Research Article

Global Solutions to the Spherically Symmetric Compressible Navier-Stokes Equations with Density-Dependent Viscosity

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We consider the exterior problem and the initial boundary value problem for the spherically symmetric isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficient in this paper. For regular initial density, we show that there exists a unique global strong solution to the exterior problem or the initial boundary value problem, respectively. In particular, the strong solution tends to the equilibrium state as $t \to +\infty$.

1. Introduction

The isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficients read as follows:

$$
\begin{aligned}
\rho_t + \mathrm{div}(\rho \mathbf{U}) &= 0, \\
(\rho \mathbf{U})_t + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla P(\rho) - \mathrm{div}(\mu(\rho) D(\mathbf{U})) - \nabla(\lambda(\rho)\,\mathrm{div}\,\mathbf{U}) &= 0,
\end{aligned}
\tag{1.1}
$$

where $t \in (0, +\infty)$ is the time and $\mathbf{x} \in R^N$, N is the spatial coordinate, $\rho > 0$ and u denote the density and velocity, respectively. Pressure function is taken as $P(\rho) = \rho^\gamma$ with $\gamma > 1$, and

$$
D(\mathbf{U}) = \frac{\nabla(\mathbf{U}) + {}^t\nabla(\mathbf{U})}{2}
\tag{1.2}
$$

is the strain tensor and $\mu(\rho)$, $\lambda(\rho)$ are the Lamé viscosity coefficients satisfying

$$\mu(\rho) > 0, \qquad \mu(\rho) + N\lambda(\rho) \geq 0. \tag{1.3}$$

There are many important progress achieved recently on the compressible Navier-Stokes equations with density-dependent viscosity coefficient. For instance, the mathematical derivations are derived in the simulation of flow surface in shallow region [1–4]. The prototype model is the viscous Saint-Venant. The well posedness of solutions to the free boundary value problem with initial finite mass and the flow density being connected with the infinite vacuum either continuously or via jump discontinuity is investigated by many authors, refer to [5–12] and references therein. Mellet and Vasseur showed the global existence of strong solutions for $\alpha \in (0, 1/2)$ [13]. The qualitative behaviors of global solutions and dynamical asymptotics of vacuum states are also made, such as the finite time vanishing of finite vacuum or asymptotical formation of vacuum in large time, the dynamical behaviors of vacuum boundary, the large time convergence to rarefaction wave with vacuum, and the stability of shock profile with large shock strength, refer to [14–17] and references therein.

In this present paper, we consider the exterior problem and the initial boundary value problem for the spherically symmetric isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficient and focus on the regularities and dynamical behaviors of global strong solution, and so forth. As $\gamma > 1$, we show that the exterior problem and the initial boundary value problem with regular initial data both admit the unique global strong solution. In particular, the strong solution tends to the equilibrium state as $t \to +\infty$.

The rest of the paper is arranged as follows. In Section 2, the main results about the dynamical behaviors of global strong solution for compressible Navier-Stokes equations are stated. Then, the theorems of the exterior problem and the initial boundary value problem are proved in Sections 3 and 4, respectively.

2. Notations and Main Results

For simplicity, we will take $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$ and $D(\mathbf{U}) = \nabla\mathbf{U}$ in (1.1). The isentropic compressible Navier-Stokes equations become

$$\rho_t + \text{div}(\rho\mathbf{U}) = 0,$$
$$(\rho\mathbf{U})_t + \text{div}(\rho\mathbf{U} \otimes \mathbf{U}) + \nabla P(\rho) - \text{div}(\rho\nabla\mathbf{U}) = 0. \tag{2.1}$$

Firstly, we consider the exterior problem, namely, the initial data and boundary conditions of (2.1) are imposed as follows:

$$(\rho, \mathbf{U})(\mathbf{x}, 0) = (\rho_0, \mathbf{U}_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$
$$\mathbf{U} = 0, \quad \text{on } \partial\Omega, \qquad \lim_{|\mathbf{x}| \to +\infty} (\rho, u)(\mathbf{x}, t) = (\overline{\rho}, 0), \quad t \in [0, T], \tag{2.2}$$

where $\Omega := R^3/\Omega_{r_-}$, Ω_{r_-} is a ball of radius r_- centered at the origin in R^3, and $\overline{\rho} > 0$ is a constant.

We are concerned with the spherically symmetric solutions of system (2.1) in an spherically symmetric exterior domain Ω. To this end, we denote that

$$|\mathbf{x}| = r, \qquad \rho(\mathbf{x},t) = \rho(r,t), \qquad \mathbf{U}(\mathbf{x},t) = u(r,t)\frac{\mathbf{x}}{r}, \tag{2.3}$$

which leads to the following system of equations for $r > 0$,

$$\rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0,$$
$$(\rho u)_t + \left(\rho u^2 + \rho^\gamma\right)_r + \frac{2\rho u^2}{r} - (\rho u_r)_r - \rho\left(\frac{2u}{r}\right)_r = 0, \tag{2.4}$$

with the initial data and boundary conditions

$$(\rho,u)(r,0) = (\rho_0,u_0)(r), \quad r \in [r_-,+\infty),$$
$$u(r_-,t) = 0, \quad \lim_{r\to+\infty}(\rho,u)(r,t) = (\overline{\rho},0) \quad t \in [0,T], \tag{2.5}$$

and the initial data satisfies for some constants $\overline{\rho}$ and $\underline{\rho} > 0$

$$\rho_0 - \overline{\rho} \in L^1([r_-,+\infty)) \cap L^2([r_-,+\infty)), \quad \inf_{\mathbf{x}\in[r_-,+\infty)}\rho_0 > \underline{\rho} > 0,$$
$$r\rho_{0r} \in L^2([r_-,+\infty)), \quad r^2 u_0 \in H^2([r_-,+\infty)). \tag{2.6}$$

Then, we define that

$$E_0 := \frac{1}{2}\int_{r_-}^{+\infty}\rho_0 u_0^2 r^2 dr + \int_{r_-}^{+\infty}\rho_0\left(\frac{1}{\gamma-1}\left(\rho_0^{\gamma-1} - \overline{\rho}^{\gamma-1}\right) + \overline{\rho}^\gamma\left(\rho_0^{-1} - \overline{\rho}^{-1}\right)\right)r^2 dr,$$
$$E_1 := \frac{1}{2}\int_{r_-}^{+\infty}\rho_0\left(u_0 + \rho_0^{-1}(\rho_0)_r\right)^2 r^2 dr + \int_{r_-}^{+\infty}\rho_0\left(\frac{1}{\gamma-1}\left(\rho_0^{\gamma-1} - \overline{\rho}^{\gamma-1}\right) + \overline{\rho}^\gamma\left(\rho_0^{-1} - \overline{\rho}^{-1}\right)\right)r^2 dr, \tag{2.7}$$

and give the main results as follows.

Theorem 2.1. *Let $\gamma > 1$. Assume that the initial data satisfies (2.6) and $E_0 + E_1 < \nu r_-^2\overline{\rho}^{\gamma/2+1/2}$, ν is a positive constant. Then, there exist two positive constants ρ_* and ρ^* and a unique global strong solution (ρ,u) to the exterior problem (2.4) and (2.5), namely, satisfying*

$$0 < \rho_* \leq \rho(r,t) \leq \rho^*, \quad (r,t) \in [r_-,+\infty) \times [0,T],$$
$$\rho - \overline{\rho} \in L^\infty\left([0,T];L^2([r_-,+\infty))\right), \quad \rho_r \in L^\infty\left([0,T];L^2([r_-,+\infty))\right), \tag{2.8}$$
$$u \in L^\infty\left([0,T];L^2([r_-,+\infty))\right), \quad u_r \in L^\infty\left([0,T];L^2([r_-,+\infty))\right).$$

Furthermore, the solution tends to the equilibrium state $(\overline{\rho},0)$

$$\left\|(\rho - \overline{\rho}, u)(\cdot, t)\right\|_{L^{\infty}([r_-, +\infty))} \longrightarrow 0, \quad t \longrightarrow +\infty. \tag{2.9}$$

Then investigates the initial boundary value problem, and the initial data and boundary conditions of (2.1) are assumed as follow:

$$(\rho, \mathbf{U})(\mathbf{x}, 0) = (\rho_0, \mathbf{U}_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$
$$\mathbf{U}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \ t \in [0, T]. \tag{2.10}$$

By (2.3), one considers (2.4) with the initial data and boundary conditions

$$(\rho, u)(r, 0) = (\rho_0, u_0)(r), \quad r \in [r_-, r_+],$$
$$u(r_-, t) = u(r_+, t) = 0, \quad t \in [0, T], \tag{2.11}$$

and the initial data satisfies for some constant $\underline{\rho} > 0$

$$\rho_0 \in L^1([r_-, r_+]) \cap W^{1,\infty}([r_-, r_+]), \quad \inf_{\mathbf{x} \in [r_-, r_+]} \rho_0 > \underline{\rho} > 0,$$
$$r^2 u_0 \in H^2([r_-, r_+]). \tag{2.12}$$

Then, can give the main results as follows.

Theorem 2.2. *Let $\gamma > 1$. Assume that the initial data satisfies* (2.12), *there exist two positive constants ρ_* and ρ^* and a unique global strong solution (ρ, u) to the initial boundary value problem* (2.4) *and* (2.11), *namely, satisfying*

$$0 < \rho_* \leq \rho(r, t) \leq \rho^*, \quad (r, t) \in [r_-, r_+] \times [0, T],$$
$$\rho \in L^{\infty}\left([0, T]; L^2([r_-, r_+])\right), \quad \rho_r \in L^{\infty}\left([0, T]; L^2([r_-, r_+])\right), \tag{2.13}$$
$$u \in L^{\infty}\left([0, T]; L^2([r_-, r_+])\right), \quad u_r \in L^{\infty}\left([0, T]; L^2([r_-, r_+])\right).$$

Furthermore, the solution (ρ, u) tends to the equilibrium state exponentially

$$\left\|(\rho - \widetilde{\rho}, u)(\cdot, t)\right\|_{L^{\infty}([r_-, r_+])} \leq C_0 e^{-C_1 t}, \tag{2.14}$$

where C_0 and C_1 are positive constants independent of time and $\widetilde{\rho} = \int_{r_-}^{r_+} \rho(r, t) r^2 dr$.

Remark 2.3. Theorems 2.1 and 2.2 hold for one-dimensional Saint-Venant's model for shallow water, that is, $\gamma = 2$, $\alpha = 1$.

3. Proof of the Exterior Problem

3.1. The A Priori Estimates

It is convenient to make use of the Lagrangian coordinates so as to establish the uniformly a priori estimates. Take the Lagrange coordinates transform

$$x = \int_{r_-}^{r} \rho(r,t)r^2 dr, \quad \tau = t, \tag{3.1}$$

which map $(r,t) \in [r_-, +\infty) \times R^+$ into $(x, \tau) \in [0, +\infty) \times R^+$. The relation between Lagrangian and Eulerian coordinates are satisfied as

$$\frac{\partial x}{\partial r} = \rho r^2, \qquad \frac{\partial x}{\partial t} = -\rho u r^2. \tag{3.2}$$

The exterior problem (2.4) and (2.5) is reformulated to

$$\rho_\tau + \rho^2 \left(r^2 u \right)_x = 0,$$
$$r^{-2} u_\tau + \left(\rho^\gamma \right)_x = \left(\rho^2 \left(r^2 u \right)_x \right)_x - \frac{2\rho_x u}{r},$$
$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in [0, +\infty),$$
$$u(0, t) = 0, \qquad \lim_{x \to +\infty} (\rho, u) = (\bar{\rho}, 0), \quad \tau \in [0, +\infty), \tag{3.3}$$

where the initial data satisfies

$$\rho_0 - \bar{\rho} \in L^1([0, +\infty)) \cap L^2([0, +\infty)), \quad \inf_{x \in [0, +\infty)} \rho_0 > \rho_- > 0,$$
$$r^2 \rho_{0x} \in L^2([0, +\infty)), \quad \frac{1}{\sqrt{r^2 \rho_0}} \left(r^2 u_0 \right) \in L^2([0, +\infty)),$$
$$\sqrt{r^2 \rho_0} \left(r^2 u_0 \right)_x \in L^2([0, +\infty)), \quad \frac{1}{\sqrt{r^2 \rho_0}} \left(r^2 \rho \left(r^2 \rho \left(r^2 u_0 \right)_x \right)_x \right) \in L^2([0, +\infty)). \tag{3.4}$$

First, we will establish the a-priori estimates for the solution (ρ, u) to the exterior problem (3.3).

Lemma 3.1. *Let $T > 0$. Under the conditions in Theorem 2.1, it holds for any solution (ρ, u) to the exterior problem (3.3) that*

$$
\frac{1}{2} \int_0^{+\infty} u^2 dx + \int_0^{+\infty} \left(\frac{1}{\gamma - 1} \left(\rho^{\gamma - 1} - \overline{\rho}^{\gamma - 1} \right) + \overline{\rho}^{\gamma} \left(\rho^{-1} - \overline{\rho}^{-1} \right) \right) dx
$$
$$
+ \int_0^{\tau} \int_0^{+\infty} \left(\frac{2u^2}{r^2} + \rho^2 u_x^2 r^4 \right) dx \, ds \leq E_0, \quad \tau \in [0, T].
\tag{3.5}
$$

Proof. Multiplying (3.3)$_2$ by $r^2 u$ and integrating the result with respect to x over $[0, +\infty)$, making use of (3.3)$_1$ and (3.4), we have

$$
\frac{1}{2} \frac{d}{d\tau} \int_0^{+\infty} u^2 dx + \frac{d}{d\tau} \int_0^{+\infty} \left(\frac{1}{\gamma - 1} \left(\rho^{\gamma - 1} - \overline{\rho}^{\gamma - 1} \right) + \overline{\rho}^{\gamma} \left(\rho^{-1} - \overline{\rho}^{-1} \right) \right) dx + \int_0^{+\infty} \rho^2 \left(r^2 u \right)_x^2 dx
$$
$$
= 2 \int_0^{+\infty} \rho \left(r u^2 \right)_x dx,
\tag{3.6}
$$

integrating (3.6) with respect to τ, we obtain

$$
\frac{1}{2} \int_0^{+\infty} u^2 dx + \int_0^{+\infty} \left(\frac{1}{\gamma - 1} \left(\rho^{\gamma - 1} - \overline{\rho}^{\gamma - 1} \right) + \overline{\rho}^{\gamma} \left(\rho^{-1} - \overline{\rho}^{-1} \right) \right) dx + \int_0^{\tau} \int_0^{+\infty} \left(\frac{2u^2}{r^2} + \rho^2 u_x^2 r^4 \right) dx \, ds
$$
$$
= \frac{1}{2} \int_0^{+\infty} u_0^2 dx + \int_0^{+\infty} \left(\frac{1}{\gamma - 1} \left(\rho_0^{\gamma - 1} - \overline{\rho}^{\gamma - 1} \right) + \overline{\rho}^{\gamma} \left(\rho_0^{-1} - \overline{\rho}^{-1} \right) \right) dx.
\tag{3.7}
$$

Lemma 3.1 can be obtained. $\square$

Lemma 3.2. *Let $T > 0$. Under the conditions in Theorem 2.1, it holds for any solution (ρ, u) to the exterior problem (3.3) that*

$$
\frac{1}{2} \int_0^{+\infty} \left(u + r^2 \rho_x \right)^2 dx + \int_0^{+\infty} \left(\frac{1}{\gamma - 1} \left(\rho^{\gamma - 1} - \overline{\rho}^{\gamma - 1} \right) + \overline{\rho}^{\gamma} \left(\rho^{-1} - \overline{\rho}^{-1} \right) \right) dx
$$
$$
+ \gamma \int_0^{\tau} \int_0^{+\infty} \rho^{\gamma - 1} \rho_x^2 r^4 dx \, ds \leq E_1, \quad \tau \in [0, T],
\tag{3.8}
$$

and there exist two constants $0 < \rho_* < \rho^*$ such that

$$
0 < \rho_* \leq \rho(x, \tau) \leq \rho^*, \quad (x, \tau) \in [0, +\infty) \times [0, T].
\tag{3.9}
$$

Proof. Differentiating (3.3)$_1$ with respect to x, we have

$$
\rho_{x\tau} + \left(\rho^2 \left(r^2 u \right)_x \right)_x = 0.
\tag{3.10}
$$

Summing (3.10) and (3.3)$_2$, we get

$$\left(r^{-2}u + \rho_x\right)_\tau + \left(\rho^\gamma\right)_x = \left(r^{-2}\right)_\tau u - \frac{2\rho_x u}{r}. \tag{3.11}$$

Note that

$$r^3(x,\tau) = r_-^3 + 3\int_0^x \frac{1}{\rho(z,\tau)}dz, \tag{3.12}$$

and so

$$\frac{\partial r}{\partial \tau} = \frac{1}{r^2}\int_0^x \left(\frac{1}{\rho}\right)_t (z,t)dz = \frac{1}{r^2}\int_0^x \left(r^2u\right)_z (z,t)dz = u(x,\tau), \tag{3.13}$$

which together with (3.11) yields

$$\left(r^{-2}u + \rho_x\right)_\tau + \left(\rho^\gamma\right)_x = -2r^{-3}u^2 - \frac{2\rho_x u}{r}. \tag{3.14}$$

Multiplying (3.14) by $(u + r^2\rho_x)r^2$ and integrating the result with respect to x and τ, we have

$$\begin{aligned}
\frac{1}{2}\int_0^{+\infty} &\left(u + r^2\rho_x\right)^2 dx + \int_0^{+\infty}\left(\frac{1}{\gamma-1}\left(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}\right) + \bar{\rho}^\gamma\left(\rho^{-1} - \bar{\rho}^{-1}\right)\right)dx \\
&+ \gamma\int_0^\tau\int_0^{+\infty}\rho^{\gamma-1}\rho_x^2 r^4 dx\, ds \\
&= \frac{1}{2}\int_0^{+\infty}\left(u_0 + r^2\rho_{0x}\right)^2 dx + \int_0^{+\infty}\left(\frac{1}{\gamma-1}\left(\rho_0^{\gamma-1} - \bar{\rho}^{\gamma-1}\right) + \bar{\rho}^\gamma\left(\rho_0^{-1} - \bar{\rho}^{-1}\right)\right)dx.
\end{aligned} \tag{3.15}$$

Let

$$\begin{aligned}
\varphi(\rho) &:= \frac{1}{\gamma-1}\left(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}\right) + \bar{\rho}^\gamma\left(\rho^{-1} - \bar{\rho}^{-1}\right), \\
\psi(\rho) &:= \int_{\bar{\rho}}^\rho \varphi(\eta)^{1/2}d\eta.
\end{aligned} \tag{3.16}$$

It follows from (3.6) and (3.13) that

$$\begin{aligned}
|\psi(\rho)| &\le \left|\int_y^{+\infty}\partial_x\psi(\rho)dx\right| \le r_-^{-2}\left|\int_y^{+\infty}\varphi(\rho)^{1/2}\rho_x r^2 dx\right| \\
&\le r_-^{-2}\left|\int_0^{+\infty}\varphi(\rho)dy\int_0^{+\infty}\left(r^2\rho_x\right)^2 dx\right|^{1/2} \le r_-^{-2}(E_0 + E_1).
\end{aligned} \tag{3.17}$$

We can verify that

(1) As $\rho \to +\infty$, it holds for $\xi = \theta\overline{\rho} + (1-\theta)\rho$, where $\theta \in (0,1)$

$$
\begin{aligned}
\lim_{\rho \to +\infty} \psi(\rho) &= \lim_{\rho \to +\infty} \int_{\overline{\rho}}^{\rho} \left((\gamma-2)\xi^{\gamma-3} + 2\overline{\rho}^{\gamma}\xi^{-3} \right)^{1/2} |\eta - \overline{\rho}| d\eta \\
&\geq \lim_{\rho \to +\infty} \left((\gamma-2)\xi^{\gamma-3} + 2\overline{\rho}^{\gamma}\xi^{-3} \right)^{1/2} \int_{\overline{\rho}}^{\rho} (\eta - \overline{\rho}) d\eta \\
&= \lim_{\rho \to +\infty} \frac{1}{2} \left((\gamma-2)(\theta\overline{\rho} + (1-\theta)\rho)^{\gamma-3} + 2\overline{\rho}^{\gamma}(\theta\overline{\rho} + (1-\theta)\rho)^{-3} \right)^{1/2} (\rho - \overline{\rho})^2 \\
&\longrightarrow +\infty.
\end{aligned}
\tag{3.18}
$$

(2) As $\rho \to 0$, it holds for $\xi = \theta\overline{\rho} + (1-\theta)\rho$, where $\theta \in (0,1)$

$$
\begin{aligned}
\lim_{\rho \to 0} \psi(\rho) &= \lim_{\rho \to 0} \int_{\overline{\rho}}^{\rho} \left((\gamma-2)\xi^{\gamma-3} + 2\overline{\rho}^{\gamma}\xi^{-3} \right)^{1/2} |\eta - \overline{\rho}| d\eta \\
&\leq -\lim_{\rho \to 0} \left((\gamma-2)\xi^{\gamma-3} + 2\overline{\rho}^{\gamma}\xi^{-3} \right)^{1/2} \int_{\overline{\rho}}^{\rho} (\eta - \overline{\rho}) d\eta \\
&= -\lim_{\rho \to 0} \frac{1}{2} \left((\gamma-2)(\theta\overline{\rho} + (1-\theta)\rho)^{\gamma-3} + 2\overline{\rho}^{\gamma}(\theta\overline{\rho} + (1-\theta)\rho)^{-3} \right)^{1/2} (\rho - \overline{\rho})^2 \\
&=: -\nu\overline{\rho}^{\gamma/2+1/2}.
\end{aligned}
\tag{3.19}
$$

Applying (3.17)–(3.19) and $E_0 + E_1 < \nu r_-^2 \overline{\rho}^{\gamma/2+1/2}$, where ν is a positive constant, we can prove (3.9).

$\square$

Lemma 3.3. *Let $T > 0$. Under the conditions in Theorem 2.1, it holds for any solution (ρ, u) to the exterior problem (3.3) that*

$$
\begin{aligned}
&\int_0^{+\infty} \left(r^2 u \right)_x^2 dx + \int_0^{+\infty} \left(r^2 u \right)_\tau^2 r^{-4} dx + \int_0^\tau \int_0^{+\infty} \left(r^2 u \right)_s^2 r^{-4} dx\, ds \\
&\quad + \int_0^\tau \int_0^{+\infty} \rho^2 \left(r^2 u \right)_{xs}^2 dx\, ds + \int_0^\tau \int_0^{+\infty} \left(r^2 u \right)_{xx}^2 dx\, ds \leq C, \quad \tau \in [0, T],
\end{aligned}
\tag{3.20}
$$

where $C > 0$ denotes a constant independent of time.

Proof. Multiplying $(3.3)_2$ by $\rho^{-2}(r^2 u)_\tau$ and integrating the result with respect to x over $[0, +\infty)$, making use of (3.4), we obtain

$$
\frac{d}{d\tau} \int_0^{+\infty} \left(\frac{1}{2} (r^2 u)_x^2 - \rho^{\gamma-2} (r^2 u)_x \right) dx + \int_0^{+\infty} \rho^{-2} (r^2 u)_\tau^2 r^{-4} dx
$$

$$
= (\gamma - 2) \int_0^{+\infty} \rho^{\gamma-1} (r^2 u)_x^2 dx - 2 \int_0^{+\infty} \rho^{\gamma-3} \rho_x (r^2 u)_\tau dx + 2 \int_0^{+\infty} \rho^{-1} \rho_x (r^2 u)_x (r^2 u)_\tau dx
$$

$$
+ 2 \int_0^{+\infty} \rho^{-2} u^2 (r^2 u)_\tau r^{-3} dx - 2 \int_0^{+\infty} \rho^{-2} \rho_x u (r^2 u)_\tau r^{-1} dx,
$$

$$(3.21)$$

which implies

$$
\int_0^{+\infty} (r^2 u)_x^2 dx + \int_0^\tau \int_0^{+\infty} (r^2 u)_s^2 r^{-4} dx\, ds
$$

$$
\leq C + C \int_0^{+\infty} \left(\rho^{\gamma-2} - \overline{\rho}^{\gamma-2} \right)^2 dx + C \int_0^\tau \int_0^{+\infty} \left(\frac{u^2}{r^2} + u_x^2 r^4 \right) dx\, ds + C \int_0^\tau \int_0^{+\infty} \rho_x^2 r^4 dx\, ds
$$

$$
+ C \int_0^\tau \int_0^{+\infty} \rho_x^2 (r^2 u)_x^2 r^4 dx\, ds + C \int_0^\tau \int_0^{+\infty} u^4 r^{-2} dx\, ds + C \int_0^\tau \int_0^{+\infty} \rho_x^2 u^2 r^2 dx\, ds
$$

$$
\leq C + C \int_0^\tau \int_0^{+\infty} \rho_x^2 (r^2 u)_x^2 r^4 dx\, ds + C \sup_{\tau \in [0,T]} \|u\|_{L^\infty}^2.
$$

$$(3.22)$$

From $(3.3)_2$, (3.5), (3.8), and (3.9), we can deduce that for some small $\epsilon \in (0,1)$

$$
\int_0^\tau \int_0^{+\infty} \rho_x^2 (r^2 u)_x^2 r^4 dx\, ds \leq \epsilon \int_0^\tau \int_0^{+\infty} \rho_x^2 (r^2 u)_x^2 r^4 dx\, ds + \epsilon \int_0^\tau \int_0^{+\infty} (r^2 u)_{xs}^2 r^{-4} dx\, ds
$$

$$
+ \epsilon \int_0^\tau \int_0^{+\infty} \rho_x^2 dx\, ds + C(\epsilon) \int_0^\tau \int_0^{+\infty} \left(\frac{u^2}{r^2} + u_x^2 r^4 \right) dx\, ds,
$$

$$(3.23)$$

$$
\sup_{\tau \in [0,T]} \|u\|_{L^\infty}^2 \leq \epsilon \sup_{\tau \in [0,T]} \int_0^{+\infty} (r^2 u)_x^2 dx + C(\epsilon) \sup_{\tau \in [0,T]} \int_0^{+\infty} u^2 dx,
$$

$$(3.24)$$

using (3.22)–(3.24), we can obtain that

$$
\int_0^{+\infty} (r^2 u)_x^2 dx + \int_0^\tau \int_0^{+\infty} (r^2 u)_s^2 r^{-4} dx\, ds \leq C + C\epsilon \int_0^\tau \int_0^{+\infty} (r^2 u)_{xs}^2 r^{-4} dx\, ds.
$$

$$(3.25)$$

Differentiating $(3.3)_2$ with respect to τ, multiplying the result by $(r^2 u)_\tau$, and integrating the result with respect to x over $[0, +\infty)$, we have

$$\frac{1}{2}\frac{d}{d\tau}\int_0^{+\infty}\left(r^2u\right)_\tau^2 r^{-4}dx + \int_0^{+\infty}\rho^2\left(r^2u\right)_{x\tau}^2 dx$$
$$= 2\int_0^{+\infty}uu_\tau\left(r^2u\right)_\tau r^{-3}dx - \frac{1}{2}\int_0^{+\infty}\left(r^{-4}\right)_\tau\left(r^2u\right)_\tau^2 dx + 2\int_0^{+\infty}u\left(r^{-1}u\right)_\tau\left(r^2u\right)_\tau r^{-2}dx$$
$$+ \int_0^{+\infty}\left(\rho^\gamma\right)_\tau\left(r^2u\right)_{x\tau}dx - \int_0^{+\infty}\left(\rho^2\right)_\tau\left(r^2u\right)_x\left(r^2u\right)_{x\tau}dx - \int_0^{+\infty}\left(\frac{2\rho_x u}{r}\right)_\tau\left(r^2u\right)_\tau dx$$
$$+ 2\int_0^{+\infty}\left(r^{-2}\right)_\tau r^{-1}u^2\left(r^2u\right)_\tau dx. \tag{3.26}$$

A complicated computation gives

$$\frac{d}{d\tau}\int_0^{+\infty}\left(r^2u\right)_\tau^2 r^{-4}dx + \int_0^{+\infty}\rho^2\left(r^2u\right)_{x\tau}^2 dx$$
$$\leq C\int_0^{+\infty}\left(r^2u\right)_\tau^2 r^{-4}dx + C\sup_{\tau\in[0,T]}\left\|\left(r^2u\right)_x^2\right\|_{L^\infty}\int_0^{+\infty}\left(r^2u\right)_x^2 dx \tag{3.27}$$
$$+ C\int_0^{+\infty}\left(\frac{u^2}{r^2} + u_x^2 r^4\right)dx\left(1 + \sup_{\tau\in[0,T]}\int_0^{+\infty}\left(r^2u\right)_x^2 dx\right),$$

integrating (3.27) with respect to τ, by means of $(3.3)_2$, (3.5), (3.8), (3.9), and (3.25), it holds that

$$\int_0^{+\infty}\left(r^2u\right)_\tau^2 r^{-4}dx + \int_0^\tau\int_0^{+\infty}\rho^2\left(r^2u\right)_{xs}^2 dx\,ds$$
$$\leq C + C\int_0^\tau\int_0^{+\infty}\left(r^2u\right)_s^2 r^{-4}dx\,ds + C\sup_{\tau\in[0,T]}\left\|\left(r^2u\right)_x^2\right\|_{L^\infty} + C\sup_{\tau\in[0,T]}\int_0^{+\infty}\left(r^2u\right)_x^2 dx$$
$$\leq C + C\int_0^\tau\int_0^{+\infty}\left(r^2u\right)_s^2 r^{-4}dx\,ds + C\sup_{\tau\in[0,T]}\left(\int_0^{+\infty}\left(r^2u\right)_x^2 dx\right)^{1/2}\left(\int_0^{+\infty}\left(r^2u\right)_{xx}^2 dx\right)^{1/2}$$
$$+ C\sup_{\tau\in[0,T]}\int_0^{+\infty}\left(r^2u\right)_x^2 dx$$
$$\leq C + C\epsilon\int_0^\tau\int_0^{+\infty}\rho^2\left(r^2u\right)_{xs}^2 dx\,ds + C\epsilon\int_0^{+\infty}\left(r^2u\right)_\tau^2 r^{-4}dx, \tag{3.28}$$

choosing the constant ϵ small sufficiently, we can complete the proof of Lemma 3.3. $\square$

Remark 3.4. By Lemmas 3.1–3.3, the following inequality holds:

$$
\begin{aligned}
\int_0^{+\infty} u^2 dx &+ \int_0^{+\infty} (\rho - \bar{\rho})^2 dx + \int_0^{+\infty} u_x^2 dx + \int_0^{+\infty} u_\tau^2 dx + \int_0^{+\infty} \rho_x^2 dx \\
&+ \int_0^\tau \int_0^{+\infty} \rho_x^2 dx\, ds + \int_0^\tau \int_0^{+\infty} u_x^2 dx\, ds + \int_0^\tau \int_0^{+\infty} u_s^2 dx\, ds \\
&+ \int_0^\tau \int_0^{+\infty} u_{xx}^2 dx\, ds + \int_0^\tau \int_0^{+\infty} u_{x\tau}^2 dx\, ds \le C.
\end{aligned}
\tag{3.29}
$$

Lemma 3.5. *Under the conditions in Theorem 2.1, it holds for any solution (ρ, u) to the exterior problem (3.3) that*

$$
\left\| (\rho - \bar{\rho}, u)(\cdot, \tau) \right\|_{L^\infty([0,+\infty))} \longrightarrow 0, \quad \tau \longrightarrow +\infty,
\tag{3.30}
$$

where $C > 0$ denotes a constant independent of time.

Proof. From Lemmas 3.1–3.3, we can obtain

$$
\int_0^\infty \left\| (\rho - \bar{\rho}, u)_x \right\|_{L^2([0,+\infty))}^2 d\tau \le r_-^{-4} \int_0^\infty \left\| (\rho - \bar{\rho}, u)_x r^2 \right\|_{L^2([0,+\infty))}^2 d\tau \le C,
\tag{3.31}
$$

$$
\begin{aligned}
\int_0^\infty &\left| \frac{d}{d\tau} \left\| (\rho - \bar{\rho}, u)_x \right\|_{L^2([0,+\infty))}^2 \right| d\tau \\
&= \int_0^\infty \left| \int_0^{+\infty} \left(-4\rho\rho_x^2 \left(r^2 u \right)_x - 2\rho^2 \rho_x \left(r^2 u \right)_{xx} \right) dx + 2\int_0^{+\infty} u_x u_{x\tau} dx \right| d\tau \\
&\le C \int_0^\infty \int_0^{+\infty} \rho_x^2 dx\, d\tau + C \int_0^\infty \int_0^{+\infty} \left(\frac{u^2}{r^2} + u_x^2 r^4 + \left(r^2 u \right)_{xx}^2 + u_{x\tau}^2 \right) dx\, d\tau \le C,
\end{aligned}
\tag{3.32}
$$

which together with (3.31) implies

$$
\left\| (\rho - \bar{\rho}, u)_x \right\|_{L^2([0,+\infty))}^2 \in W^{1,1}(R^+).
\tag{3.33}
$$

It holds from Gagliardo-Nirenberg-Sobolev inequality that

$$
\left\| (\rho - \bar{\rho}, u) \right\|_{L^\infty([0,+\infty))} \le \left\| (\rho - \bar{\rho}, u) \right\|_{L^2([0,+\infty))}^{1/2} \left\| (\rho - \bar{\rho}, u)_x \right\|_{L^2([0,+\infty))}^{1/2},
\tag{3.34}
$$

which together with (3.5), (3.9), and (3.33) implies this lemma. $\qquad\square$

3.2. Proof of Theorem 2.1

Proof. The global existence of unique strong solution to the exterior problem as (2.4) and (2.5) can be established in terms of the short-time existence carried out as in [6], the uniform

a-priori estimates and the analysis of regularities, which indeed follow from Lemmas 3.1–3.3. We omit the details. The large time behaviors follow from Lemma 3.5 directly. The proof of Theorem 2.1 is completed.   □

4. Proof of the Initial Boundary Value Problem

4.1. The A-Priori Estimates

Take the Lagrange coordinates transform

$$x = \int_{r_-}^{r} \rho(r,t)r^2 dr, \quad \tau = t. \tag{4.1}$$

By (4.1) and the conservation of mass for (ρ, u)

$$\int_{r_-}^{r_+} \rho(r,t)r^2 dr = \int_{r_-}^{r_+} \rho_0(r)r^2 dr =: 1, \tag{4.2}$$

the Lagrange coordinates transform (4.1) map $(r,t) \in [r_-, r_+] \times R^+$ into $(x, \tau) \in [0,1] \times R^+$.

The relation between Lagrangian and Eulerian coordinates are satisfied as

$$\frac{\partial x}{\partial r} = \rho r^2, \qquad \frac{\partial x}{\partial t} = -\rho u r^2, \tag{4.3}$$

and the initial boundary value problem's (2.4) and (2.11) are reformulated to

$$
\begin{aligned}
\rho_\tau + \rho^2 \left(r^2 u\right)_x &= 0, \\
r^{-2} u_\tau + (\rho^\gamma)_x &= \left(\rho^2 \left(r^2 u\right)_x\right)_x - \frac{2\rho_x u}{r}, \\
(\rho, u)(x, 0) &= (\rho_0, u_0)(x), \quad x \in [0,1], \\
u(0,t) = u(1,t) &= 0, \quad \tau \in [0, +\infty),
\end{aligned} \tag{4.4}
$$

where the initial data satisfies

$$\rho_0 \in L^1([0,1]) \cap W^{1,\infty}([0,1]), \quad \inf_{x \in [0,1]} \rho_0 > \underline{\rho} > 0,$$

$$\frac{1}{\sqrt{r^2 \rho_0}} \left(r^2 u_0\right) \in L^2([0,1]), \quad \sqrt{r^2 \rho_0} \left(r^2 u_0\right)_x \in L^2([0,1]),$$

$$\frac{1}{\sqrt{r^2 \rho_0}} \left(r^2 \rho \left(r^2 \rho \left(r^2 u_0\right)_x\right)_x\right) \in L^2([0,1]). \tag{4.5}$$

Then, we will establish the a-priori estimates for the solution (ρ, u) to the initial boundary value problem (4.4).

Lemma 4.1. *Let $T > 0$. Under the conditions in Theorem 2.2, it holds for any solution (ρ, u) to the initial boundary value problem* (4.4) *that*

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma - 1}\rho^{\gamma-1} \right) dx + \int_0^\tau \int_0^1 \left(\frac{2u^2}{r^2} + \rho^2 u_x^2 r^4 \right) dx\, ds$$
$$= \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma - 1}\rho_0^{\gamma-1} \right) dx, \quad \tau \in [0, T]. \tag{4.6}$$

Proof. Multiplying (4.4)$_2$ by $r^2 u$ and integrating the result with respect to x over $[0, 1]$, using (4.4)$_1$ and (4.5), we obtain

$$\frac{d}{d\tau} \int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma - 1}\rho^{\gamma-1} \right) dx + \int_0^1 \rho^2 \left(r^2 u \right)_x^2 dx = 2 \int_0^1 \rho \left(r u^2 \right)_x dx, \tag{4.7}$$

and integrating (4.7) with respect to τ, we obtain

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma - 1}\rho^{\gamma-1} \right) dx + \int_0^\tau \int_0^1 \left(\frac{2u^2}{r^2} + \rho^2 u_x^2 r^4 \right) dx\, ds = \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma - 1}\rho_0^{\gamma-1} \right) dx. \tag{4.8}$$

Lemma 4.1 can be obtained. $\qquad\square$

Lemma 4.2. *Let $T > 0$. Under the conditions in Theorem 2.2, it holds for any solution (ρ, u) to the initial boundary value problem* (4.4) *that*

$$\frac{1}{2} \int_0^1 \left(u + r^2 \rho_x \right)^2 dx + \frac{1}{\gamma - 1} \int_0^1 \rho^{\gamma-1} dx + \gamma \int_0^\tau \int_0^1 \rho^{\gamma-1} \rho_x^2 r^4 dx\, ds$$
$$= \frac{1}{2} \int_0^1 \left(u_0 + r^2 \rho_{0x} \right)^2 dx + \frac{1}{\gamma - 1} \int_0^1 \rho_0^{\gamma-1} dx, \quad \tau \in [0, T], \tag{4.9}$$

where C is a positive constant independent of time.

Proof. Differentiating (4.4)$_1$ with respect to x, we have

$$\rho_{x\tau} + \left(\rho^2 \left(r^2 u \right)_x \right)_x = 0, \tag{4.10}$$

which together with (4.4)$_2$ and $\partial r / \partial \tau = u$ gives

$$\left(r^{-2} u + \rho_x \right)_\tau + \left(\rho^\gamma \right)_x = -2r^{-3}u^2 - \frac{2\rho_x u}{r}. \tag{4.11}$$

Multiplying (4.11) by $(u + r^2\rho_x)r^2$, and integrating the result with respect to x and τ, it holds that

$$
\begin{aligned}
&\frac{1}{2} \int_0^1 \left(u + r^2\rho_x\right)^2 dx + \frac{1}{\gamma - 1} \int_0^1 \rho^{\gamma-1} dx + \gamma \int_0^\tau \int_0^1 \rho^{\gamma-1} \rho_x^2 r^4 dx\, ds \\
&\quad = \frac{1}{2} \int_0^1 \left(u_0 + r^2\rho_{0x}\right)^2 dx + \frac{1}{\gamma - 1} \int_0^1 \rho_0^{\gamma-1} dx.
\end{aligned}
\tag{4.12}
$$

The proof of (4.9) is completed. $\qquad\square$

Lemma 4.3. *Let $T > 0$. Under the conditions in Theorem 2.2, there exists a constant $\rho^* > 0$ such that*

$$
0 < \rho(x,\tau) \le \rho^*, \quad (x,\tau) \in [0,1] \times [0,T].
\tag{4.13}
$$

Proof. It follows from (4.6) and (4.9) that

$$
\begin{aligned}
\rho(x,\tau) = \rho(r,t) &\le \int_{r_-}^{r_+} \rho(r,t) dr + \int_{r_-}^{r_+} |\rho_r(r,t)| dr \\
&\le r_-^{-2} \int_{r_-}^{r_+} \rho(r,t) r^2 dr + r_-^{-2} \int_{r_-}^{r_+} \sqrt{\rho} \frac{|\rho_r(r,t)|}{\sqrt{\rho}} r^2 dr \\
&\le C + C \int_{r_-}^{r_+} \frac{1}{\rho} |\rho_r(r,t)|^2 r^2 dr \\
&\le C + C \int_0^1 |\rho_x(x,\tau)|^2 r^4 dx \le C =: \rho^*.
\end{aligned}
\tag{4.14}
$$

$\qquad\square$

Lemma 4.4. *Let $T > 0$. Under the conditions in Theorem 2.2, it holds for any solution (ρ, u) to the initial boundary value problem (4.4) that*

$$
\int_0^1 u^{2n} dx + \int_0^\tau \int_0^1 \left(\frac{u^{2n}}{r^2} + \rho^2 u^{2n-2} u_x^2 r^4\right) dx\, ds \le C(T),
\tag{4.15}
$$

for any positive integer $n \in N$, where $C(T)$ is a positive constant dependent of time.

Proof. Multiplying $(4.4)_2$ with u^{2n-1}, integrating by parts over $[0,1]$, we have

$$
\frac{1}{2n} \frac{d}{d\tau} \int_0^1 u^{2n} dx + \int_0^1 \rho^2 \left(r^2 u\right)_x \left(r^2 u^{2n-1}\right)_x dx = \int_0^1 \left(\rho^\gamma \left(r^2 u^{2n-1}\right)_x + \rho\left(2ru^{2n}\right)_x\right) dx.
\tag{4.16}
$$

Since it holds that

$$
\begin{aligned}
\left(r^2 u\right)_x \left(r^2 u^{2n-1}\right)_x &= \left(\frac{2u}{\rho r} + r^2 u_x\right)\left(\frac{2u^{2n-1}}{\rho r} + (2n-1)r^2 u^{2n-2} u_x\right) \\
&= \frac{4u^{2n}}{\rho^2 r^2} + (2n-1)u^{2n-2} u_x^2 r^4 + \frac{4n u^{2n-1} u_x r}{\rho},
\end{aligned}
\tag{4.17}
$$

it follows from (4.16) that

$$
\begin{aligned}
\frac{d}{d\tau}\int_0^1 \frac{u^{2n}}{2n}dx + 2\int_0^1 \frac{u^{2n}}{r^2}dx &+ (2n-1)\int_0^1 \rho^2 u^{2n-2} u_x^2 r^4 dx \\
&= 2\int_0^1 \frac{\rho^{\gamma-1} u^{2n-1}}{r}dx + (2n-1)\int_0^1 \rho^\gamma u^{2n-2} u_x r^2 dx \\
&\leq \int_0^1 \frac{u^{2n}}{r^2}dx + \int_0^1 \rho^2 u^{2n-2} u_x^2 r^4 dx + C\|\rho\|_{L^\infty}^{2(\gamma-1)}\int_0^1 u^{2n-2}dx,
\end{aligned}
\tag{4.18}
$$

which together with (4.13) and Young's inequality yields

$$
\frac{d}{d\tau}\int_0^1 u^{2n}dx + \int_0^1 \frac{u^{2n}}{r^2}dx + \int_0^1 \rho^2 u^{2n-2} u_x^2 r^4 dx \leq C + \int_0^1 u^{2n}dx,
\tag{4.19}
$$

and by applying the Gronwall's inequality to (4.19), we can obtain (4.15). □

Lemma 4.5. *Let $T > 0$, for $n \in \mathbb{N}$, and $n > 1/2(\gamma-1)$. Under the conditions in Theorem 2.2, it holds for any solution (ρ, u) to the initial boundary value problem (4.4) that*

$$
\int_0^\tau \left\|\rho^{2n(\gamma-1)} u^{2n}\right\|_{L^\infty([0,1])} ds \leq C(T),
\tag{4.20}
$$

$$
\int_0^\tau \left\|\left|(\rho^\gamma)_x r^2\right|^{2n}\right\|_{L^\infty([0,1])} ds \leq C(T), \quad \tau \in [0,T],
\tag{4.21}
$$

where $C(T)$ is a positive constant dependent of time.

Proof. By means of Sobolev imbedding theorem and Cauchy-Schwarz inequality, applying (4.6), (4.13), and (4.15), we get

$$
\begin{aligned}
\int_0^\tau & \left\| \rho^{2n(\gamma-1)} u^{2n} \right\|_{L^\infty([0,1])} ds \\
&\leq \int_0^\tau \int_0^1 \left| \rho^{2n(\gamma-1)} u^{2n} \right| dx\, ds + \int_0^\tau \int_0^1 \left| \left(\rho^{2n(\gamma-1)} u^{2n} \right)_x \right| dx\, ds \\
&\leq C(T) + C \int_0^\tau \int_0^1 \rho^{2n(\gamma-1)-1} |\rho_x| u^{2n} dx\, ds + C \int_0^\tau \int_0^1 \rho^{2n(\gamma-1)} \left| u^{2n-1} u_x \right| dx\, ds \\
&\leq C(T) + C \int_0^\tau \int_0^1 |\rho_x|^2 r^4 dx\, ds + C \int_0^\tau \int_0^1 \rho^{2(2n(\gamma-1)-1)} u^{4n} r^{-4} dx\, ds \\
&\quad + \int_0^\tau \int_0^1 \rho^{2(2n(\gamma-1)-1)} u^{2n} r^{-4} dx\, ds + C \int_0^\tau \int_0^1 \rho^2 u^{2n-2} u_x^2 r^4 dx\, ds \leq C(T).
\end{aligned}
\tag{4.22}
$$

Next, we find that

$$
\rho_x r^2 = \rho_{0x} r_0^2 + u_0 - u - \int_0^\tau (\rho^\gamma)_x r^2 ds,
\tag{4.23}
$$

which together with (4.13), (4.15), and (4.22) gives

$$
\begin{aligned}
\int_0^\tau & \left\| \left| (\rho^\gamma)_x r^2 \right|^{2n} \right\|_{L^\infty} ds \\
&= \gamma^{2n} \int_0^\tau \left\| \rho^{2n(\gamma-1)} \left| \rho_x r^2 \right|^{2n} \right\|_{L^\infty} ds \\
&= \gamma^{2n} \int_0^\tau \left\| \rho^{2n(\gamma-1)} \left| \rho_{0x} r_0^2 + u_0 - u - \int_0^s (\rho^\gamma)_x r^2 dl \right|^{2n} \right\|_{L^\infty} ds \\
&\leq C \int_0^\tau \left\| \rho^{2n(\gamma-1)} \left(\rho_{0x}^{2n} + u_0^{2n} + u^{2n} \right) \right\|_{L^\infty} ds + C \int_0^\tau \left\| \rho^{2n(\gamma-1)} \left(\int_0^s (\rho^\gamma)_x r^2 dl \right)^{2n} \right\|_{L^\infty} ds \\
&\leq C(T) + C \int_0^\tau \left\| \rho^{2n(\gamma-1)} u^{2n} \right\|_{L^\infty} ds + C(T) \int_0^\tau \int_0^s \left\| \left| (\rho^\gamma)_x r^2 \right|^{2n} \right\|_{L^\infty} dl\, ds,
\end{aligned}
\tag{4.24}
$$

applying the Gronwall's inequality to (4.24), we obtain (4.21). $\qquad\square$

Lemma 4.6. *Let $T > 0$. Under the conditions in Theorem 2.2, there exists a constant $\rho_* > 0$ such that*

$$
\rho(x, \tau) \geq \rho_* > 0, \quad (x, \tau) \in [0, 1] \times [0, T].
\tag{4.25}
$$

Proof. It is easy to verify that

$$
\begin{aligned}
\overline{\rho^{(\gamma+1)/2}}(\tau) =: \int_0^1 \rho^{(\gamma+1)/2}(x,\tau)dx &\geq \frac{r_-^2}{r_+^2}\left(\int_{r_-}^{r_+}\rho(r,t)r^2dr\right)^{1+(\gamma+1)/2} \\
&= \frac{r_-^2}{r_+^2}\left(\int_{r_-}^{r_+}\rho_0(r)r^2dr\right)^{1+(\gamma+1)/2} > 0,
\end{aligned}
\tag{4.26}
$$

$$
\left\|\rho^{(\gamma+1)/2} - \overline{\rho^{(\gamma+1)/2}}(\tau)\right\|_{L^2([0,1])}^2 \in W^{1,1}([0,T]).
\tag{4.27}
$$

Indeed, it holds that

$$
\int_0^\tau \left\|\rho^{(\gamma+1)/2} - \overline{\rho^{(\gamma+1)/2}}(s)\right\|_{L^2([0,1])}^2 ds \leq C\int_0^\tau \left\|\left(\rho^{(\gamma+1)/2}\right)_x r^2\right\|_{L^2([0,1])}^2 ds \leq C,
\tag{4.28}
$$

which together with (4.4), (4.9), and (4.13) gives

$$
\begin{aligned}
\int_0^\tau &\left|\frac{d}{ds}\left\|\rho^{(\gamma+1)/2} - \overline{\rho^{(\gamma+1)/2}}(s)\right\|_{L^2([0,1])}^2\right|ds \\
&\leq (\gamma+1)\int_0^\tau\left|\int_0^1\left(\rho^{(\gamma+1)/2} - \overline{\rho^{(\gamma+1)/2}}(s)\right)\rho^{(\gamma+1)/2-1}\rho_s dx\right|ds \\
&\quad + 2\int_0^\tau\left|\int_0^1\left(\rho^{(\gamma+1)/2} - \overline{\rho^{(\gamma+1)/2}}(s)\right)\left(\overline{\rho^{(\gamma+1)/2}}(s)\right)_s dx\right|ds \\
&\leq C\int_0^\tau\int_0^1\left(\rho^{(\gamma+1)/2} - \overline{\rho^{(\gamma+1)/2}}(s)\right)^2 dx\,ds + \int_0^\tau\int_0^1\rho^2 u_x^2 r^4 dx\,ds \leq C.
\end{aligned}
\tag{4.29}
$$

By Gagliardo-Nirenberg-Sobolev inequality and (4.27), it follows that

$$
\left\|\rho^{(\gamma+1)/2} - \overline{\rho^{(\gamma+1)/2}}(\tau)\right\|_{L^\infty([0,1])}^2 \longrightarrow 0 \quad \text{as } \tau \longrightarrow +\infty.
\tag{4.30}
$$

Thus, there is a $T_0 > 0$ and a constant $\rho_1 > 0$ such that

$$
\rho(x,\tau) \geq \rho_1, \quad x \in [0,1], \ \tau \in [T_0, +\infty).
\tag{4.31}
$$

For $\tau \in [0, T_0]$, denote

$$
v(x,\tau) =: \frac{1}{\rho(x,\tau)r^2(x,\tau)},
\tag{4.32}
$$

then from (4.4), we can obtain

$$v_\tau = \frac{\left(r^2 u\right)_x}{r^2} - \frac{2vu}{r}, \tag{4.33}$$

multiplying (4.33) by $4v^3$ and integrating the result over $[0,1] \times [0,T_0]$, we have

$$
\begin{aligned}
\int_0^1 v^4 dx &= \int_0^1 v_0^4 dx + 4 \int_0^\tau \int_0^1 v^3 \frac{\left(r^2 u\right)_x}{r^2} dx\, ds - 8 \int_0^\tau \int_0^1 \frac{v^4 u}{r} dx\, ds \\
&= \int_0^1 v_0^4 dx + 12 \int_0^\tau \int_0^1 v^4 u \left(\rho_x r^2\right) dx\, ds + 24 \int_0^\tau \int_0^1 v^4 \frac{u}{r} dx\, ds \\
&= \int_0^1 v_0^4 dx + 12 \int_0^\tau \int_0^1 v^4 u \left(\rho_{0x} r_0^2 + u_0 - u - \int_0^s \left(\rho^\gamma\right)_x r^2 dl\right) dx\, ds \\
&\quad + 24 \int_0^\tau \int_0^1 v^4 \frac{u}{r} dx\, ds.
\end{aligned}
\tag{4.34}
$$

From (4.21), it holds that

$$
\begin{aligned}
&\int_0^1 v^4 dx + 12 \int_0^\tau \int_0^1 v^4 u^2 dx\, ds \\
&\quad = \int_0^1 v_0^4 dx + 12 \int_0^\tau \int_0^1 v^4 u \rho_{0x} r_0^2 dx\, ds + 12 \int_0^\tau \int_0^1 v^4 u u_0 dx\, ds \\
&\qquad - 12 \int_0^\tau \int_0^1 v^4 u \left(\int_0^s \left(\rho^\gamma\right)_x r^2 dl\right) dx\, ds + 24 \int_0^\tau \int_0^1 v^4 \frac{u}{r} dx\, ds \\
&\quad \leq 6 \int_0^\tau \int_0^1 v^4 u^2 dx\, ds + C(T_0) \int_0^\tau \int_0^1 v^4 dx\, ds,
\end{aligned}
\tag{4.35}
$$

which yields to

$$\int_0^1 v^4 dx \leq C + C(T_0) \int_0^\tau \int_0^1 v^4 dx\, ds, \tag{4.36}$$

where $C(T_0)$ is a positive constant dependent of time T_0. By Gronwall's inequality, (4.36) leads to

$$\int_0^1 v^4 dx = \int_0^1 \frac{1}{\rho^4 r^8} dx \leq C(T_0). \tag{4.37}$$

It holds for $(x, \tau) \in ([0, 1]) \times [0, T_0]$ that

$$
\begin{aligned}
\frac{1}{\rho} &= \int_0^1 \frac{1}{\rho} dx + \int_0^1 \left| \left(\frac{1}{\rho} \right)_x \right| dx \\
&\leq C + C \int_0^1 \frac{1}{\rho^4 r^8} dx + C \left(\int_0^1 \frac{1}{\rho^4 r^8} dx \right)^{1/2} \left(\int_0^1 \rho_x^2 r^4 dx \right)^{1/2} \leq C(T_0),
\end{aligned}
\tag{4.38}
$$

namely,

$$
\rho(x, \tau) \geq C(T_0) =: \rho_2, \quad (x, \tau) \in ([0, 1]) \times [0, T_0]. \tag{4.39}
$$

Therefore, we can choose

$$
\rho_* = \min\{\rho_1, \rho_2\}, \tag{4.40}
$$

to get

$$
\rho \geq \rho_*, \quad (x, \tau) \in ([0, 1]) \times [0, +\infty). \tag{4.41}
$$

$\square$

Lemma 4.7. *Let $T > 0$. Under the conditions in Theorem 2.2, it holds for any solution (ρ, u) to the initial boundary value problem (4.4) that*

$$
\begin{aligned}
\int_0^1 \left(r^2 u \right)_x^2 dx &+ \int_0^1 \left(r^2 u \right)_\tau^2 r^{-4} dx + \int_0^\tau \int_0^1 \left(r^2 u \right)_s^2 r^{-4} dx\, ds \\
&+ \int_0^\tau \int_0^1 \rho^2 \left(r^2 u \right)_{xs}^2 dx\, ds + \int_0^\tau \int_0^1 \left(r^2 u \right)_{xx}^2 dx\, ds \leq C, \quad \tau \in [0, T],
\end{aligned}
\tag{4.42}
$$

where $C > 0$ denotes a constant independent of time.

Proof. The proof is similar to Lemma 3.3. We omit here. $\square$

Remark 4.8. By Lemmas 4.1–4.7, the following inequality holds:

$$
\begin{aligned}
\int_0^1 u^2 dx &+ \int_0^1 (\rho - \bar{\rho})^2 dx + \int_0^1 u_x^2 dx + \int_0^1 u_\tau^2 dx + \int_0^1 \rho_x^2 dx \\
&+ \int_0^\tau \int_0^1 \rho_x^2 dx\, ds + \int_0^\tau \int_0^1 u_x^2 dx\, ds + \int_0^\tau \int_0^1 u_s^2 dx\, ds \\
&+ \int_0^\tau \int_0^1 u_{xx}^2 dx\, ds + \int_0^\tau \int_0^1 u_{x\tau}^2 dx\, ds \leq C.
\end{aligned}
\tag{4.43}
$$

Lemma 4.9. *Under the conditions in Theorem 2.2, it holds for any solution (ρ, u) to the initial boundary value problems (2.4) and (2.11) that*

$$\left\|(\rho - \tilde{\rho}, u)(\cdot, t)\right\|_{L^\infty([r_-, r_+])} \le C_0 e^{-C_1 t}, \quad t > 0, \tag{4.44}$$

where C_0 and C_1 denote the constants independent of time and $\tilde{\rho} = \int_{r_-}^{r_+} \rho(r, t) r^2 dr$.

Proof. In a similar argument to show (4.8) and (4.12) with modification, we can obtain the following

$$\frac{d}{dt} \int_{r_-}^{r_+} \left(\frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \left(\rho^\gamma - \tilde{\rho}^\gamma - \gamma \tilde{\rho}^{\gamma-1} (\rho - \tilde{\rho}) \right) \right) r^2 dr + \int_{r_-}^{r_+} \left(2\rho u^2 + \rho u_r^2 r^2 \right) dr = 0, \tag{4.45}$$

$$\frac{1}{2} \frac{d}{dt} \int_{r_-}^{r_+} \rho \left(u + \rho^{-1} \rho_r \right)^2 r^2 dr + \frac{1}{\gamma - 1} \frac{d}{dt} \int_{r_-}^{r_+} \left(\rho^\gamma - \tilde{\rho}^\gamma - \gamma \tilde{\rho}^{\gamma-1} (\rho - \tilde{\rho}) \right) r^2 dr$$
$$+ \gamma \int_{r_-}^{r_+} \rho^{\gamma-2} \rho_r^2 r^2 dr = 0. \tag{4.46}$$

It holds from (4.9), (4.13), and (4.25) that

$$\int_{r_-}^{r_+} (\rho - \tilde{\rho})^2 r^2 dr \le C \int_{r_-}^{r_+} \rho^{\gamma-2} \rho_r^2 r^2 dr. \tag{4.47}$$

Denote

$$E(t) := \int_{r_-}^{r_+} \rho \left(u^2 + \left(u + \rho^{-1} \rho_r \right)^2 \right) r^2 dr + \int_{r_-}^{r_+} \left(\rho^\gamma - \tilde{\rho}^\gamma - \gamma \tilde{\rho}^{\gamma-1} (\rho - \tilde{\rho}) \right) r^2 dr. \tag{4.48}$$

By (4.45)–(4.48), a complicated computation gives rise to

$$\frac{d}{dt} E(t) + \overline{C} E(t) \le 0, \tag{4.49}$$

which gives

$$E(t) \le E(0) e^{-\overline{C} t}. \tag{4.50}$$

By the fact that

$$E(t) \ge c \left(\|u\|_{L^2([r_-, r_+])}^2 + \|\rho - \tilde{\rho}\|_{L^2([r_-, r_+])}^2 + \|\rho_r\|_{L^2([r_-, r_+])}^2 \right), \tag{4.51}$$

where $c > 0$ is a constant independent of time and Gagliardo-Nirenberg-Sobolev inequality

$$\left\| (\rho - \tilde{\rho}, u) \right\|_{L^\infty([r_-, r_+])} \leq C \left\| (\rho - \tilde{\rho}, u) \right\|_{L^2([r_-, r_+])}^{1/2} \left\| (\rho - \tilde{\rho}, u)_r \right\|_{L^2([r_-, r_+])}^{1/2}, \tag{4.52}$$

we obtain (4.44). □

4.2. Proof of Theorem 2.2

Proof. The proof of Theorem 2.2 is similar to Theorem 2.1. We omit the details. □

Acknowledgments

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