Research Article

Stability and Bifurcation in a Delayed Holling-Tanner Predator-Prey System with Ratio-Dependent Functional Response

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We analyze a delayed Holling-Tanner predator-prey system with ratio-dependent functional response. The local asymptotic stability and the existence of the Hopf bifurcation are investigated. Direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are studied by deriving the equation describing the flow on the center manifold. Finally, numerical simulations are presented for the support of our analytical findings.

1. Introduction

Predator-prey dynamics has long been and will continue to be of interest to both applied mathematicians and ecologists due to its universal existence and importance [1]. Although the early Lotka-Volterra model has given way to more sophisticated models from both a mathematical and biological point of view, it has been challenged by ecologists for its functional response suffers from paradox of enrichment and biological control paradox. The ratio-dependent models are discussed as a solution to these difficulties and found to be a more reasonable choice for many predator-prey interactions [2–4]. One type of the ratio-dependent models which plays a special role in view of the interesting dynamics it possesses is the ratio-dependent Holling-Tanner predator-prey system [5, 6]. A ratio-dependent Holling-Tanner predator-prey system takes the form of

$$
\begin{aligned}
\frac{dN}{dt} &= rN(t)\left(1 - \frac{N(t)}{k}\right) - \frac{mN(t)P(t)}{N(t) + qP(t)}, \\
\frac{dP}{dt} &= P(t)\left[s\left(1 - h\frac{P(t)}{N(t)}\right)\right],
\end{aligned}
\tag{1.1}
$$

where $N(t)$ and $P(t)$ represent the population of prey species and predator species at time t. It is assumed that in the absence of the predator, the prey grows logistically with carrying k and intrinsic growth rate r. The predator growth equation is of logistic type with a modification of the conventional one. The parameter m represents the maximal predator per capita consumption rate, and q is the half capturing saturation constant. The parameter s is the intrinsic growth rate of the predator and h is the number of prey required to support one predator at equilibrium, when y equals x/h. All the parameters are assumed to be positive.

Liang and Pan [6] established the sufficient conditions for the global stability of positive equilibrium of system (1.1) by constructing Lyapunov function. Considering the effect of time delays on the system, Saha and Chakrabarti [7] considered the following delayed system

$$
\begin{aligned}
\frac{dN}{dt} &= rN(t)\left(1 - \frac{N(t-\tau)}{k}\right) - \frac{mN(t)P(t)}{N(t) + qP(t)}, \\
\frac{dP}{dt} &= P(t)\left[s\left(1 - h\frac{P(t)}{N(t)}\right)\right],
\end{aligned}
\tag{1.2}
$$

where τ is the negative feedback delay of the prey. Saha and Chakrabarti [7] proved that the system (1.2) is permanent under certain conditions and obtained the conditions for the local and global stability of the positive equilibrium. It is well known that studies on dynamical systems not only involve a discussion of stability and persistence, but also involve many dynamical behaviors such as periodic phenomenon, bifurcation, and chaos [8–10]. In particular, the Hopf bifurcation has been studied by many authors [11–13]. Based on this consideration and since both species are growing logistically, we consider the Hopf bifurcation of the following system with two delays:

$$
\begin{aligned}
\frac{dN}{dt} &= rN(t)\left(1 - \frac{N(t-\tau_1)}{k}\right) - \frac{mN(t)P(t)}{N(t) + qP(t)}, \\
\frac{dP}{dt} &= P(t)\left[s\left(1 - h\frac{P(t-\tau_2)}{N(t-\tau_2)}\right)\right],
\end{aligned}
\tag{1.3}
$$

where τ_1 and τ_2 represent the negative feedbacks in prey and predator growth.

Before proceeding further we nondimensionalize our model system (1.3) with the following scaling $rt \rightarrow t$, $r\tau_1 \rightarrow \tau_1$, $r\tau_2 \rightarrow \tau_2$, $N(t)/k \rightarrow x(t)$, $mP(t)/rk \rightarrow y(t)$. Then we get the nondimensional form of system (1.3):

$$
\begin{aligned}
\frac{dx}{dt} &= x(t)(1 - x(t-\tau_1)) - \frac{x(t)y(t)}{x(t) + \alpha y(t)}, \\
\frac{dy}{dt} &= \beta y(t)\left(\delta - \frac{y(t-\tau_2)}{x(t-\tau_2)}\right),
\end{aligned}
\tag{1.4}
$$

where $\alpha = qr/m$, $\beta = sh/m$, $\delta = m/hr$.

This paper is organized as follows. In the next section, we will consider the local stability of the positive equilibrium and the existence of Hopf bifurcation of system (1.4). In Section 3, we can determine the direction of the Hopf bifurcations and the stability of

the bifurcating periodic solutions. Some numerical simulations are also given to illustrate the theoretical prediction in Section 4.

2. Local Stability and Hopf Bifurcation

Considering the ecological significance of system (1.4), we are interested only in the positive equilibrium of system (1.4). It is not difficult to verify that system (1.4) has a unique positive equilibrium $E(x_0, y_0)$, where $x_0 = (1 + \alpha\delta - \delta)/(1 + \alpha\delta)$, $y_0 = \delta x_0$ if $(H_1): 1 + \alpha\delta > \delta$ holds.

Let $x(t) = u_1(t) + x_0$, $y(t) = u_2(t) + y_0$, and we still denote $u_1(t)$ and $u_2(t)$ by $x(t)$ and $y(t)$, respectively, then system (1.4) can be rewritten as

$$\frac{dx(t)}{dt} = a_{11}x(t) + a_{12}y(t) + b_{11}x(t - \tau_1) + \sum_{i+j+k\geq 2} f_1^{ijk} x^i y^j x^k (t - \tau_1),$$

$$\frac{dy(t)}{dt} = b_{21}x(t - \tau_2) + b_{22}y(t - \tau_2) + \sum_{i+j+k\geq 2} f_2^{ijk} y^i x^j (t - \tau_2) y^k (t - \tau_2),$$

(2.1)

where

$$a_{11} = 1 - x_0 - \frac{\alpha y_0^2}{(x_0 + \alpha y_0)^2}, \qquad a_{12} = -\frac{x_0^2}{(x_0 + \alpha y_0)^2},$$

$$b_{11} = -x_0, \qquad b_{21} = \frac{\beta y_0^2}{x_0^2}, \qquad b_{22} = -\frac{\beta y_0}{x_0},$$

$$f_1^{ijk} = \frac{1}{i!j!k!} \frac{\partial^{i+j+k} f_1}{\partial x^i \partial y^j \partial x^k (t - \tau_1)} \bigg|_{(x_0, y_0)}, \qquad f_2^{ijk} = \frac{1}{i!j!k!} \frac{\partial^{i+j+k} f_2}{\partial y^i \partial x^j (t - \tau_2) \partial y^k (t - \tau_2)} \bigg|_{(x_0, y_0)},$$

$$f_1 = x(t)(1 - x(t - \tau_1)) - \frac{x(t)y(t)}{x(t) + \alpha y(t)}, \qquad f_2 = \beta y(t) \left(\delta - \frac{y(t - \tau_2)}{x(t - \tau_2)} \right).$$

(2.2)

Then the linearized system of (2.1) is

$$\frac{dx(t)}{dt} = a_{11}x(t) + a_{12}y(t) + b_{11}x(t - \tau_1),$$

$$\frac{dy(t)}{dt} = b_{21}x(t - \tau_2) + b_{22}y(t - \tau_2).$$

(2.3)

The characteristic equation of (2.3) is

$$\lambda^2 + A\lambda + B\lambda e^{-\lambda\tau_1} + (C\lambda + D)e^{-\lambda\tau_2} + Ee^{-\lambda(\tau_1 + \tau_2)} = 0,$$

(2.4)

where

$$A = -a_{11}, \qquad B = -b_{11}, \qquad C = -b_{22}, \qquad D = a_{11}b_{22} - a_{12}b_{21}, \qquad E = b_{11}b_{22}.$$

(2.5)

Case 1 ($\tau_1 = \tau_2 = 0$). Equation (2.4) reduces to

$$\lambda^2 + (A + B + C)\lambda + D + E = 0. \tag{2.6}$$

If the condition (H_2): $A + B + C > 0$ and $D + E > 0$ holds, it is clear that roots of (2.6) must have negative real parts.

Case 2 ($\tau_1 > 0$, $\tau_2 = 0$). Equation (2.4) becomes

$$\lambda^2 + (A + C)\lambda + D + (B\lambda + E)e^{-\lambda\tau_1} = 0. \tag{2.7}$$

Let $\lambda = i\omega_1$ ($\omega_1 > 0$) be a root of (2.7). Then, we have

$$\begin{aligned} B\omega_1 \sin \tau_1\omega_1 + E \cos \tau_1\omega_1 = \omega_1^2 - D, \\ B\omega_1 \cos \tau_1\omega_1 - E \sin \tau_1\omega_1 = -(A + C)\omega_1. \end{aligned} \tag{2.8}$$

From (2.8), we can get

$$\omega_1^4 + \left((A + C)^2 - B^2 - 2D\right)\omega_1^2 + D^2 - E^2. \tag{2.9}$$

If the condition (H_3): $D^2 - E^2 < 0$ holds, then (2.9) has a unique positive root ω_{10}

$$\omega_{10} = \sqrt{\frac{-\left((A + C)^2 - B^2 - 2D\right) + \sqrt{\left((A + C)^2 - B^2 - 2D\right)^2 - 4(D^2 - E^2)}}{2}}. \tag{2.10}$$

The corresponding critical value of time delay τ_{10} is

$$\tau_{10} = \frac{1}{\omega_{10}}\arccos\frac{(E - AB - BC)\omega_{10}^2 - DE}{B^2\omega_{10}^2 + E^2} + \frac{2n\pi}{\omega_{10}}, \quad n = 0, 1, 2, \ldots. \tag{2.11}$$

Next, differentiating (2.7) with respect to τ_1 and substituting $\tau_1 = \tau_{10}$, then we get

$$\operatorname{Re}\left\{\frac{d\lambda}{d\tau_1}\right\}^{-1}_{\tau_1 = \tau_{10}} = \frac{2\omega_{10}^2 + (A + C)^2 - B^2 - 2D}{(A + C)^2\omega_{10}^2 + (\omega_{10}^2 - D)^2}. \tag{2.12}$$

From (2.10) and (2.12), we have

$$\operatorname{Re}\left\{\frac{d\lambda}{d\tau_1}\right\}^{-1}_{\tau_1 = \tau_{10}} = \frac{\sqrt{\left((A + C)^2 - B^2 - 2D\right)^2 - 4(D^2 - E^2)}}{(A + C)^2\omega_{10}^2 + (\omega_{10}^2 - D)^2}. \tag{2.13}$$

Therefore, if the condition (H_3): $D^2 - E^2 < 0$ holds, then $\text{Re}\{d\lambda/d\tau_1\}^{-1}_{\tau_1=\tau_{10}} > 0$. Thus, we have the following results.

Theorem 2.1. *For system (1.4), if the conditions (H_2)-(H_3) hold, then the positive equilibrium $E(x_0, y_0)$ of system (1.4) is asymptotically stable for $\tau \in [0, \tau_{10})$ and unstable when $\tau > \tau_{10}$, system (1.4) undergoes a Hopf bifurcation at $E(x_0, y_0)$ when $\tau_1 = \tau_{10}$.*

Case 3 $(\tau_1 = 0, \tau_2 > 0)$. Equation (2.4) becomes

$$\lambda^2 + (A + B)\lambda + (C\lambda + D + E)e^{-\lambda\tau_2} = 0. \tag{2.14}$$

Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be a root of (2.14). Then, we get

$$C\omega_2 \sin \tau_2\omega_2 + (D + E)\cos \tau_2\omega_2 = \omega_2^2,$$
$$C\omega_2 \cos \tau_2\omega_2 - (D + E)\sin \tau_2\omega_2 = -(A + B)\omega_2. \tag{2.15}$$

It follows that

$$\omega_2^4 + \left((A + B)^2 - C^2\right)\omega_2^2 - (D + E)^2. \tag{2.16}$$

If the condition (H_2) holds, then $-(D + E)^2 < 0$. Thus, (2.16) has a unique positive root ω_{20},

$$\omega_{20} = \sqrt{\frac{-\left((A + B)^2 - C^2\right) + \sqrt{\left((A + B)^2 - C^2\right)^2 + 4(D + E)^2}}{2}}. \tag{2.17}$$

The corresponding critical value of time delay τ_{20} is

$$\tau_{20} = \frac{1}{\omega_{20}}\arccos\frac{(D + E - AC - BC)\omega_{10}^2}{C^2\omega_{20}^2 + (D + E)^2} + \frac{2n\pi}{\omega_{20}}, \quad n = 0, 1, 2, \ldots. \tag{2.18}$$

Similar as in Case 2, we know that if the condition (H_2) holds, then we have

$$\text{Re}\left\{\frac{d\lambda}{d\tau_2}\right\}^{-1}_{\tau_2=\tau_{20}} = \frac{\sqrt{\left((A + B)^2 - C^2\right)^2 + 4(D + E)^2}}{\omega_{20}^4 + (A + B)^2\omega_{20}^2} > 0. \tag{2.19}$$

In conclusion, we have the following results.

Theorem 2.2. *For system (1.4), if the condition (H_2) holds, then the positive equilibrium $E(x_0, y_0)$ of system (1.4) is asymptotically stable for $\tau \in [0, \tau_{20})$ and unstable when $\tau > \tau_{20}$, system (1.4) undergoes a Hopf bifurcation at $E(x_0, y_0)$ when $\tau_1 = \tau_{20}$.*

Case 4 ($\tau_1 = \tau_2 = \tau > 0$). Equation (2.4) becomes

$$\lambda^2 + A\lambda + ((B+C)\lambda + D)e^{-\lambda\tau} + Ee^{-2\lambda\tau} = 0. \tag{2.20}$$

Multiplying $e^{\lambda\tau}$ on both sides of (2.20), we have

$$(B+C)\lambda + D + \left(\lambda^2 + A\lambda\right)e^{\lambda\tau} + Ee^{-\lambda\tau} = 0. \tag{2.21}$$

Let $\lambda = i\omega$ ($\omega > 0$) be a root of (2.21). Then, we get

$$\begin{aligned} A\omega \sin\tau\omega - \left(E - \omega^2\right)\cos\tau\omega &= D, \\ A\omega \cos\tau\omega - \left(E + \omega^2\right)\sin\tau\omega &= -(B+C)\omega. \end{aligned} \tag{2.22}$$

Then, we can get

$$\sin\tau\omega = \frac{m_3\omega^3 + m_1\omega}{\omega^4 + n_2\omega^2 + n_0}, \qquad \cos\tau\omega = \frac{m_2\omega^2 + m_0}{\omega^4 + n_2\omega^2 + n_0}, \tag{2.23}$$

where

$$\begin{aligned} m_0 &= DE, \qquad m_1 = AD - E(B+C), \qquad m_2 = D - A(B+C), \\ m_3 &= B + C, \qquad n_0 = -E^2, \qquad n_2 = A^2. \end{aligned} \tag{2.24}$$

Thus, we can obtain

$$\omega^8 + A_3\omega^6 + A_2\omega^4 + A_1\omega^2 + A_0 = 0, \tag{2.25}$$

with

$$\begin{aligned} A_0 &= n_0^2 - m_0^2, \qquad A_1 = 2n_0n_2 - m_1^2 - 2m_0m_2, \\ A_2 &= 2n_0 - 2m_1m_3 - m_2^2, \qquad A_3 = 2n_2 - m_3^2. \end{aligned} \tag{2.26}$$

Let $e = \omega^2$, then (2.25) can be transformed into the following form

$$e^4 + A_3e^3 + A_2e^2 + A_1e + A_0 = 0. \tag{2.27}$$

Next, we suppose that (H_4): (2.27) has at least one positive root. Without loss of generality, we suppose that it has four positive roots which are denoted as e_1, e_2, e_3, and e_4. Thus, (2.25)

has four positive roots $\omega_k = \sqrt{e_k}$, $k = 1, 2, 3, 4$. The corresponding critical value of time delay $\tau_k^{(j)}$ is

$$\tau_k^{(j)} = \frac{1}{\omega_k} \arccos \frac{m_2 \omega^2 + m_0}{\omega^4 + n_2 \omega^2 + n_0} + \frac{2j\pi}{\omega_k}, \quad k = 1, 2, 3, 4, \ j = 0, 1, 2 \ldots. \tag{2.28}$$

Let $\tau_0 = \min\{\tau_k^{(0)}\}$, $k \in \{1, 2, 3, 4\}$, $\omega_0 = \omega_{k_0}$.

Differentiating (2.21) regarding τ and substituting $\tau = \tau_0$, we obtain

$$\operatorname{Re}\left\{\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_0}^{-1} = \frac{M_R N_R + M_I N_I}{N_R^2 + N_I^2}, \tag{2.29}$$

where

$$M_R = B + C + A\cos\tau_0\omega_0 - 2\omega_0\sin\tau_0\omega_0, \qquad M_I = A\sin\tau_0\omega_0 + 2\omega_0\cos\tau_0\omega_0,$$

$$N_R = \left(A\omega_0^2 + E\right)\cos\tau_0\omega_0 - \omega_0^3\sin\tau_0\omega_0, \qquad N_I = \left(A\omega_0^2 - E\right)\sin\tau_0\omega_0 + \omega_0^3\cos\tau_0\omega_0. \tag{2.30}$$

Obviously, if the condition (H_5): $M_R N_R + M_I N_I \neq 0$ holds, then $\operatorname{Re}\{d\lambda/d\tau\}_{\tau=\tau_0}^{-1} \neq 0$. Namely, the transversality condition is satisfied if H_5 holds. From the above analysis, we have the following theorem.

Theorem 2.3. *For system* (1.4)*, if the conditions* (H_2)*,* (H_4)*, and* (H_5) *hold, then the positive equilibrium* $E(x_0, y_0)$ *of system* (1.4) *is asymptotically stable for* $\tau \in [0, \tau_0)$ *and unstable when* $\tau > \tau_0$*, system* (1.4) *undergoes a Hopf bifurcation at* $E(x_0, y_0)$ *when* $\tau = \tau_0$*.*

Case 5 $(\tau_1 \neq \tau_2$ *and* $\tau_1 > 0$*,* $\tau_2 > 0)$*.* We consider (2.4) with τ_2 in its stable interval and τ_1 is considered as a parameter. Let $\lambda = i\omega_1'$ $(\omega_1' > 0)$ be the root of (2.4). Then we have

$$\begin{aligned} M\sin\tau_1\omega_1' + N\cos\tau_1\omega_1' = P, \\ M\cos\tau_1\omega_1' - N\sin\tau_1\omega_1' = Q, \end{aligned} \tag{2.31}$$

where

$$M = B\omega_1' - E\sin\tau_2\omega_1', \qquad N = E\cos\tau_2\omega_1',$$

$$P = \left(\omega_1'\right)^2 - C\omega_1'\sin\tau_2\omega_1' - D\cos\tau_2\omega_1', \tag{2.32}$$

$$Q = -A\omega_1' - C\omega_1'\cos\tau_2\omega_1' + D\sin\tau_2\omega_1'.$$

It follows that

$$f_1(\omega_1') + f_2(\omega_1')\cos\tau_2\omega_1' + f_3(\omega_1')\sin\tau_2\omega_1' = 0, \tag{2.33}$$

where

$$f_1(\omega_1') = (\omega_1')^4 + \left(A^2 - B^2 + C^2\right)(\omega_1')^2 + D^2 - E^2,$$

$$f_2(\omega_1') = 2(AC - D)(\omega_1')^2, \qquad f_3(\omega_1') = -2C(\omega_1')^3 + (2BE - 2AD)\omega_1'. \tag{2.34}$$

Suppose that (H_6): (2.33) has at least finite positive roots.

If the condition (H_6) holds, we denote the roots of (2.33) as $\omega_{11}', \omega_{12}', \dots \omega_{1k}'$. Then, for every fixed ω_{1i}' $(i = 1, 2, \dots k)$, the corresponding critical value of time delay $\{\tau_{1i}^{(j)} \mid j = 0, 1, 2, \dots\}$ is

$$\tau_{1i}^{(j)} = \frac{1}{\omega_i}\arccos\frac{MQ + PN}{M^2 + N^2} + \frac{2j\pi}{\omega_i}, \quad i = 1, 2, \dots k, \ j = 0, 1, 2, \dots \tag{2.35}$$

Let $\tau_{1*} = \min\{\tau_{1i}^{(j)} \mid i = 1, 2, \dots, k, \ j = 0, 1, 2, \dots\}$. The corresponding purely imaginary roots of (2.33) are denoted as $\pm i\omega_*$. Next, we give the following assumption. (H_7): $\{d\operatorname{Re}(\lambda)/d\tau_1\}_{\tau_1=\tau_{1*}}^{-1} \neq 0$. Hence, we have the following theorem.

Theorem 2.4. *Suppose that the conditions (H_2), (H_6), and (H_7) hold and $\tau_2 \in (0, \tau_{20})$. The positive equilibrium $E(x_0, y_0)$ of system (1.4) is asymptotically stable for $\tau_1 \in [0, \tau_{1*})$ and unstable when $\tau_1 > \tau_{1*}$, system (1.4) undergoes a Hopf bifurcation at $E(x_0, y_0)$ when $\tau_1 = \tau_0$.*

3. Direction and Stability of Bifurcated Periodic Solutions

In this section, we will employ the normal form method and center manifold theorem introduced by Hassard [14] to determine the direction of Hopf bifurcation and stability of bifurcating periodic solutions of system (1.4) at $\tau_1 = \tau_2 = \tau = \tau_0$.

We denote τ as $\tau = \tau_0 + \mu$, $\mu \in R$, $t = s\tau$, Then $\mu = 0$ is the Hopf bifurcation value of system (1.4). For convenience, we first rescale the time by $s \to t/\tau$, $x(s\tau) = x_1(s)$, $y(s\tau) = x_2(s)$ and still denote $s = t$, then system (1.4) can be transformed to the following form:

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{3.1}$$

where

$$u(t) = (x(t), y(t))^T \in C\left([-1, 0], R^2\right),$$

$$u_t(\theta) = u(t + \theta) = (x(t + \theta), y(t + \theta))^T \in C\left([-1, 0], R^2\right) \tag{3.2}$$

and L_μ, F are given by

$$L_\mu\phi = (\tau_0 + \mu)\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_0 + \mu)\begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix},$$

$$F(\mu, \phi) = (\tau_0 + \mu)(F_1, F_2)^T, \tag{3.3}$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C([-1,0], R^2)$, and

$$
\begin{aligned}
F_1 &= a_{13}\phi_1^2(0) - \phi_1(0)\phi_1(-1) + a_{14}\phi_1(0)\phi_2(0) + a_{15}\phi_2^2(0) + a_{16}\phi_1^3(0) \\
&\quad + a_{17}\phi_1^2(0)\phi_2(0) + a_{18}\phi_1(0)\phi_2^2(0) + a_{19}\phi_2^3(0) + \cdots, \\
F_2 &= b_{23}\phi_1^2(-1) + b_{24}\phi_1(-1)\phi_2(0) + b_{25}\phi_1(-1)\phi_2(-1) + b_{26}\phi_2(0)\phi_2(-1) \\
&\quad + b_{27}\phi_1^3(-1)\phi_3(-1) + b_{28}\phi_1^2(-1)\phi_2(0) + b_{29}\phi_1^2(-1)\phi_2(-1) + \cdots,
\end{aligned}
\tag{3.4}
$$

with

$$
\begin{aligned}
&a_{13} = \frac{\alpha y_0^2}{(x_0 + \alpha y_0)^3}, \qquad a_{14} = -\frac{2\alpha x_0 y_0}{(x_0 + \alpha y_0)^3}, \qquad a_{15} = \frac{\alpha x_0^2}{(x_0 + \alpha y_0)^3}, \\
&a_{16} = -\frac{\alpha y_0^2}{(x_0 + \alpha y_0)^4}, \qquad a_{17} = \frac{2\alpha x_0 y_0 - \alpha^2 y_0^2}{(x_0 + \alpha y_0)^4}, \qquad a_{18} = \frac{2\alpha^2 x_0 y_0 - \alpha x_0^2}{(x_0 + \alpha y_0)^4}, \\
&a_{19} = -\frac{\alpha^2 y_0^2}{(x_0 + \alpha y_0)^4}, \qquad b_{23} = -\frac{\beta\delta^2}{x_0}, \qquad b_{24} = \frac{\beta\delta}{x_0}, \qquad b_{25} = \frac{\beta\delta}{x_0}, \\
&b_{26} = -\frac{\beta}{x_0}, \qquad b_{27} = \frac{\beta\delta^2}{x_0^2}, \qquad b_{28} = -\frac{\beta\delta}{x_0^2}, \qquad b_{29} = -\frac{\beta\delta}{x_0^2}.
\end{aligned}
\tag{3.5}
$$

Hence, by the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \mu)$: $[-1, 0] \to R^2$ whose elements are of bounded variation such that

$$
L_\mu \phi = \int_{-1}^{0} [d\eta(\theta, \mu)]\phi(\theta), \quad \phi \in C\left([-1,0], R_+^3\right).
\tag{3.6}
$$

In fact, we choose

$$
\eta(\theta, \mu) = (\tau_0 + \mu)\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}\delta(\theta) + (\tau_0 + \mu)\begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}\delta(\theta + 1),
\tag{3.7}
$$

where δ is the Dirac delta function, then (3.3) is satisfied.

For $\phi \in C([-1, 0], R^2)$, we define

$$
A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\[2mm] \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases}
\tag{3.8}
$$

$$
R(\mu)\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases}
$$

Then system (3.1) can be transformed into the following operator equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{3.9}$$

The adjoint operator A^* of $A(0)$ is defined by

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\[2mm] \displaystyle\int_{-1}^{0} d\eta^T(\xi, 0)\varphi(-\xi), & s = 0, \end{cases} \tag{3.10}$$

associated with a bilinear form

$$\langle \varphi(s), \phi(\theta) \rangle = \overline{\varphi}^T(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \overline{\varphi}^T(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.11}$$

where $\eta(\theta) = \eta(\theta, 0)$.

From the above discussion, we know that that $\pm i w_0 \tau_0$ are eigenvalues of $A(0)$ and they are also eigenvalues of A^*.

We assume that $q(\theta) = (1, q_2)^T e^{i\tau_0 w_0 \theta}$ are the eigenvectors of $A(0)$ belonging to the eigenvalue $i\tau_0 w_0$ and $q^*(\theta) = D(1, q_2^*)e^{i\tau_0 w_0 \theta}$ are the eigenvectors of A^* belonging to $-i\tau_0 w_0$. Thus,

$$A(0)q(\theta) = i\tau_0 w_0 q(\theta), \qquad A^*(0)q^*(\theta) = -i\tau_0 w_0 q^*(\theta). \tag{3.12}$$

Then, we can obtain

$$q_2 = \frac{i w_0 - a_{11} - b_{11}}{a_{12}}, \qquad q_2^* = -\frac{i w_0 + a_{11} + b_{11}}{b_{21}} b_{21} e^{i\tau_0 w_0},$$

$$\overline{D} = \left[1 + q_2 q_2^* + (b_{11} + b_{21}q_2^* + b_{22}q_2 q_2^*)e^{i\tau_0 w_0}\right]^{-1}. \tag{3.13}$$

Next, we get the coefficients used in determining the important quantities of the periodic solution by using a computation process similar to that in [15]:

$$g_{20} = 2\tau_0 \overline{D}\left\{ a_{13} + a_{14}q^{(2)}(0) + a_{15}\left(q^{(2)}(0)\right)^2 - q^{(1)}(-1) \right.$$

$$+ \overline{q}_2^*\left(b_{23}\left(q^{(1)}(-1)\right)^2 + b_{24}q^{(1)}(-1)q^{(2)}(0) \right.$$

$$\left. +b_{25}q^{(1)}(-1)q^{(2)}(-1) + b_{26}q^{(2)}(0)q^{(2)}(-1)\right)\right\}$$

$$g_{11} = \tau_0 \overline{D}\left\{2a_{13} + a_{14}\left(q^{(2)}(0) + \overline{q}^{(2)}(0)\right) + 2a_{15}q^{(2)}(0)\overline{q}^{(2)}(0) - q^{(1)}(-1) \right.$$

$$- \overline{q}^{(1)}(-1) + \overline{q}_2^*\left(2b_{23}q^{(1)}(-1)\overline{q}^{(1)}(-1)\right.$$

$$+ b_{24}\left(q^{(1)}(-1)\overline{q}^{(2)}(0) + \overline{q}^{(1)}(-1)q^{(2)}(0)\right)$$

$$+ b_{25}\left(q^{(1)}(-1)\overline{q}^{(2)}(-1) + \overline{q}^{(1)}(-1)q^{(2)}(-1)\right)$$

$$+ b_{26}\left(q^{(2)}(0)\overline{q}^{(2)}(-1) + \overline{q}^{(2)}(0)q^{(2)}(-1)\right)\Big)\Big\},$$

$$g_{02} = 2\tau_0\overline{D}\Big\{a_{13} + a_{14}\overline{q}^{(2)}(0) + a_{15}\left(\overline{q}^{(2)}(0)\right)^2 - \overline{q}^{(1)}(-1)$$

$$+ \overline{q}_2^*\left(b_{23}\left(\overline{q}^{(1)}(-1)\right)^2 + b_{24}\overline{q}^{(1)}(-1)\overline{q}^{(2)}(0)\right.$$

$$\left.+ b_{25}\overline{q}^{(1)}(-1)\overline{q}^{(2)}(-1) + b_{26}\overline{q}^{(2)}(0)\overline{q}^{(2)}(-1)\right)\Big\},$$

$$g_{21} = 2\tau_0\overline{D}\Big\{a_{13}\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right)$$

$$+ a_{14}\left(\frac{1}{2}W_{20}^{(2)}(0) + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}^{(2)}(0) + W_{11}^{(1)}(0)q^{(2)}(0)\right)$$

$$+ a_{15}\left(W_{20}^{(2)}(0)\overline{q}^{(2)}(0) + 2W_{11}^{(2)}(0)q^{(2)}(0)\right)$$

$$- \left(W_{11}^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(-1) + W_{11}^{(1)}(0)q^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}^{(1)}(-1)\right) + 3a_{16}$$

$$+ a_{17}\left(2q^{(2)}(0) + \overline{q}^{(2)}(0)\right) + a_{18}\left(2q^{(2)}(0)\overline{q}^{(2)}(0) + \left(q^{(2)}(0)\right)^2\right)$$

$$+ 3a_{19}\left(q^{(2)}(0)\right)^2\overline{q}^{(2)}(0)$$

$$+ \overline{q}_2^*\left(b_{23}\left(W_{20}^{(1)}(-1)\overline{q}^{(1)}(-1) + 2W_{11}^{(1)}(-1)q^{(1)}(-1)\right)\right.$$

$$+ b_{24}\left(\frac{1}{2}W_{20}^{(2)}(0)\overline{q}^{(1)}(-1) + W_{11}^{(2)}(0)q^{(1)}(-1)\right.$$

$$\left.+ \frac{1}{2}W_{20}^{(1)}(-1)\overline{q}^{(2)}(0) + W_{11}^{(1)}(-1)q^{(2)}(0)\right)$$

$$+ b_{25}\left(\frac{1}{2}W_{20}^{(1)}(-1)\overline{q}^{(2)}(-1) + W_{11}^{(1)}(-1)q^{(2)}(-1)\right.$$

$$\left.+ \frac{1}{2}W_{20}^{(2)}(-1)\overline{q}^{(1)}(-1) + W_{11}^{(2)}(-1)q^{(1)}(-1)\right)$$

$$+ b_{26}\left(\frac{1}{2}W_{20}^{(2)}(0)\overline{q}^{(2)}(-1) + W_{11}^{(2)}(0)q^{(2)}(-1)\right.$$

$$\left.+ \frac{1}{2}W_{20}^{(2)}(-1)\overline{q}^{(2)}(0) + W_{11}^{(2)}(-1)q^{(2)}(0)\right)$$

$$+ 3b_{27}\left(q^{(1)}(-1)\right)^2\overline{q}^{(1)}(-1)$$

$$+ b_{28} \left(\left(q^{(1)}(-1) \right)^2 \overline{q}^{(2)}(0) + 2 q^{(1)}(-1) \overline{q}^{(1)}(-1) q^{(2)}(0) \right)$$

$$+ b_{29} \left(\left(q^{(1)}(-1) \right)^2 \overline{q}^{(2)}(-1) + 2 q^{(1)}(-1) \overline{q}^{(1)}(-1) q^{(2)}(-1) \right) \right) \Bigg\},$$

$$(3.14)$$

with

$$W_{20}(\theta) = \frac{i g_{20} q(0)}{\tau_0 \omega_0} e^{i \tau_0 \omega_0 \theta} + \frac{i \overline{g}_{02} \overline{q}(0)}{3 \tau_0 \omega_0} e^{-i \tau_0 \omega_0 \theta} + E_1 e^{2 i \tau_0 \omega_0 \theta},$$

$$W_{11}(\theta) = -\frac{i g_{11} q(0)}{\tau_0 \omega_0} e^{i \tau_0 \omega_0 \theta} + \frac{i \overline{g}_{11} \overline{q}(0)}{\tau_0 \omega_0} e^{-i \tau_0 \omega_0 \theta} + E_2,$$

$$(3.15)$$

where E_1 and E_2 can be computed as the following equations, respectively

$$E_1 = 2 \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} \times \begin{pmatrix} 2 i \omega_0 - a_{11} - b_{11} & -a_{12} \\ -c_{21} e^{-2 i \tau_0 \omega_0} & 2 i \omega_0 - c_{22} e^{-2 i \tau_0 \omega_0} \end{pmatrix}^{-1},$$

$$E_2 = -\begin{pmatrix} N_{11} \\ N_{21} \end{pmatrix} \times \begin{pmatrix} a_{11} + b_{11} & a_{12} \\ c_{21} & c_{22} \end{pmatrix}^{-1}$$

$$(3.16)$$

with

$$M_{11} = a_{13} + a_{14} q^{(2)}(0) + a_{15} \left(q^{(2)}(0) \right)^2 - q^{(1)}(-1),$$

$$M_{21} = b_{23} \left(q^{(1)}(-1) \right)^2 + b_{24} q^{(1)}(-1) q^{(2)}(0) + b_{25} q^{(1)}(-1) q^{(2)}(-1) + b_{26} q^{(2)}(0) q^{(2)}(-1),$$

$$N_{11} = 2 a_{13} + a_{14} \left(q^{(2)}(0) + \overline{q}^{(2)}(0) \right) + 2 a_{15} q^{(2)}(0) \overline{q}^{(2)}(0) - q^{(1)}(-1) - \overline{q}^{(1)}(-1),$$

$$N_{21} = 2 b_{23} q^{(1)}(-1) \overline{q}^{(1)}(-1) + b_{24} \left(q^{(1)}(-1) \overline{q}^{(2)}(0) + \overline{q}^{(1)}(-1) q^{(2)}(0) \right)$$

$$+ b_{25} \left(q^{(1)}(-1) \overline{q}^{(2)}(-1) + \overline{q}^{(1)}(-1) q^{(2)}(-1) \right)$$

$$+ b_{26} \left(q^{(2)}(0) \overline{q}^{(2)}(-1) + \overline{q}^{(2)}(0) q^{(2)}(-1) \right).$$

$$(3.17)$$

Therefore, we can calculate the following values:

$$C_1(0) = \frac{i}{2 \tau_0 \omega_0} \left(g_{11} g_{20} - 2 |g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}},$$

$$\beta_2 = 2 \mathrm{Re}\{C_1(0)\}.$$

$$(3.18)$$

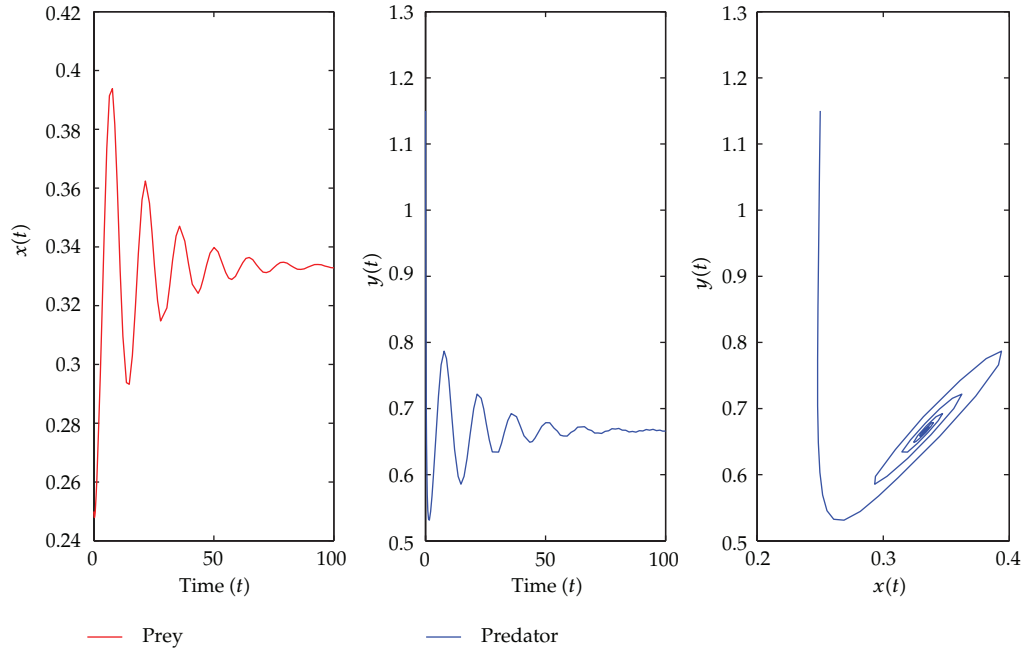Based on the discussion above, we can obtain the following results.

Figure 1: $E(0.3333, 0.6666)$ is asymptotically stable when $\tau_1 = 3.15 < \tau_{10} = 3.7457$ with initial value 0.25, 1.15.

Theorem 3.1. *The direction of the Hopf bifurcation is determined by the sign of μ_2: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical). The stability of bifurcating periodic solutions is determined by the sign of β_2: if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable).*

4. Numerical Example

In this section, to illustrate the analytical results obtained in the previous sections, we present some numerical simulations. Let $\alpha = 1$, $\beta = 1$, $\delta = 2$, then we have the following particular case of system (1.4):

$$\frac{dx}{dt} = x(t)(1 - x(t - \tau_1)) - \frac{x(t)y(t)}{x(t) + y(t)},$$

$$\frac{dy}{dt} = y(t)\left(2 - \frac{y(t - \tau_2)}{x(t - \tau_2)}\right). \tag{4.1}$$

Obviously, $1 + \alpha\delta = 3 > \delta = 2$. Thus, the condition (H_1) holds. Then we can get the unique positive equilibrium $E(0.3333, 0.6666)$ of system (4.1). By a simple computation, $A + B + C = 2.1110 > 0$, $D + E = 0.5998 > 0$. Namely, the condition (H_2) holds.
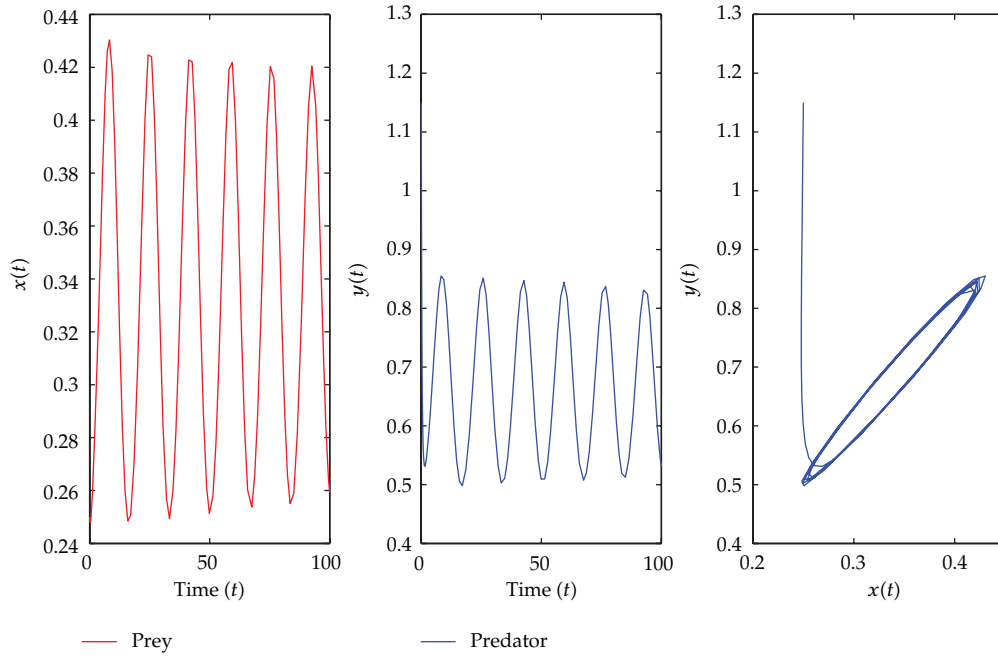
Figure 2: $E(0.3333, 0.6666)$ is unstable when $\tau_1 = 4.05 > \tau_{10} = 3.7457$ with initial value 0.25, 1.15.
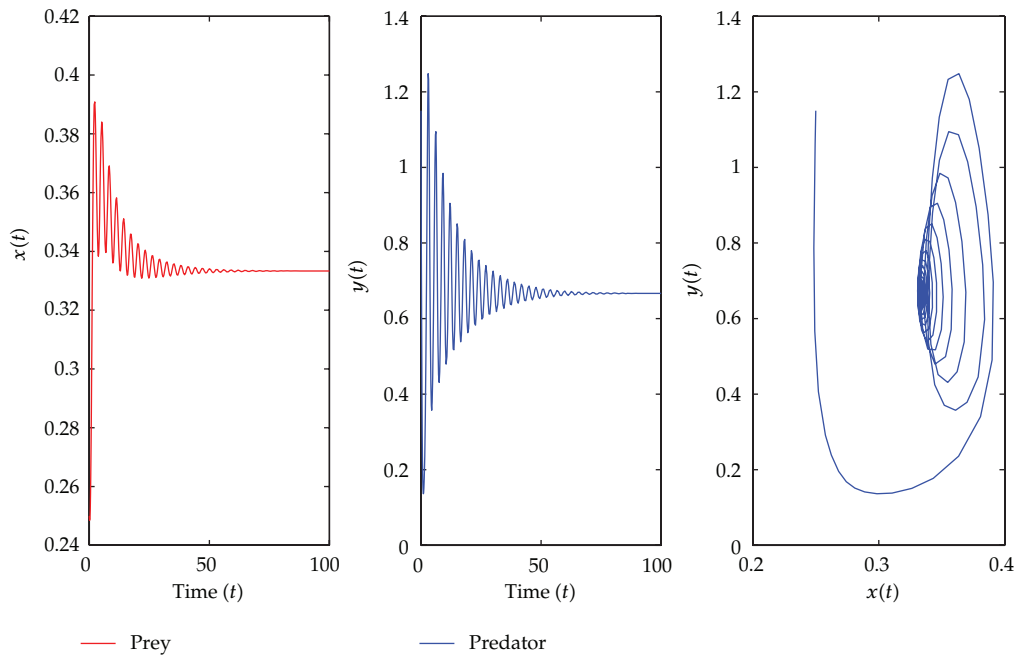

Figure 3: $E(0.3333, 0.6666)$ is asymptotically stable when $\tau_2 = 0.68 < \tau_{20} = 0.7311$ with initial value 0.25, 1.15.
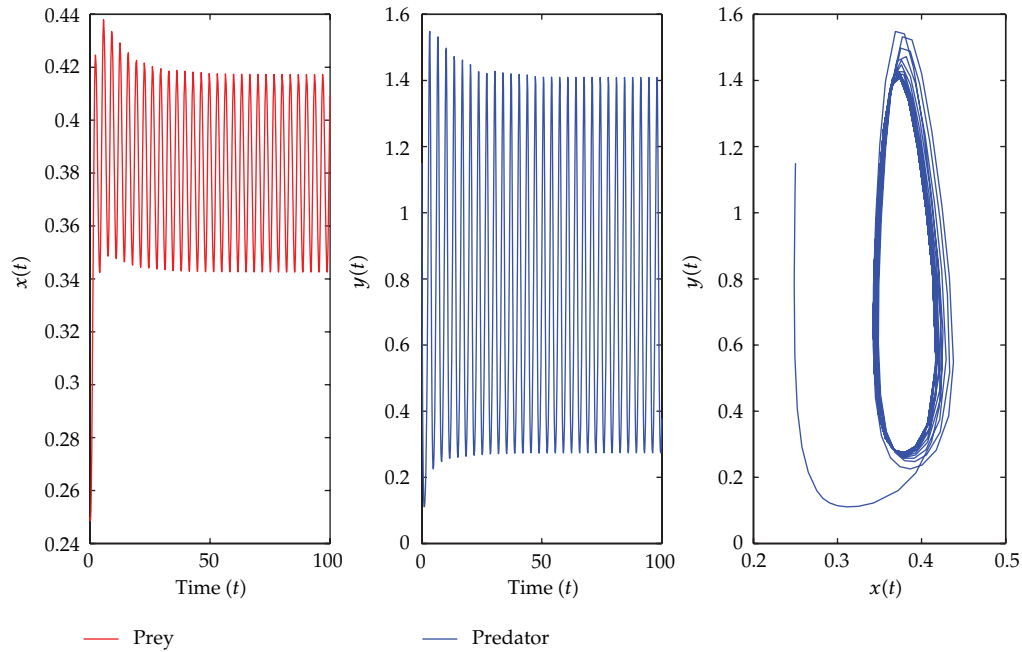
Figure 4: $E(0.3333, 0.6666)$ is unstable when $\tau_2 = 0.75 > \tau_{10} = 0.7311$ with initial value 0.25, 1.15.

Firstly, we can obtain that $D^2 - E^2 = -0.4444 < 0$. Namely, the condition (H_3) is satisfied for $\tau_1 > 0$, $\tau_2 = 0$. Further, we have $\omega_{10} = 0.3734$, $\tau_{10} = 3.7457$. By Theorem 2.1, we can know that the positive equilibrium $E(0.3333, 0.6666)$ is asymptotically stable for $\tau_1 \in [0, \tau_{10})$ and unstable when $\tau_1 > \tau_{10}$. Let $\tau_1 = 3.15 \in [0, \tau_{10})$, then the positive equilibrium $E(0.3333, 0.6666)$ is asymptotically stable, which can be seen from Figure 1. When $\tau_1 = 4.05 > \tau_{10} = 3.7457$, it can be seen from Figure 2 that the positive equilibrium $E(0.3333, 0.6666)$ is unstable and a Hopf bifurcation occurs. Similarly, we have $\omega_{20} = 2.0189$, $\tau_{20} = 0.7311$. For $\tau_2 = 0.68 \in [0, \tau_{20})$, the positive equilibrium $E(0.3333, 0.6666)$ is asymptotically stable from Theorem 2.2 and this property can be shown in Figure 3. If $\tau_2 = 0.75 > \tau_{20} = 0.7311$, the positive equilibrium $E(0.3333, 0.6666)$ is unstable and a Hopf bifurcation occurs, and the corresponding waveform and phase plots are shown in Figure 4.

Secondly, we can get $\omega_0 = 2.2937$, $\tau_0 = 0.7015$, and $\lambda'(\tau_0) = 0.9911 + 1.7808i$ for $\tau_1 = \tau_2 = \tau > 0$. From Theorem 2.3, we know that $E(0.3333, 0.6666)$ is asymptotically stable for $\tau \in [0, \tau_0)$, which can be illustrated by Figure 5. As can be seen from Figure 5 that when $\tau = 0.64 \in [0, \tau_0)$ the positive equilibrium $E(0.3333, 0.6666)$ is asymptotically stable. However, if $\tau = 0.74 > \tau_0 = 0.7015$, then the positive equilibrium $E(0.3333, 0.6666)$ becomes unstable and a family of bifurcated periodic solutions occur, which is illustrated by Figure 6. In addition, from (3.18), we get $\mu_2 = 139.3 > 0$, $\beta_2 = -276.12 < 0$. Thus, by Theorem 3.1, we know that the Hopf bifurcation is supercritical and the bifurcated periodic solutions are stable.

Lastly, regard τ_1 as a parameter and let $\tau_2 = 0.5 \in (0, \tau_{20})$, we can obtain that $\omega_* = 0.3758$. Further we have $\tau_{1*} = 3.7688$. Let $\tau_1 = 3.25 \in [0, \tau_{1*})$, we can know that the positive equilibrium $E(0.3333, 0.6666)$ is asymptotically stable from Theorem 2.4, which can be shown by Figure 7. When $\tau_1 = 4.15 > \tau_{1*} = 3.7688$ then the positive equilibrium $E(0.3333, 0.6666)$ becomes unstable and a Hopf bifurcation occurs, which can be illustrated in Figure 8.
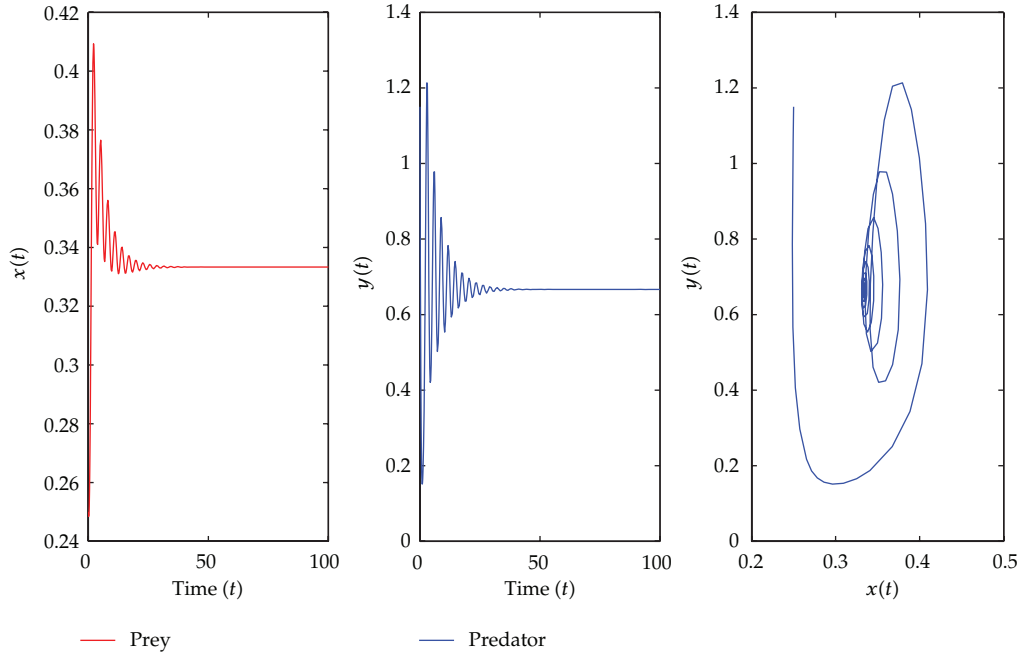
Figure 5: $E(0.3333, 0.6666)$ is asymptotically stable when $\tau = 0.64 < \tau_0 = 0.7015$ with initial value 0.25, 1.15.
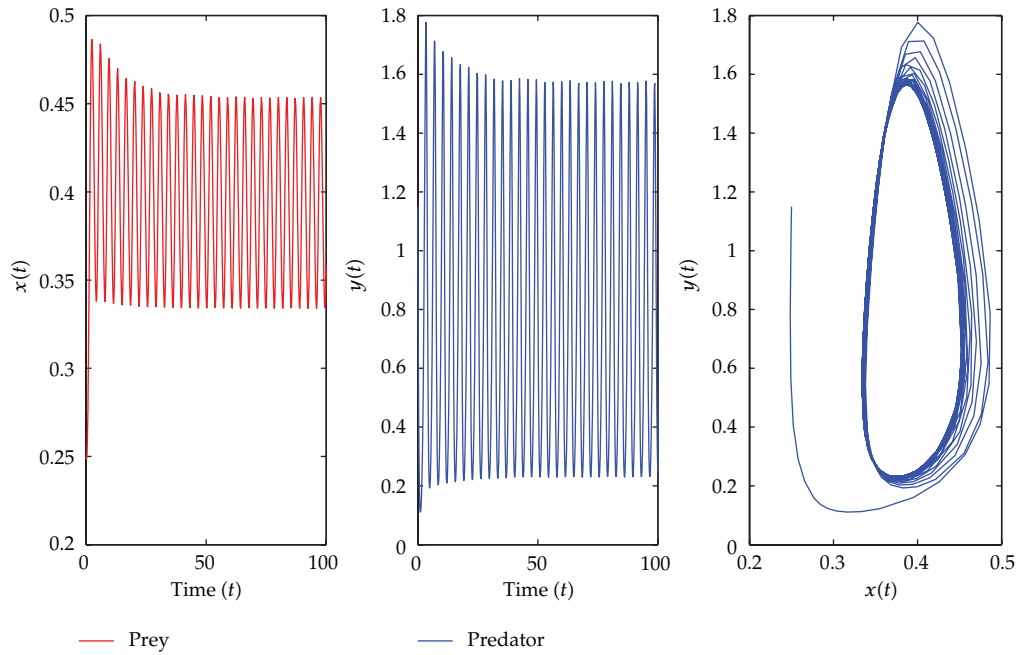

Figure 6: $E(0.3333, 0.6666)$ is unstable when $\tau = 0.74 > \tau_0 = 0.7015$ with initial value 0.25, 1.15.
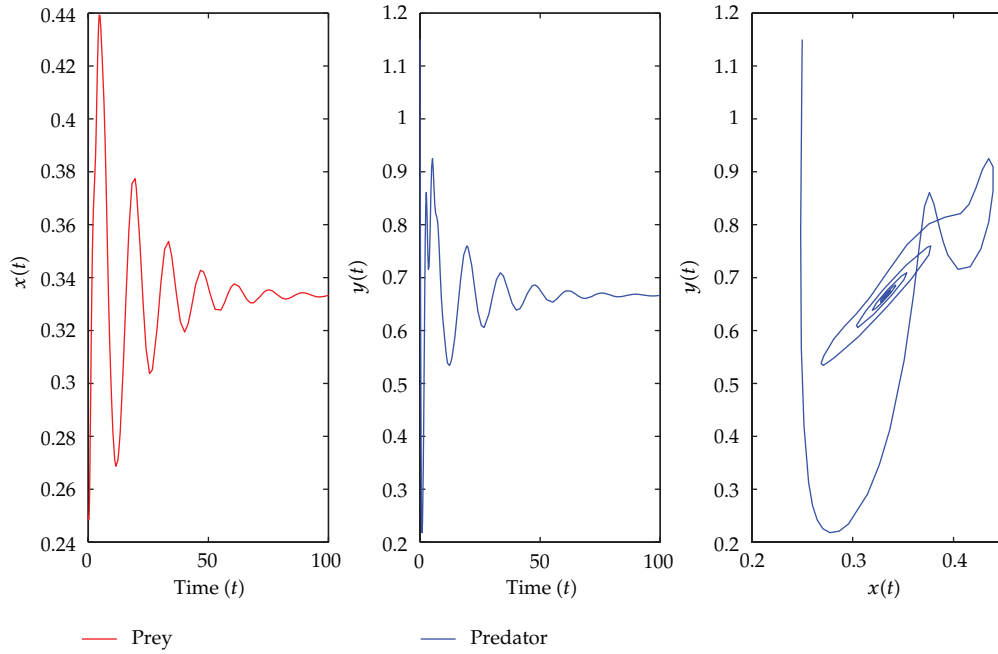
Figure 7: $E(0.3333, 0.6666)$ is asymptotically stable when $\tau_1 = 3.25 < \tau_{1*} = 3.7457$ with $\tau_2 = 0.5$ and initial value 0.25, 1.15.
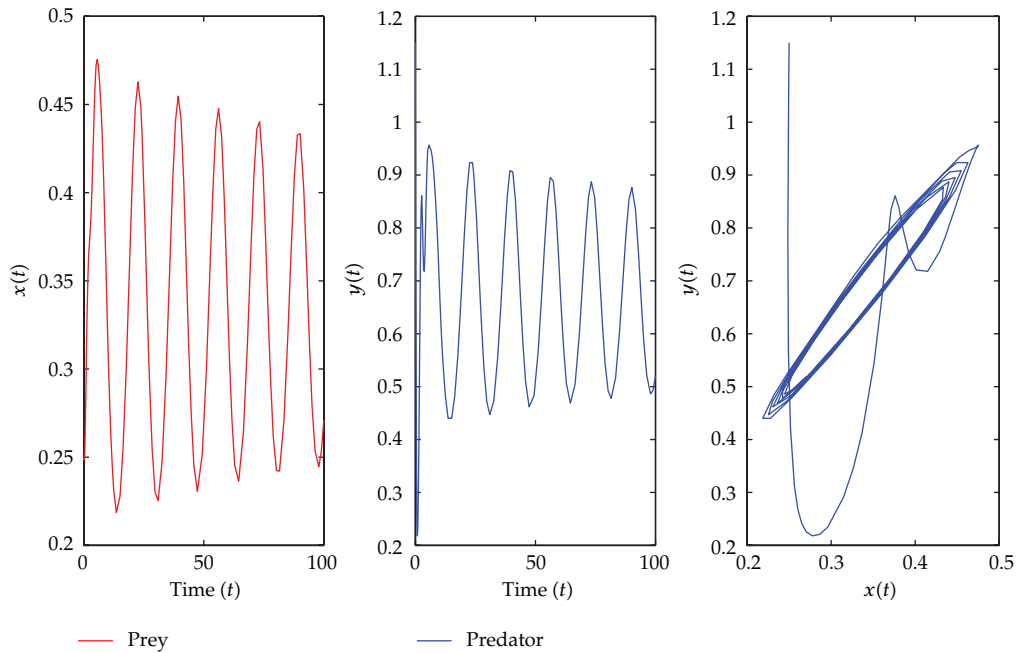

Figure 8: $E(0.3333, 0.6666)$ is unstable when $\tau_1 = 4.15 > \tau_{1*} = 3.7457$ with with $\tau_2 = 0.5$ and initial value 0.25, 1.15.

5. Conclusion

In the present paper, a Holling-Tanner predator-prey system with ratio-dependent functional response and two delays is investigated. We prove that the system is asymptotically stable under certain conditions. Compared with the system considered in [7], we not only consider the feedback delay of the prey but also the feedback delay of the predator. By choosing the delay as a bifurcation parameter, we show that the Hopf bifurcations can occur as the delay crosses some critical values. Furthermore, we get that the two species could also coexist with some available delays of the prey and the predator. This is valuable from the view of biology. In addition, Saha and Chakrabarti [7] only considered the stability of the system. It is well known that there are also some other behaviors for dynamical systems. Based on this consideration, we investigate the Hopf bifurcation and properties of the bifurcated periodic solutions of the system. The direction and the stability of the bifurcated periodic solutions are determined by applying the normal theory and the center manifold theorem. If the bifurcated periodic solutions are stable, then the two species may coexist in an oscillatory mode from the viewpoint of biology. Some numerical simulations supporting the theoretical results are also included.
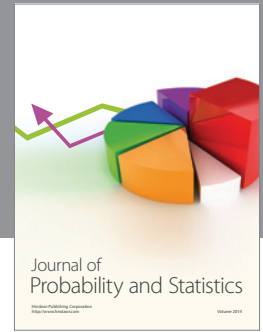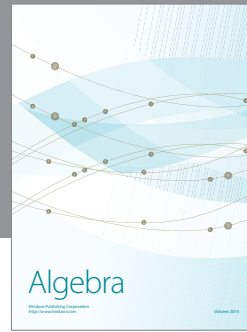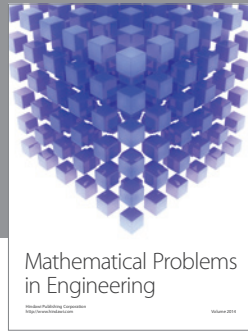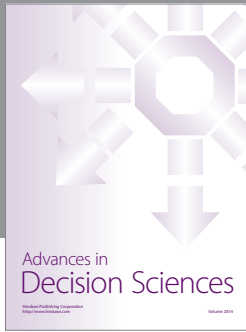
Acknowledgments

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