. Kou, "Option pricing under a mixed-exponential jump diffusion model," *Management Science*, vol. 57, no. 11, pp. 2067–2081, 2011.

[19] H. Albrecher, M. M. Claramunt, and M. Mármol, "On the distribution of dividend payments in a Sparre Andersen model with generalized Erlang(n) interclaim times," *Insurance: Mathematics and Economics*, vol. 37, no. 2, pp. 324–334, 2005.

[20] E. C. K. Cheung and S. Drekic, "Dividend moments in the dual risk model: exact and approximate approaches," *Astin Bulletin*, vol. 38, no. 2, pp. 399–422, 2008.