

*Research Article*

# **Exact Solutions of Generalized Boussinesq-Burgers Equations and (2+1)-Dimensional Davey-Stewartson Equations**

**Isaiah Elvis Mhlanga and Chaudry Masood Khalique**

*International Institute for Symmetry Analysis and Mathematical Modelling,  
Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046,  
Mmabatho 2735, South Africa*

Correspondence should be addressed to Isaiah Elvis Mhlanga, isaiah.mhlanga@nwu.ac.za

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We study two coupled systems of nonlinear partial differential equations, namely, generalized Boussinesq-Burgers equations and (2+1)-dimensional Davey-Stewartson equations. The Lie symmetry method is utilized to obtain exact solutions of the generalized Boussinesq-Burgers equations. The travelling wave hypothesis approach is used to find exact solutions of the (2+1)-dimensional Davey-Stewartson equations.

## **1. Introduction**

Most nonlinear physical phenomena that appear in many areas of scientific fields such as plasma physics, solid state physics, fluid dynamics, optical fibers, mathematical biology, and chemical kinetics can be modelled by nonlinear partial differential equations (NLPDEs). The investigation of exact travelling wave solutions of these NLPDEs is important for the understanding of most nonlinear physical phenomena and possible applications. To address this issue, various methods for finding travelling wave solutions to NLPDEs have been proposed. Some of the most important methods include homogeneous balance method [1], the ansatz method [2, 3], variable separation approach [4], inverse scattering transform method [5], Bäcklund transformation [6], Darboux transformation [7], Hirota's bilinear method [8], the  $(G'/G)$ -expansion method [9], the reduction mKdV equation method [10], the tri-function method [11, 12], the projective Riccati equation method [13], the sine-cosine method [14, 15], the Jacobi elliptic function expansion method [16, 17], the  $F$ -expansion method [18], the exp-function expansion method [19], and Lie symmetry method [20–24].

In this paper, we study two systems of nonlinear partial differential equations, namely, generalized Boussinesq-Burgers equations and (2+1)-dimensional Davey-Stewartson equations. We employ the Lie symmetry method and the travelling wave variable approach to obtain exact solutions to both these systems.

The Lie symmetry approach is one of the most effective methods to determine solutions of nonlinear partial differential equations. Sophus Lie (1842–1899), with the inspiration from Galois' theory for solving algebraic equations, discovered this method which is known today as Lie group analysis. He showed that many of the known methods of integration of ordinary differential equations could be derived in a systematic manner using his theory of continuous transformation groups. In the past few decades, a considerable amount of development has been made in symmetry methods for differential equations. This is evident by the number of research papers, books, and many new symbolic softwares devoted to the subject [20–24]. The travelling wave variable approach converts the NLPDEs into a nonlinear ordinary differential equations and is often useful in obtaining exact solutions of the partial differential equations.

## 2. The Generalized Boussinesq-Burgers Equations

We first consider the generalized Boussinesq-Burgers equations [25] given by

$$u_t + auu_x + bv_x = 0, \quad (2.1)$$

$$v_t + c(uv)_x + du_{xxx} = 0, \quad (2.2)$$

where  $a, b, c$ , and  $d$  are real nonzero constants. These equations arise in the study of fluid flow and describe the propagation of shallow water waves, where  $x$  and  $t$  represent the normalized space and time, respectively. Here  $u(x, t)$  represents the horizontal velocity and at the leading order it is the depth averaged horizontal field, while  $v(x, t)$  denotes the height of the water surface above the horizontal level at the bottom [25].

The Boussinesq-Burgers equations given by (2.1)-(2.2) will be solved by the Lie symmetry approach. The symmetry group of the generalized Boussinesq-Burgers equations (2.1)-(2.2) will be generated by the vector field given by

$$X = \xi^1(t, x, u, v) \frac{\partial}{\partial t} + \xi^2(t, x, u, v) \frac{\partial}{\partial x} + \eta^1(t, x, u, v) \frac{\partial}{\partial u} + \eta^2(t, x, u, v) \frac{\partial}{\partial v}. \quad (2.3)$$

Applying the third prolongation  $\text{pr}^{(3)}X$  [21] to (2.1)-(2.2) and solving the resultant overdetermined system of linear partial differential equations, one obtains the following three Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= -2t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2v \frac{\partial}{\partial v}. \end{aligned} \quad (2.4)$$

We now consider the symmetry  $X_1 + \nu X_2$ , where  $\nu$  is an arbitrary constant. This symmetry gives rise to the group-invariant solution as

$$u = F(z), \quad v = G(z), \quad (2.5)$$

where  $z = x - \nu t$  is an invariant of the symmetry  $X_1 + \nu X_2$ . Substitution of (2.5) into (2.1)-(2.2) results in the system of ordinary differential equations as

$$aF(z)F'(z) - \nu F'(z) + bG'(z) = 0, \quad (2.6)$$

$$dF'''(z) + cG(z)F'(z) + cF(z)G'(z) - \nu G'(z) = 0. \quad (2.7)$$

Integration of (2.6) with respect to  $z$  yields

$$\frac{1}{2}aF(z)^2 - \nu F(z) + bG(z) = 0, \quad (2.8)$$

where the constant of integration is chosen to be zero, since we are looking for a soliton solution. Solving for  $G(z)$ , we obtain

$$G(z) = \frac{2\nu F(z) - aF(z)^2}{2b}. \quad (2.9)$$

Substituting this value of  $G(z)$  into (2.7) gives the third-order nonlinear ordinary differential equation as

$$2bdF'''(z) - 3acF(z)^2F'(z) + 2\nu(a + 2c)F(z)F'(z) - 2\nu^2F'(z) = 0, \quad (2.10)$$

which can be integrated twice to obtain

$$F'(z)^2 - \frac{ac}{4bd}F(z)^4 + \frac{\nu(a + 2c)}{3bd}F(z)^3 - \frac{\nu^2}{bd}F(z)^2 = 0. \quad (2.11)$$

Here again the constants of integration are taken to be zero for the same reason as given above. Integrating (2.11) and reverting back to our original variables, we obtain

$$u(x, t) = \frac{12\nu}{2(a + 2c) + \alpha \cosh(\gamma z + \delta) - \beta \sinh(\gamma z + \delta)}, \quad (2.12)$$

$$v(x, t) = \frac{12\nu^2[-6a + 2(a + 2c) + \alpha \cosh(\gamma z + \delta) - \beta \sinh(\gamma z + \delta)]}{b[2(a + 2c) + \alpha \cosh(\gamma z + \delta) - \beta \sinh(\gamma z + \delta)]^2}, \quad (2.13)$$

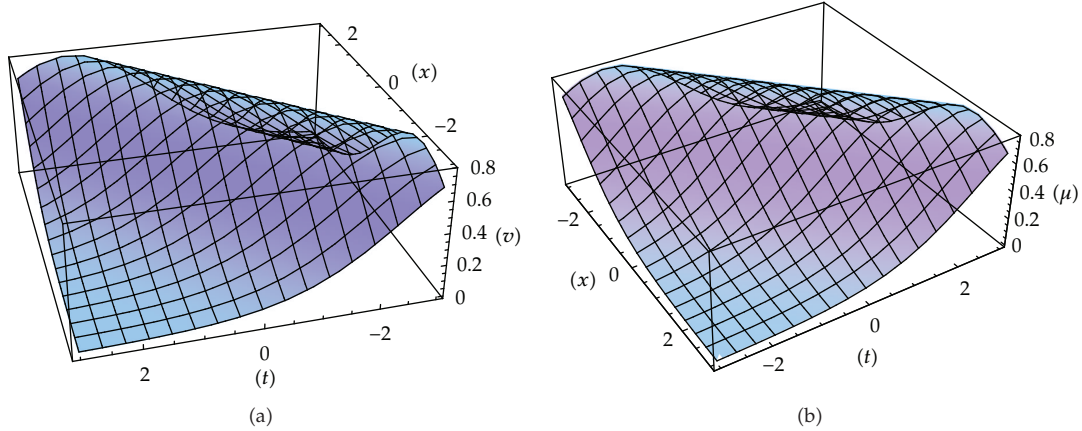


Figure 1: Profile of the solitary wave solution (2.12)-(2.13).

where  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  are constants given by

$$\begin{aligned}\alpha &= a^2 - 5ac + 4c^2 + 1, \\ \beta &= a^2 - 5ac + 4c^2 - 1, \\ \gamma &= \frac{v}{\sqrt{bd}}, \\ \delta &= 2\sqrt{3}cv, \\ z &= x - vt.\end{aligned}\tag{2.14}$$

A profile of the solution (2.12)-(2.13) is given in Figure 1.

### 3. The (2+1)-Dimensional Davey-Stewartson Equations

The (2+1)-dimensional Davey-Stewartson equations

$$iu_t + u_{xx} - u_{yy} - 2|u|^2u - 2uv = 0,\tag{3.1}$$

$$v_{xx} + v_{yy} + 2(|u|^2)_{xx} = 0\tag{3.2}$$

were first introduced by Davey and Stewartson in 1974 [26]. This system of equations is completely integrable and is often used to describe the long-time evolution of a two-dimensional wave packet [27–29].

We first transform the (2+1)-dimensional Davey-Stewartson equations (3.1)-(3.2) to a system of nonlinear ordinary differential equations in order to derive its exact solutions.

We make the following transformation:

$$u = e^{i\theta}u(\xi), \quad v = v(\xi), \quad \theta = px + qy + rt, \quad \xi = kx + cy + dt,\tag{3.3}$$

where  $p, q, r, k, c,$  and  $d$  are real constants. Using this transformation, the (2+1)-dimensional Davey-Stewartson equations (3.1)-(3.2) transform to

$$\left(-r - p^2 + q^2\right)u + i(d + 2pk - 2cq)u' + \left(k^2 - c^2\right)u'' - 2u^3 - 2uv = 0, \quad (3.4)$$

$$\left(k^2 + c^2\right)v'' + 2\left(u^2\right)'' = 0. \quad (3.5)$$

Integration of (3.5) twice and taking the constants of integration to be zero, one obtains

$$v = \frac{-2u^2}{k^2 + c^2}. \quad (3.6)$$

Now substituting (3.6) into (3.4), we get

$$u'' = \left(\frac{r + p^2 - q^2}{k^2 - c^2}\right)u + \left(\frac{2k^2 + 2c^2 - 4}{k^4 - c^4}\right)u^3, \quad (3.7)$$

which can be written in the following form:

$$u'' = Au + Bu^3, \quad (3.8)$$

where

$$A = \frac{r + p^2 - q^2}{k^2 - c^2}, \quad B = \frac{2k^2 + 2c^2 - 4}{k^4 - c^4}. \quad (3.9)$$

Solving (3.8), with the aid of Mathematica, we obtain the following solution:

$$u = \pm \frac{1}{P_2} i \operatorname{sn}(P_1 | \omega), \quad (3.10)$$

where  $\operatorname{sn}(P_1 | \omega)$  is a Jacobian elliptic function of the *sine-amplitude* [30], and

$$P_1 = \frac{\sqrt{\left(\sqrt{A^2 - 2Bc_1} - A\right)(z + c_2)^2}}{\sqrt{2}}, \quad P_2 = \sqrt{\frac{B}{\sqrt{A^2 - 2Bc_1} + A}}, \quad (3.11)$$

$$\omega = \frac{-Bc_1 + A\left(\sqrt{A^2 - 2Bc_1} + A\right)}{Bc_1}$$

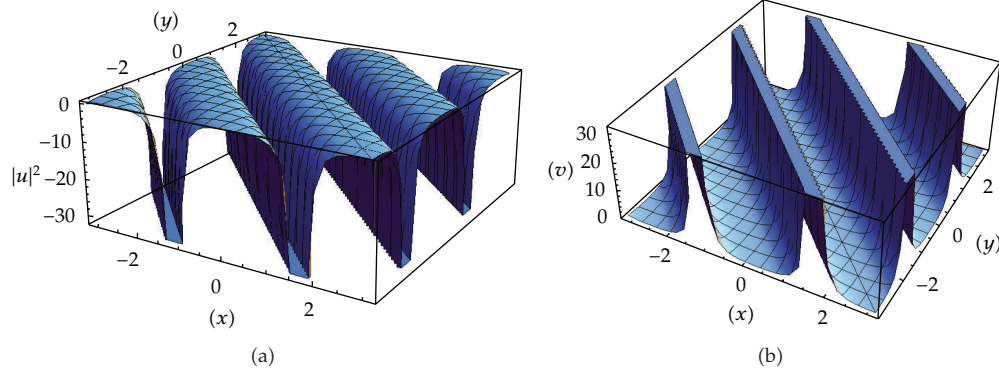


Figure 2: Profile of the solution (3.12).

is the modulus of the elliptic function with  $0 < \omega < 1$ . Here  $c_1$  and  $c_2$  are constants of integration. Reverting back to our original variables, we can now write the solution of our (2+1)-dimensional Davey-Stewartson equations as

$$u(x, y, t) = \pm \frac{1}{P_2} i \operatorname{sn}(P_1 | \omega), \quad (3.12)$$

where

$$P_1 = \frac{\sqrt{(\sqrt{A^2 - 2Bc_1} - A)(kx + cy + dt + c_2)^2}}{\sqrt{2}}, \quad (3.13)$$

$\omega$ , and  $P_2$  are as above.

Now  $v(x, y, t)$  can be obtained from (3.6).

It should be noted that the solution (3.12) is valid for  $0 < \omega < 1$  and as  $\omega$  approaches zero, the solution becomes the normal sine function,  $\sin z$ , and as  $\omega$  approaches 1, the solution tends to the tanh function,  $\tanh z$ .

The profile of the solution (3.12) is given in Figure 2.

#### 4. Conclusion

In this paper, we studied two systems of nonlinear partial differential equations. Firstly, we obtained exact solutions of the generalized Boussinesq-Burgers equations given by (2.1)-(2.2) using the Lie symmetry method. The solutions obtained were travelling wave solutions. Secondly, we found exact solutions of the (2+1)-dimensional Davey-Stewartson equations (3.1)-(3.2) using the travelling wave hypothesis. The Davey-Stewartson system was first transformed to a system of nonlinear ordinary differential equations, which was then solved to obtain the exact solutions.

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