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Research Article

Existence and Algorithm for Solving the System of Mixed Variational Inequalities in Banach Spaces

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The purpose of this paper is to study the existence and convergence analysis of the solutions of the system of mixed variational inequalities in Banach spaces by using the generalized f projection operator. The results presented in this paper improve and extend important recent results of Zhang et al. (2011) and Wu and Huang (2007) and some recent results.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, let C be a nonempty closed and convex subset of E, and let E^* denote the dual of E. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing of E^* and E. If E is a Hilbert space, $\langle\cdot,\cdot\rangle$ denotes an inner product on E. It is well known that the metric projection operator $P_C : E \to C$ plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, and complementarity problems, and so forth (see, e.g., [1, 2] and the references therein). In 1993, Alber [3] introduced and studied the generalized projections $\pi_C : E^* \to C$ and $\Pi_E : E \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Moreover, Alber [1] presented some applications of the generalized projections to approximately solving variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [2] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solving the variational inequality in Banach spaces. Later, Wu and Huang [4] introduced a new generalized f-projection operator in Banach spaces which extended the definition of the generalized projection operators introduced by Abler [3] and proved some properties of the generalized f-projection operator.

As an application, they studied the existence of solution for a class of variational inequalities in Banach spaces. In 2007, Wu and Huang [5] proved some properties of the generalized f-projection operator and proposed iterative method of approximating solutions for a class of generalized variational inequalities in Banach spaces. In 2009, Fan et al. [6] presented some basic results for the generalized f-projection operator and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces. In 2011, Zhang et al. [7] introduced and considered the system of mixed variational inequalities in Banach spaces. Using the generalized f-projection operator technique, they introduced some iterative methods for solving the system of mixed variational inequalities and proved the convergence of the proposed iterative methods under suitable conditions in Banach spaces. Recently, many authors studied methods for solving the system of generalized (mixed) variational inequalities and the system of nonlinear variational inequalities problems (see, e.g., [8–17] and references therein).

We first introduce and consider the *system of mixed variational inequalities* (SMVI) which is to find $\hat{x}, \hat{y}, \hat{z} \in C$ such that

$$\begin{aligned}
&\langle \delta_1 T_1 \hat{z} + J\hat{x} - J\hat{z}, y - \hat{x} \rangle + f_1(y) - f_1(\hat{x}) \geq 0, \quad \forall y \in C,\\
&\langle \delta_2 T_2 \hat{x} + J\hat{y} - J\hat{x}, y - \hat{y} \rangle + f_2(y) - f_2(\hat{y}) \geq 0, \quad \forall y \in C,\\
&\langle \delta_3 T_3 \hat{y} + J\hat{z} - J\hat{y}, y - \hat{z} \rangle + f_3(y) - f_3(\hat{z}) \geq 0, \quad \forall y \in C,
\end{aligned} \tag{1.1}$$

where $\delta_j > 0$, $T_j : C \to E^*$, $f_j : C \to \mathbb{R} \cup \{+\infty\}$ for $j = 1, 2, 3$ are mappings and J is the normalized duality mapping from E to E^*.

As special case of the problem (1.1), we have the following.

If $f_j(x) = 0$ for $j = 1, 2, 3$, for all $x \in C$, (1.1) is equivalent to find $\hat{x}$, $\hat{y}$ and $\hat{z} \in C$ such that

$$\begin{aligned}
&\langle \delta T_1 \hat{z} + J\hat{x} - J\hat{z}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C,\\
&\langle \delta_2 T_2 \hat{x} + J\hat{y} - J\hat{x}, y - \hat{y} \rangle \geq 0, \quad \forall y \in C,\\
&\langle \delta_3 T_3 \hat{y} + J\hat{z} - J\hat{y}, y - \hat{z} \rangle \geq 0, \quad \forall y \in C.
\end{aligned} \tag{1.2}$$

The problem (1.2) is called the *system of variational inequalities* we denote by (SVI).

If $T_2 = T_3$, $f_2(x) = f_3(x)$, for all $x \in C$ and $\hat{y} = \hat{z}$, then (1.1) is reduced to find $\hat{x}, \hat{y} \in C$ such that

$$\begin{aligned}
&\langle \delta_1 T_1 \hat{y} + J\hat{x} - J\hat{y}, y - \hat{x} \rangle + f_1(y) - f_1(\hat{x}) \geq 0, \quad \forall y \in C,\\
&\langle \delta_2 T_2 \hat{x} + J\hat{y} - J\hat{x}, y - \hat{y} \rangle + f_2(y) - f_2(\hat{y}) \geq 0, \quad \forall y \in C,
\end{aligned} \tag{1.3}$$

which is studied by Zhang et al. [7].

If $T = T_1 = T_2 = T_3$, $f_1(x) = f_2(x) = f_3(x)$, for all $x \in C$ and $\hat{x} = \hat{y} = \hat{z}$, (1.1) is reduced to find $\hat{x}$ such that

$$\langle T\hat{x}, y - \hat{x} \rangle + f_1(y) - f_1(\hat{x}) \geq 0, \quad \forall y \in C. \tag{1.4}$$

This iterative method is studied by Wu and Huang [5].

If $f_1(x) = 0$, for all $x \in C$, (1.4) is reduced to find $\widehat{x}$ such that

$$\langle T\widehat{x}, y - \widehat{x} \rangle \geq 0, \quad \forall y \in C, \tag{1.5}$$

which is studied by Alber [1, 18], Li [2], and Fan [19]. If $E = H$ is a Hilbert space, (1.5) holds which is known as the *classical variational inequality* introduced and studied by Stampacchia [20].

If $E = H$ is a Hilbert space, then (1.1) is reduced to find $\widehat{x}, \widehat{y}, \widehat{z} \in C$ such that

$$\begin{aligned}
&\langle \delta_1 T_1 \widehat{z} + \widehat{x} - \widehat{z}, y - \widehat{x} \rangle + f_1(y) - f_1(\widehat{x}) \geq 0, \quad \forall y \in C,\\
&\langle \delta_2 T_2 \widehat{x} + \widehat{y} - \widehat{x}, y - \widehat{y} \rangle + f_2(y) - f_2(\widehat{y}) \geq 0, \quad \forall y \in C,\\
&\langle \delta_3 T_3 \widehat{y} + \widehat{z} - \widehat{y}, y - \widehat{z} \rangle + f_3(y) - f_3(\widehat{z}) \geq 0, \quad \forall y \in C.
\end{aligned} \tag{1.6}$$

If $f_j(x) = 0$ for $j = 1, 2, 3$, for all $x \in C$, (1.6) reduces to the following (SVI):

$$\begin{aligned}
&\langle \delta_1 T_1 \widehat{z} + \widehat{x} - \widehat{z}, y - \widehat{x} \rangle \geq 0, \quad \forall y \in C,\\
&\langle \delta_2 T_2 \widehat{x} + \widehat{y} - \widehat{x}, y - \widehat{y} \rangle \geq 0, \quad \forall y \in C,\\
&\langle \delta_3 T_3 \widehat{y} + \widehat{z} - \widehat{y}, y - \widehat{z} \rangle \geq 0, \quad \forall y \in C.
\end{aligned} \tag{1.7}$$

The purpose of this paper is to study the existence and convergence analysis of solutions of the system of mixed variational inequalities in Banach spaces by using the generalized f-projection operator. The results presented in this paper improve and extend important recent results in the literature.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then, a Banach space E is said to be *smooth* if the limit $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$. Let E be a Banach space. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by $\rho_E(t) = \sup\{((\|x + y\| + \|x - y\|)/2) - 1 : \|x\| = 1, \|y\| \leq t\}$. The *modulus of convexity* of E is the function $\eta_E : [0, 2] \to [0, 1]$ defined by $\eta_E(\varepsilon) = \inf\{1 - \|(x + y)/2\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}$. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}$. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

If E is a reflexive smooth and strictly convex Banach space and $J^* : E^* \to 2^E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_E^* are the identity mappings on E and E^*. If E is a uniformly smooth and uniformly convex Banach space, then J is uniformly norm-to-norm continuous on bounded subsets of E and J^* is also uniformly norm-to-norm continuous on bounded subsets of E^*.

Let E and F be Banach spaces, $T : D(T) \subset E \to F$, the operator T is said to be *compact* if it is continuous and maps the bounded subsets of $D(T)$ onto the relatively compact subsets of F; the operator T is said to be *weak to norm continuous* if it is continuous from the weak topology of E to the strong topology of F.

We also need the following lemmas for the proof of our main results.

Lemma 2.1 (Xu [21]). *Let $q > 1$ and $r > 0$ be two fixed real numbers. Let E be a q-uniformly convex Banach space if and only if there exists a continuous strictly increasing and convex function $g : [0, +\infty) \to [0, +\infty)$, $g(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^q \leq \lambda\|x\|^q + (1-\lambda)\|y\|^q - \varsigma_q(\lambda)g(\|x-y\|) \tag{2.1}$$

for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $\lambda \in [0,1]$, where $\varsigma_q(\lambda) = \lambda(1-\lambda)^q + \lambda^q(1-\lambda)$.

For case $q = 2$, we have

$$\|\lambda x + (1-\lambda)y\|^q \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|). \tag{2.2}$$

Lemma 2.2 (Chang [22]). *Let E be a uniformly convex and uniformly smooth Banach space. The following holds:*

$$\|\phi + \Phi\|^2 \leq \|\phi\|^2 + 2\langle \Phi, J^*(\phi + \Phi)\rangle, \quad \forall \phi, \Phi \in E^*. \tag{2.3}$$

Next we recall the concept of the generalized f-projection operator. Let $G : E^* \times C \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, x) = \|\xi\|^2 - 2\langle \xi, x\rangle + \|x\|^2 + 2\rho f(x), \tag{2.4}$$

where $\xi \in E^*$, ρ is positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous. From definitions of G and f, it is easy to see the following properties:

(1) $(\|\xi\| - \|x\|)^2 + 2\rho f(x) \leq G(\xi, x) \leq (\|\xi\| + \|x\|)^2 + 2\rho f(x)$;

(2) $G(\xi, x)$ is convex and continuous with respect to x when ξ is fixed;

(3) $G(\xi, x)$ is convex and lower semicontinuous with respect to ξ when x is fixed.

Definition 2.3. Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. It is said that $\Pi_C^f : E^* \to 2^C$ is the *generalized f-projection operator* if

$$\prod_C^f \xi = \left\{u \in C : G(\xi, u) = \inf_{y \in C} G(\xi, y)\right\}, \quad \forall \xi \in E^*. \tag{2.5}$$

In this paper, we fixed $\rho = 1$, we have

$$G(\xi, x) = \|\xi\|^2 - 2\langle \xi, x\rangle + \|x\|^2 + 2f(x). \tag{2.6}$$

For the generalized f-projection operator, Wu and Hung [5] proved the following basic properties.

Lemma 2.4 (Wu and Hung [4]). *Let E be a reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. The following statements hold:*

(1) $\Pi_C^f \xi$ *is nonempty closed convex subset of C for all $\xi \in E^*$;*

(2) *if E is smooth, then for all $\xi \in E^*$, $x \in \Pi_C^f \xi$ if and only if*

$$\langle \xi - Jx, x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \tag{2.7}$$

(3) *if E is smooth, then for any $\xi \in E^*$, $\Pi_C^f \xi = (J + \rho \partial f)^{-1} \xi$, where ∂f is the subdifferential of the proper convex and lower semicontinuous functional f.*

Lemma 2.5 (Wu and Hung [4]). *If $f(x) \geq 0$ for all $x \in C$, then for any $\rho > 0$,*

$$G(Jx, y) \leq G(\xi, y) + 2\rho f(y), \quad \forall \xi \in E^*, \ y \in C, \ x \in \prod_C^f \xi. \tag{2.8}$$

Lemma 2.6 (Fan et al. [6]). *Let E be a reflexive strictly convex Banach space with its dual E^* and C is a nonempty closed convex subset of E. If $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous, then*

(1) $\Pi_C^f : E^* \to C$ *is single valued and norm to weak continuous;*

(2) *if E has the property (h), that is, for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, imply that $x_n \to x$, then $\Pi_C^f : E^* \to C$ is continuous.*

Defined the functional $G_2 : E \times C \to \mathbb{R} \cup \{+\infty\}$ by

$$G_2(x, y) = G(Jx, y), \quad \forall x \in E, \ y \in C. \tag{2.9}$$

3. Generalized Projection Algorithms

Proposition 3.1. *Let C be a nonempty closed and convex subset of a reflexive strictly convex and smooth Banach space E. If $f_j : C \to \mathbb{R} \cup \{+\infty\}$ for $j = 1, 2, 3$ is proper, convex, and lower semicontinuous, then $(\widehat{x}, \widehat{y}, \widehat{z})$ is a solution of (SMVI) equivalent to finding $\widehat{x}, \widehat{y}, \widehat{z}$ such that*

$$\begin{aligned} \widehat{x} &= \prod_C^{f_1} (J\widehat{z} - \delta_1 T_1 \widehat{z}), \\ \widehat{y} &= \prod_C^{f_2} (J\widehat{x} - \delta_2 T_1 \widehat{x}), \\ \widehat{z} &= \prod_C^{f_3} (J\widehat{y} - \delta_3 T_1 \widehat{y}). \end{aligned} \tag{3.1}$$

Proof. From Lemma 2.4 (2) and E is a reflexive strictly convex and smooth Banach space, we known that J is single valued and $\Pi_C^{f_j}$ for $j = 1, 2, 3$ is well defined and single valued. So, we can conclude that Proposition 3.1 holds. □

For solving the system of mixed variational inequality (1.1), we defined some projection algorithms as follow.

Algorithm 3.2. For an initial point $x_0, z_0 \in C$*, define the sequences* $\{x_n\}, \{y_n\}$*, and* $\{z_n\}$ *as follows:*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} &= \prod_C^{f_2}(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} &= \prod_C^{f_3}(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{aligned} \tag{3.2}$$

where $0 < a \le \alpha_n \le b < 1$.

If $f_j(x) = 0$, $j = 1,2,3$, for all $x \in C$, then Algorithm 3.2 reduces to the following iterative method for solving the system of variational inequalities (1.2).

Algorithm 3.3. For an initial point $x_0, z_0 \in C$*, define the sequences* $\{x_n\}, \{y_n\}$*, and* $\{z_n\}$ *as follows:*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \prod_C(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} &= \prod_C(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} &= \prod_C(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{aligned} \tag{3.3}$$

where $0 < a \le \alpha_n \le b < 1$.

For solving the problem (1.6), we defined the algorithm as follows:

If $E = H$ is a Hilbert space, then Algorithm 3.2 reduces to the following.

Algorithm 3.4. For an initial point $x_0, z_0 \in C$*, define the sequences* $\{x_n\}, \{y_n\}$*, and* $\{z_n\}$ *as follows:*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} &= \prod_C^{f_2}(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} &= \prod_C^{f_3}(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{aligned} \tag{3.4}$$

where $0 < a \le \alpha_n \le b < 1$.

If $f_j(x) = 0$, $j = 1,2,3$, for all $x \in C$, then Algorithm 3.4 reduces to the following iterative method for solving the problem (1.7) as follows.

Algorithm 3.5. For an initial point $x_0, z_0 \in C$, define the sequences $\{x_n\}, \{y_n\},$ and $\{z_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C(Jz_n - \delta_1 T_1 z_n), \\ y_{n+1} &= P_C(Jx_{n+1} - \delta_2 T_2 x_{n+1}), \\ z_{n+1} &= P_C(Jy_{n+1} - \delta_3 T_3 y_{n+1}), \end{aligned} \tag{3.5}$$

where $0 < a \leq \alpha_n \leq b < 1$.

4. Existence and Convergence Analysis

Theorem 4.1. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E with dual space E^*. If the mapping $T_j : C \to E^*$ and $f_j : C \to \mathbb{R} \cup \{+\infty\}$ which is convex lower semicontinuous mappings for $j = 1,2,3$ satisfying the following conditions:*

(i) *$\langle T_j x, J^*(Jx - \delta_j T_j x)\rangle \geq 0$, for all $x \in C$ for $j = 1,2,3$;*

(ii) *$(J - \delta_j T_j)$ are compact for $j = 1,2,3$;*

(iii) *$f_j(0) = 0$ and $f_j(x) \geq 0$, for all $x \in C$ and $j = 1,2,3$;*

then the system of mixed variational inequality (1.1) *has a solution $(\hat{x}, \hat{y}, \hat{z})$ and sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ defined by Algorithm 3.2 have convergent subsequences $\{x_{n_i}\}, \{y_{n_i}\}$, and $\{z_{n_i}\}$ such that*

$$\begin{aligned} x_{n_i} &\longrightarrow \hat{x}, \quad i \longrightarrow \infty, \\ y_{n_i} &\longrightarrow \hat{y}, \quad i \longrightarrow \infty, \\ z_{n_i} &\longrightarrow \hat{z}, \quad i \longrightarrow \infty. \end{aligned} \tag{4.1}$$

Proof. Since E is a uniformly convex and uniform smooth Banach space, we known that J is bijection from E to E^* and uniformly continuous on any bounded subsets of E. Hence, $\Pi_C^{f_j}$ for $j = 1,2,3$ is well-defined and single-value implies that $\{x_n\}, \{y_n\}$, and $\{z_n\}$ are well defined. Let $G_2(x,y) = G(Jx, y)$, for any $x \in C$ and $y = 0$, we have

$$\begin{aligned} G_2(x,0) &= G(Jx,0) \\ &= \|Jx\|^2 - 2\langle Jx, 0\rangle + 2f(0) \\ &= \|Jx\|^2 \\ &= \|x\|^2. \end{aligned} \tag{4.2}$$

By (4.2) and Lemma 2.5, we have

$$\begin{aligned} G_2\left(\prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n), 0\right) &= G\left(J\left(\prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right), 0\right) \\ &\leq G(Jz_n - \delta_1 T_1 z_n, 0) \\ &= \|Jz_n - \delta_1 T_1 z_n\|^2. \end{aligned} \tag{4.3}$$

From Lemma 2.2, and for all $x \in C$, $\langle T_1 x, J^*(Jx - \delta_1 T_1 x)\rangle \geq 0$, so for $z_n \in C$, we obtain

$$\begin{aligned}
\|Jz_n - \delta_1 T_1 z_n\|^2 &\leq \|Jz_n\|^2 - 2\langle \delta_1 T_1 z_n, J^*(Jz_n - \delta_1 T_1 z_n)\rangle \\
&\leq \|Jz_n\|^2 \\
&\leq \|z_n\|^2.
\end{aligned} \tag{4.4}$$

Again by Lemma 2.2, for all $x \in C$, $\langle T_2 x, J^*(Jx - \delta_2 T_2 x)\rangle \geq 0$, and for $x_{n+1} \in C$, we have

$$\begin{aligned}
\left\|y_{n+1}\right\|^2 &= G_2(y_{n+1}, 0) \\
&= G(Jy_{n+1}, 0) \\
&= G\left(J\prod_{C}^{f_2}(Jx_{n+1} - \delta_2 T_2 x_{n+1}), 0\right) \\
&\leq G(Jx_{n+1} - \delta_2 T_2 x_{n+1}, 0) \\
&\leq \|Jx_{n+1} - \delta_2 T_2 x_{n+1}\|^2 \\
&\leq \|Jx_{n+1}\|^2 - 2\langle \delta_2 T_2 x_{n+1}, J^*(Jx_{n+1} - \delta_2 T_2 x_{n+1})\rangle \\
&\leq \|Jx_{n+1}\|^2 \\
&\leq \|x_{n+1}\|^2.
\end{aligned} \tag{4.5}$$

In similar way, for all $x \in C$, $\langle T_3 x, J^*(Jx - \delta_3 T_3 x)\rangle \geq 0$, and $z_{n+1} \in C$, we also have

$$\begin{aligned}
\|z_{n+1}\|^2 &= G(Jz_{n+1}, 0) \\
&\leq G(Jy_{n+1} - \delta_3 T_3 y_{n+1}, 0) \\
&= \left\|Jy_{n+1} - \delta_3 T_3 y_{n+1}\right\|^2 \\
&\leq \left\|Jy_{n+1}\right\|^2 - 2\langle \delta_3 T_3 y_{n+1}, J^*(Jy_{n+1} - \delta_3 T_3 y_{n+1})\rangle \\
&\leq \left\|y_{n+1}\right\|^2.
\end{aligned} \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\|z_{n+1}\|^2 \leq \|x_{n+1}\|^2, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

From (4.5) and (4.6), we compute

$$\begin{aligned}
\|x_{n+1}\|^2 &\leq (1 - \alpha_n)\|x_n\| + \alpha_n \left\|\prod_{C}^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\| \\
&\leq (1 - \alpha_n)\|x_n\| + \alpha_n\|z_n\| \\
&\leq (1 - \alpha_n)\|x_n\| + \alpha_n\left\|y_n\right\| \\
&\leq (1 - \alpha_n)\|x_n\| + \alpha_n\|x_n\| \\
&= \|x_n\|.
\end{aligned} \tag{4.8}$$

This implies that the sequences $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\}$ are bounded. For a positive number r such that $\{x_n\}, \{y_n\}, \{z_n\}, \{\Pi_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\} \in B_r$, by Lemma 2.1, for $q = 2$, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for $\alpha_n \in [0, 1]$, we have

$$\begin{aligned}
\|x_{n+1}\|^2 &= \left\|(1-\alpha_n)x_n + \alpha_n \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\|^2 \\
&\leq (1-\alpha_n)\|x_n\|^2 + \alpha_n \left\|\prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\|^2 \\
&\quad - \alpha_n(1-\alpha_n) g\left\|x_n - \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\| \\
&= (1-\alpha_n)\|x_n\|^2 + \alpha_n G_2\left(\prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n, 0)\right) \\
&\quad - \alpha_n(1-\alpha_n) g\left\|x_n - \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\|.
\end{aligned} \tag{4.9}$$

Applying (4.3), (4.4), and (4.7), we have

$$\begin{aligned}
\alpha_n(1-\alpha_n) g\left\|x_n - \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\| &\leq (1-\alpha_n)\|x_n\|^2 - \|x_{n+1}\|^2 \\
&\quad + \alpha_n G_2\left(\prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n), 0\right) \\
&\leq (1-\alpha_n)\|x_n\|^2 - \|x_{n+1}\|^2 + \alpha_n\|x_n\|^2 \\
&= \|x_n\|^2 - \|x_{n+1}\|^2.
\end{aligned} \tag{4.10}$$

Summing (4.10), for $n = 0, 1, 2, 3, \ldots, k$, we have

$$\sum_{n=0}^{k} \alpha_n(1-\alpha_n) g\left\|x_n - \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\| \leq \|x_0\|^2 - \|x_{k+1}\|^2 \leq \|x_0\|^2, \tag{4.11}$$

taking $k \to \infty$, we get

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) g\left\|x_n - \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n)\right\| \leq \|x_0\|^2. \tag{4.12}$$

This shows that series (4.12) is converge, we obtain that

$$\lim_{n \to \infty} \alpha_n (1 - \alpha_n) g \left\| x_n - \prod_C^{f_1} (J z_n - \delta_1 T_1 z_n) \right\| = 0. \tag{4.13}$$

From $0 < a \leq \alpha_n \leq b < 1$ for all n, thus $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) > 0$ and (4.13), we have

$$\lim_{n \to \infty} g \left\| x_n - \prod_C^{f_1} (J z_n - \delta_1 T_1 z_n) \right\| = 0. \tag{4.14}$$

By property of functional g, we have

$$\lim_{n \to \infty} \left\| x_n - \prod_C^{f_1} (J z_n - \delta_1 T_1 z_n) \right\| = 0. \tag{4.15}$$

Since $\{z_n\}$ is bounded sequence and $(J - \delta_1 T_1)$ is compact on C, then sequence $\{J z_n - \delta_1 T_1 z_n\}$ has a convergence subsequence such that

$$\{J z_{n_i} - \delta_1 T_1 z_{n_i}\} \longrightarrow w_0 \in E^* \quad \text{as } i \longrightarrow \infty. \tag{4.16}$$

By the continuity of the $\Pi_C^{f_1}$, we have

$$\lim_{i \to \infty} \prod_C^{f_1} (J z_{n_i} - \delta_1 T_1 z_{n_i}) = \prod_C^{f_1} (w_0). \tag{4.17}$$

Again since $\{x_n\}, \{y_n\}$ are bounded and $(J - \delta_2 T_2), (J - \delta_3 T_3)$ are compact on C, then sequences $\{J x_n - \delta_2 T_2 x_n\}$ and $\{J y_n - \delta_3 T_3 y_n\}$ have convergence subsequences such that

$$\begin{aligned} &\{J x_{n_i} - \delta_2 T_2 x_{n_i}\} \longrightarrow u_0 \in E^* \quad \text{as } i \longrightarrow \infty, \\ &\{J y_{n_i} - \delta_3 T_3 y_{n_i}\} \longrightarrow v_0 \in E^* \quad \text{as } i \longrightarrow \infty. \end{aligned} \tag{4.18}$$

By the continuity of $\Pi_C^{f_2}$ and $\Pi_C^{f_3}$, we have

$$\lim_{i\to\infty}\prod_C^{f_2}\left(Jx_{n_i}-\delta_2T_2x_{n_i}\right)=\prod_C^{f_2}(u_0), \tag{4.19}$$

$$\lim_{i\to\infty}\prod_C^{f_3}\left(Jy_{n_i}-\delta_3T_3y_{n_i}\right)=\prod_C^{f_3}(v_0). \tag{4.20}$$

Let

$$\prod_C^{f_1}(w_0)=\widehat{x}, \tag{4.21}$$

$$\prod_C^{f_2}(u_0)=\widehat{y}, \tag{4.22}$$

$$\prod_C^{f_3}(v_0)=\widehat{z}. \tag{4.23}$$

By using the triangle inequality, we have

$$\|x_{n_i}-\widehat{x}\|\le\left\|x_{n_i}-\prod_C^{f_1}\left(Jz_{n_i}-\delta_1T_1z_{n_i}\right)\right\|+\left\|\prod_C^{f_1}\left(Jz_{n_i}-\delta_1T_1z_{n_i}\right)-\widehat{x}\right\|. \tag{4.24}$$

From (4.15) and (4.17), we have

$$\lim_{i\to\infty}x_{n_i}=\widehat{x}. \tag{4.25}$$

By definition of z_{n_i}, we get

$$\|z_{n_i}-\widehat{z}\|\le\left\|\prod_C^{f_3}\left(Jy_{n_i}-\delta_3T_3y_{n_i}\right)-\widehat{z}\right\|. \tag{4.26}$$

It follows by (4.20) and (4.23), we obtain

$$\lim_{i\to\infty}z_{n_i}=\widehat{z}. \tag{4.27}$$

In the same way, we also have

$$\lim_{i\to\infty}y_{n_i}=\widehat{y}. \tag{4.28}$$

By the continuity properties of $(J-\delta_1 T_1), (J-\delta_2 T_2), (J-\delta_3 T_3)$, and $\Pi_C^{f_j}$ for $j = 1,2,3$. We conclude that

$$\begin{aligned} \widehat{x} &= \prod_C^{f_1}(J\widehat{z} - \delta_1 T_1 \widehat{z}), \\ \widehat{y} &= \prod_C^{f_2}(J\widehat{x} - \delta T_2 \widehat{x}), \\ \widehat{z} &= \prod_C^{f_3}(J\widehat{y} - \delta_3 T_3 \widehat{y}). \end{aligned} \tag{4.29}$$

This completes of proof. □

Theorem 4.2. *Let C be a nonempty compact and convex subset of a uniformly convex and uniformly smooth Banach space E with dual space E^*. If the mapping $T_j : C \to E^*$ and $f_j : C \to \mathbb{R} \cup \{+\infty\}$ which is convex lower semicontinuous mappings for $j = 1,2,3$ satisfy the following conditions:*

(i) $\langle T_j x, J^*(Jx - \delta_j T_j x)\rangle \geq 0$, *for all $x \in C$ for $j = 1,2,3$;*

(ii) $f_j(0) = 0$ *and* $f_j(x) \geq 0$, *for all $x \in C$ for $j = 1,2,3$;*

then the system of mixed variational inequality (1.1) *has a solution $(\widehat{x}, \widehat{y}, \widehat{z})$ and sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ defined by Algorithm 3.2 have a convergent subsequences $\{x_{n_i}\}, \{y_{n_i}\}$, and $\{z_{n_i}\}$ such that*

$$\begin{aligned} x_{n_i} &\longrightarrow \widehat{x}, \quad i \longrightarrow \infty, \\ y_{n_i} &\longrightarrow \widehat{y}, \quad i \longrightarrow \infty, \\ z_{n_i} &\longrightarrow \widehat{z}, \quad i \longrightarrow \infty. \end{aligned} \tag{4.30}$$

Proof. In the same way to the proof in Theorem 4.1, we have

$$\lim_{n\to\infty} \left\| x_n - \prod_C^{f_1}(Jz_n - \delta_1 T_1 z_n) \right\| = 0. \tag{4.31}$$

Hence, there exist subsequences $\{x_{n_i}\} \subset \{x_n\}$ and $\{z_{n_i}\} \subset \{z_n\}$ such that

$$\lim_{i\to\infty} \left\| x_{n_i} - \prod_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) \right\| = 0. \tag{4.32}$$

From the compactness of C, we have that

$$\begin{aligned} \{x_{n_i}\} &\longrightarrow \widehat{x} \quad \text{as } i \longrightarrow \infty, \\ \{z_{n_i}\} &\longrightarrow \widehat{z} \quad \text{as } i \longrightarrow \infty, \end{aligned} \tag{4.33}$$

where $\widehat{x}, \widehat{z}$ are points in C. Also, for a sequence $\{y_n\} \supset \{y_{n_i}\} \to \widehat{y}$ as $i \to \infty$, where $\widehat{y}$ is a points in C. By the continuity properties of $J, T_2, T_3\Pi_C^{f_2}$, and $\Pi_C^{f_3}$, we obtain that

$$\begin{aligned}\widehat{y} &= \prod_C^{f_2}(J\widehat{x} - \delta_2 T_2\widehat{x}),\\ \widehat{z} &= \prod_C^{f_3}(J\widehat{y} - \delta_3 T_3\widehat{y}).\end{aligned} \tag{4.34}$$

From definition of x_{n+1}, we get

$$\begin{aligned}&\left\|\prod_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) - \widehat{x}\right\|\\ &\quad= \left\|\prod_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) - \widehat{x} + x_{n_i+1} - (1-\alpha_n)x_{n_i} - \alpha_n\prod_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i})\right\|\\ &\quad= \left\|x_{n_i+1} - \widehat{x} + (1-\alpha_n)\prod_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i}) \ - x_{n_i}\right\|\\ &\quad\le \|x_{n_i+1} - \widehat{x}\| + (1-\alpha_n)\left\|x_{n_i} - \prod_C^{f_1}(Jz_{n_i} - \delta_1 T_1 z_{n_i})\right\|.\end{aligned} \tag{4.35}$$

By (4.25) and (4.31), we have

$$\widehat{x} = \prod_C^{f_1}(J\widehat{z} - \delta_1 T_1\widehat{z}). \tag{4.36}$$

This completes of proof. □

Corollary 4.3. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E with dual space E^*. If the mapping $T_j : C \to E^*$ for $j = 1,2,3$ satisfy the following conditions:*

(i) $\langle T_j x, J^*(Jx - \delta_j T_j x)\rangle \ge 0$, *for all $x \in C$ for $j = 1,2,3$;*

(ii) $(J - \delta_j T_j)$ *are compact for $j = 1,2,3$;*

then the system of mixed variational inequality (1.2) *has a solution $(\widehat{x}, \widehat{y}, \widehat{z})$ and sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ defined by Algorithm 3.3 have convergent subsequences $\{x_{n_i}\}, \{y_{n_i}\}$, and $\{z_{n_i}\}$ such that $x_{n_i} \to \widehat{x}$, $i \to \infty$, $y_{n_i} \to \widehat{y}$, $i \to \infty$, and $z_{n_i} \to \widehat{z}$, $i \to \infty$.*

If $E = H$ is a Hilbert space, then $H^* = H, J^* = J = I$, so one obtains the following corollary.

Corollary 4.4. *Let C be a nonempty closed and convex subset of a Hilbert space H. If the mapping $T_j : C \to H$ and $f_j : C \to \mathbb{R} \cup \{+\infty\}$ which is convex lower semicontinuous mappings for $j = 1,2,3$ satisfy the following conditions:*

(i) $\langle T_j x, x - \delta_j T_j x\rangle \geq 0$ *for* $j = 1, 2, 3$;

(ii) $f_j(0) = 0$ *and* $f_j(x) \geq 0$ *for all* $x \in C$ *for* $j = 1, 2, 3$;

then the system of mixed variational inequality (1.6) *has a solution* $(\widehat{x}, \widehat{y}, \widehat{z})$ *and sequences* $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *defined by Algorithm 3.4 have a convergent subsequences* $\{x_{n_i}\}, \{y_{n_i}\}$, *and* $\{z_{n_i}\}$ *such that* $x_{n_i} \to \widehat{x}$, $i \to \infty$, $y_{n_i} \to \widehat{y}$, $i \to \infty$, *and* $z_{n_i} \to \widehat{z}$, $i \to \infty$.

Corollary 4.5. *Let* C *be a nonempty closed and convex subset of a Hilbert space* H. *If the mapping* $T_j : C \to H$ *for* $j = 1, 2, 3$ *satisfy the conditions:* $\langle T_j x, x-\delta_j T_j x\rangle \geq 0$ *for* $j = 1, 2, 3$; *then the system of mixed variational inequality* (1.7) *has a solution* $(\widehat{x}, \widehat{y}, \widehat{z})$ *and sequences* $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *defined by Algorithm 3.5 have a convergent subsequences* $\{x_{n_i}\}, \{y_{n_i}\}$, *and* $\{z_{n_i}\}$ *such that* $x_{n_i} \to \widehat{x}$, $i \to \infty$, $y_{n_i} \to \widehat{y}$, $i \to \infty$, *and* $z_{n_i} \to \widehat{z}$, $i \to \infty$.

Remark 4.6. Theorems 4.1 and 4.2 and Corollary 4.3 extend and improve the results of Zhang et al. [7] and Wu and Huang [5].

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