Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 434938, 18 pages
doi:10.1155/2012/434938

Research Article

Existence of Subharmonic Solutions for a Class of Second-Order p-Laplacian Systems with Impulsive Effects

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Received 14 April 2011; Accepted 1 July 2011

Academic Editor: Meng Fan

By using minimax methods in critical point theory, a new existence theorem of infinitely many periodic solutions is obtained for a class of second-order p-Laplacian systems with impulsive effects. Our result generalizes many known works in the literature.

1. Introduction

Consider the following p-Laplacian system with impulsive effects:

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - L(t)|u(t)|^{p-2}u(t) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R},$$
$$u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0,$$
$$\Delta\left(\left|\dot{u}(t_j)\right|^{p-2}\dot{u}(t_j)\right) = \left|\dot{u}\left(t_j^+\right)\right|^{p-2}\dot{u}\left(t_j^+\right) - \left|\dot{u}\left(t_j^-\right)\right|^{p-2}\dot{u}\left(t_j^-\right) = \nabla I_j(u(t_j)), \quad j = 1,2,\dots,m, \tag{1.1}$$

where $p > 1$, $T > 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, and $\nabla I_j : \mathbb{R}^N \to \mathbb{R}^N$ $(j = 1,2,\dots,m)$ are continuous and $F : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is T-periodic in t for all $u \in \mathbb{R}^N$, $\nabla F(t,u)$ is the gradient of $F(t,u)$ with respect to u. $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a T-periodic positive definite symmetric matrix.

Throughout this paper, we always assume the following condition holds.

(A) $F(t,x)$ is measurable in t for all $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0,T];\mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \qquad |\nabla F(t,x)| \le a(|x|)b(t) \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

For the sake of convenience, in the sequel, we define $B = \{1,2,\ldots,m\}$.

When $p = 2$, $\nabla I_j \equiv 0$, $j \in B$, problem (1.1) becomes the following second-order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}. \tag{1.3}$$

There are many papers concerning the existence of periodic solutions or homoclinic solutions for problem (1.3) by minimax methods. Here for identifying a few, we only mention [1–8].

For $\nabla I_j \neq 0$, $j \in B$, problem (1.1) involves impulsive effects. Impulsive differential equations are suitable for the mathematical simulation of evolutionary processes in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change (that is jumps) in their values. Since these processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the processes, it is natural to suppose that these perturbations act instantaneously, that is, in the form of impulse. Processes of this type are often investigated in various fields of science and technology, for example, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, and so on. For more details of impulsive differential equations, we refer the readers to the books [9, 10].

There are many methods for finding periodic solutions of impulsive differential equations, such as the monotone-iterative technique, a numerical-analytical method, the method of upper and lower solutions, and the method of bilateral approximations. For more information about periodic solutions of impulsive differential equations, one can refer to the papers [11–18]. However, there are few papers [19–25] concerning periodic solutions of impulsive differential equations by variational methods. So it is a novel method to employ variational methods to investigate the existence of periodic solutions for impulsive differential equations.

Motivated by the above papers, we study the existence of subharmonic solutions for problem (1.1) by applying minimax methods in critical point theory. Our result is new, which seems not to be found in the literature.

Throughout this paper, let $q \in (1, +\infty)$ satisfy $1/p + 1/q = 1$.

2. Preliminaries

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we construct a variational structure. With this variational structure, we can reduce the problem of finding solutions of problem (1.1) to that of seeking the critical points of the corresponding functional.

Let k be a positive integer and $W_{kT}^{1,p}$ the Sobolev space defined by

$$W_{kT}^{1,p} = \left\{u : \mathbb{R} \to \mathbb{R}^N \mid u \text{ is absolutely continuous, } u(t) = u(t+kT),\ \dot{u} \in L^p\left([0,kT];\mathbb{R}^N\right)\right\} \tag{2.1}$$

with the norm

$$\|u\| = \left(\int_0^{kT} |u(t)|^p dt + \int_0^{kT} |\dot{u}(t)|^p dt\right)^{1/p}. \tag{2.2}$$

Take $v \in W_{kT}^{1,p}$ and multiply the two sides of the equality

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - L(t)|u(t)|^{p-2}u(t) + \nabla F(t,u(t)) = 0 \tag{2.3}$$

by v and integrate from 0 to kT; we have

$$\int_0^{kT}\left(\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right)', v(t)\right)dt = \int_0^{kT}\left(L(t)|u(t)|^{p-2}u(t), v(t)\right)dt - \int_0^{kT}(\nabla F(t,u(t)), v(t))dt. \tag{2.4}$$

Moreover, by $\dot{u}(0) = \dot{u}(T)$, one has

$$\begin{aligned}
&\int_0^{kT}\left(\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right)', v(t)\right)dt \\
&\quad = k\int_0^{T}\left(\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right)', v(t)\right)dt \\
&\quad = k\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\left(\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right)', v(t)\right)dt \\
&\quad = k\sum_{j=0}^{m}\left[\left|\dot{u}\left(t_{j+1}^-\right)\right|^{p-2}\dot{u}\left(t_{j+1}^-\right)v\left(t_{j+1}^-\right) - \left|\dot{u}\left(t_j^+\right)\right|^{p-2}\dot{u}\left(t_j^+\right)v\left(t_j^+\right) - \int_{t_j}^{t_{j+1}}\left(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t)\right)dt\right] \\
&\quad = k\sum_{j=0}^{m}\left(\left|\dot{u}\left(t_{j+1}^-\right)\right|^{p-2}\dot{u}\left(t_{j+1}^-\right)v\left(t_{j+1}^-\right) - \left|\dot{u}\left(t_j^+\right)\right|^{p-2}\dot{u}\left(t_j^+\right)v\left(t_j^+\right)\right) - \int_0^{kT}\left(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t)\right)dt \\
&\quad = k|\dot{u}(T)|^{p-2}\dot{u}(T)v(T) - k|\dot{u}(0)|^{p-2}\dot{u}(0)v(0) - k\sum_{j=1}^{m}\nabla I_j(u(t_j))v(t_j)
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{kT}\left(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t)\right)dt\\
&=-k\sum_{j=1}^{m}\nabla I_j(u(t_j))v(t_j)-\int_0^{kT}\left(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t)\right)dt.
\end{aligned}
\tag{2.5}
$$

Together with (2.4), we get

$$
\begin{aligned}
&\int_0^{kT}\left(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t)\right)dt+k\sum_{j=1}^{m}\nabla I_j(u(t_j))v(t_j)+\int_0^{kT}\left(L(t)|u(t)|^{p-2}u(t),v(t)\right)dt\\
&\quad=\int_0^{kT}(\nabla F(t,u(t)),v(t))dt.
\end{aligned}
\tag{2.6}
$$

Definition 2.1. We say that a function $u\in W_{kT}^{1,p}$ is a weak solution of problem (1.1) if the identity

$$
\begin{aligned}
&\int_0^{kT}\left(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t)\right)dt+k\sum_{j=1}^{m}\nabla I_j(u(t_j))v(t_j)+\int_0^{kT}\left(L(t)|u(t)|^{p-2}u(t),v(t)\right)dt\\
&\quad=\int_0^{kT}(\nabla F(t,u(t)),v(t))dt
\end{aligned}
\tag{2.7}
$$

holds for any $v\in W_{kT}^{1,p}$.

Define the functional ϕ_k on $W_{kT}^{1,p}$ by

$$
\begin{aligned}
\phi_k(u)&=\frac{1}{p}\int_0^{kT}\left[|\dot{u}(t)|^p+\left(L(t)|u(t)|^{p-2}u(t),u(t)\right)\right]dt-\int_0^{kT}F(t,u(t))dt+k\sum_{j=1}^{m}I_j(u(t_j))\\
&=\varphi_k(u)+\psi_k(u),\quad u\in W_{kT}^{1,p},
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{gathered}
\varphi_k(u)=\frac{1}{p}\int_0^{kT}\left[|\dot{u}(t)|^p+\left(L(t)|u(t)|^{p-2}u(t),u(t)\right)\right]dt-\int_0^{kT}F(t,u(t))dt,\\
\psi_k(u)=k\sum_{j=1}^{m}I_j(u(t_j)).
\end{gathered}
\tag{2.9}
$$

It follows from assumption (A) that the functional φ_k is continuously differentiable on $W_{kT}^{1,p}$ and

$$
\langle\varphi_k'(u),v\rangle=\int_0^{kT}\left[\left(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t)\right)+\left(L(t)|u(t)|^{p-2}u(t),v(t)\right)-(\nabla F(t,u(t)),v(t))\right]dt
\tag{2.10}
$$

for $u, v \in W_{kT}^{1,p}$. By the continuity of ∇I_j, $j \in B$, one has that $\psi_k \in (W_{kT}^{1,p}, \mathbb{R})$. Hence, $\phi_k(u) \in (W_{kT}^{1,p}, \mathbb{R})$. For any $v \in W_{kT}^{1,p}$, we have

$$\langle \phi_k'(u), v \rangle = \int_0^{kT} \left[\left(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t) \right) + \left(L(t)|u(t)|^{p-2}u(t), v(t) \right) - (\nabla F(t, u(t)), v(t)) \right] dt + k\sum_{j=1}^{m} \nabla I_j(u(t_j))v(t_j). \tag{2.11}$$

By Definition 2.1, the weak solutions of problem (1.1) correspond to the critical points of the functional ϕ_k.

For $u \in W_{kT}^{1,p}$, let $\overline{u} = (1/kT)\int_0^{kT} u(t)dt$ and $\widetilde{u}(t) = u(t) - \overline{u}$; then it follows from Proposition 1.1 in [26] that

$$\|u\|_\infty := \max_{t\in[0,kT]} |u(t)| \le \left((kT)^{-1/p} + (kT)^{1/q} \right) \|u\| = d_k \|u\|, \tag{2.12}$$

where $d_k = (kT)^{-1/p} + (kT)^{1/q}$, and if $(1/kT)\int_0^{kT} u(t)dt = 0$, then

$$\|\widetilde{u}\|_\infty := \max_{t\in[0,kT]} |\widetilde{u}(t)| \le (kT)^{1/q} \|\dot{u}\|_{L^p}, \tag{2.13}$$

$$\|\widetilde{u}\|_{L^p}^p \le (kT)^p \|\dot{u}\|_{L^p}^p, \tag{2.14}$$

where $1/p + 1/q = 1$. Let $\widetilde{W}_{kT}^{1,p} = \{u \in W_{kT}^{1,p} \mid \overline{u} = 0\}$; then $W_{kT}^{1,p} = \widetilde{W}_{kT}^{1,p} \oplus \mathbb{R}^N$. We will use the following lemma to prove our main results.

Lemma 2.2 (see [27]). *Let E be a real Banach space with $E = X_1 \oplus X_2$, where X_1 is finite dimensional. Suppose that $\varphi \in C^1(E, \mathbb{R})$ satisfies the (PS) condition, and*

(a) *there exist constants $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho \bigcap X_2} \ge \alpha$, where $B_\rho := \{u \in E \mid \|u\| \le \rho\}$, and ∂B_ρ denotes the boundary of B_ρ;*

(b) *there exists an $e \in \partial B_1 \bigcap X_2$ and $L > \rho$ such that if $Q \equiv (\overline{B}_L \bigcap X_1) \oplus \{re \mid 0 \le r \le L\}$, then $\varphi|_{\partial Q} \le 0$.*

Then φ possesses a critical value $c \ge \alpha$ which can be characterized as $c = \inf_{h\in\Gamma} \max_{u\in Q} \varphi(h(u))$, where $\Gamma = \{h \in C(\overline{Q}, E) \mid h = \mathrm{id} \text{ on } \partial Q\}$.

It is well known that a deformation lemma can be proved with the weaker condition (C) replacing the usual (PS) condition. So Lemma 2.2 holds true under condition (C).

3. Main Result and Proof

Theorem 3.1. *Assume that (A) holds and F, I_j satisfy the following conditions:*

(I1) *there exists $c_j > 0$ such that*

$$0 \leq I_j(x) \leq \frac{c_j}{k}|x|^p, \quad j \in B, \ \forall x \in \mathbb{R}^N; \tag{3.1}$$

(I2) *for any $j \in B$,*

$$\nabla I_j(x)x \leq pI_j(x), \quad \forall x \in \mathbb{R}^N; \tag{3.2}$$

(H1) $\int_0^T F(t,x)dt \geq 0$, *for all $x \in \mathbb{R}^N$;*

(H2) $\lim_{|x|\to 0}(F(t,x)/|x|^p) = 0$ *uniformly for a.e. $t \in [0,T]$;*

(H3) $\lim_{|x|\to\infty}(F(t,x)/|x|^p) = +\infty$ *uniformly for a.e. $t \in [0,T]$;*

(H4) *there exists a positive constant M such that* $\limsup_{|x|\to\infty}(F(t,x)/|x|^r) \leq M$ *uniformly for a.e. $t \in [0,T]$;*

(H5) *there exists $M_1 > 0$ such that* $\liminf_{|x|\to\infty}((\nabla F(t,x),x)-pF(t,x))/|x|^\mu \geq M_1$ *uniformly for a.e. $t \in [0,T]$,*

where $r > p$ and $\mu > r-p$. Then problem (1.1) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

Remark 3.2. As far as we know, there is no paper considering subharmonic solutions of impulsive differential equations. Our result is new.

Proof. The proof is divided into three steps. In the following, C_i $(i = 1,\dots)$ denote different positive constants.

Step 1. The functional ϕ_k satisfies condition (C). Let $\{u_n\} \subset W^{1,p}_{kT}$ satisfying $(1 + \|u_n\|)\|\phi'_k(u_n)\| \to 0$ as $n \to \infty$ and $\phi_k(u_n)$ is bounded; then, there exists a constant C_1 such that

$$\left|\phi_k(u_n)\right| \leq C_1, \qquad (1 + \|u_n\|)\left\|\phi'_k(u_n)\right\| \leq C_1. \tag{3.3}$$

From (H4), there exists $M_2 > 0$ such that

$$F(t,x) \leq M|x|^r \quad \forall |x| \geq M_2, \ \text{a.e. } t \in [0,T]. \tag{3.4}$$

By assumption (A), for $|x| \leq M_2$, there exists $C_2 = \max_{|x|\leq M_2} a(|x|) > 0$ such that

$$|F(t,x)| \leq C_2 b(t), \tag{3.5}$$

which together with (3.4) implies that

$$F(t,x) \leq M|x|^r + C_2 b(t), \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T]. \tag{3.6}$$

By (3.3) and (3.6), we have

$$\begin{aligned}\phi_k(u_n) + \int_0^{kT} F(t,u_n)dt &\leq C_1 + \int_0^{kT} \left(M|u_n(t)|^r + C_2 b(t)\right)dt \\ &= C_1 + C_2 k\|b\|_{L^1} + M\int_0^{kT} |u_n(t)|^r dt \\ &= C_3 + M\int_0^{kT} |u_n(t)|^r dt.\end{aligned} \tag{3.7}$$

Since $L(t)$ is continuous T-periodic positive definite symmetric matrix on $[0,T]$, there exist constants $c_1, c_2 > 0$ such that

$$c_1|x|^p \leq \left(L(t)|x|^{p-2}x, x\right) \leq c_2|x|^p, \quad \forall x \in \mathbb{R}^N. \tag{3.8}$$

It follows from (3.8) and (I1) that

$$\begin{aligned}\phi_k(u_n) + \int_0^{kT} F(t,u_n)dt &= \frac{1}{p}\int_0^{kT}\left[|\dot{u}_n(t)|^p + \left(L(t)|u_n(t)|^{p-2}u_n(t), u_n(t)\right)\right]dt + k\sum_{j=1}^{m} I_j(u(t_j)) \\ &\geq \frac{1}{p}\int_0^{kT}\left[|\dot{u}_n(t)|^p + c_1|u_n(t)|^p\right]dt \\ &\geq \min\left\{\frac{1}{p}, \frac{c_1}{p}\right\}\|u_n\|^p \\ &= C_4\|u_n\|^p.\end{aligned} \tag{3.9}$$

By (3.7) and (3.9), we get

$$C_4\|u_n\|^p \leq C_3 + M\int_0^{kT} |u_n(t)|^r dt. \tag{3.10}$$

From (H5), there exists $M_3 > 0$ such that

$$(\nabla F(t,x), x) - pF(t,x) \geq M_1|x|^\mu \quad \text{for } |x| \geq M_3, \text{ a.e. } t \in [0,T]. \tag{3.11}$$

By assumption (A), for $|x| \leq M_3$, there exists $C_5 = \max_{|x|\leq M_3} a(|x|) > 0$ such that

$$\left|(\nabla F(t,x), x) - pF(t,x)\right| \leq C_5(p + M_3)b(t). \tag{3.12}$$

Thus, from (3.11) and (3.12), we have

$$(\nabla F(t,x),x) - pF(t,x) \geq M_1|x|^{\mu} - M_1 M_3^{\mu} - C_5(p+M_3)b(t) \quad \text{for } x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T], \tag{3.13}$$

which together with (3.3) and (I2) implies that

$$\begin{aligned}(p+1)C_1 &\geq p\phi_k(u_n) - \langle \phi_k'(u_n), u_n\rangle \\ &= \int_0^{kT} \left[(\nabla F(t,u_n),u_n) - pF(t,u_n)\right]dt + pk\sum_{j=1}^{m} I_j(u_n(t_j)) \\ &\quad - k\sum_{j=1}^{m} \nabla I_j(u_n(t_j))u_n(t_j) \\ &\geq M_1 \int_0^{kT} |u_n(t)|^{\mu}dt - C_5(p+M_3)\int_0^{kT} b(t)dt - M_1 M_3^{\mu} kT \\ &= M_1 \int_0^{kT} |u_n(t)|^{\mu}dt - C_6.\end{aligned} \tag{3.14}$$

Hence, $\int_0^{kT} |u_n(t)|^{\mu}dt$ is bounded. If $\mu > r$, we have

$$\int_0^{kT} |u_n(t)|^{r}dt \leq (kT)^{(\mu-r)/\mu}\left(\int_0^{kT} |u_n(t)|^{\mu}dt\right)^{r/\mu}, \tag{3.15}$$

which together with (3.10) implies that $\|u_n\|$ is bounded. If $\mu \leq r$, then from (2.12), we get

$$\int_0^{kT} |u_n(t)|^{r}dt \leq \|u_n\|_{\infty}^{r-\mu}\left(\int_0^{kT} |u_n(t)|^{\mu}dt\right)^{r/\mu} \leq d_k^{r-\mu}\|u_n\|^{r-\mu}\left(\int_0^{kT} |u_n(t)|^{\mu}dt\right)^{r/\mu}. \tag{3.16}$$

Since $\mu > r-p$, it follows from (3.10) that $\|u_n\|$ is bounded too. Therefore, $\|u_n\|$ is bounded in $W_{kT}^{1,p}$. Hence, there exists a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W_{kT}^{1,p}, \tag{3.17}$$

$$u_n \longrightarrow u_0 \quad \text{strongly in } C\left([0,kT];\mathbb{R}^N\right), \tag{3.18}$$

$$u_n \longrightarrow u_0 \quad \text{strongly in } L^p\left([0,kT];\mathbb{R}^N\right). \tag{3.19}$$

From (2.11), we have

$$\langle \phi_k'(u_n), u_n - u_0 \rangle = \int_0^{kT} \Big[\Big(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t) \Big) + \Big(L(t)|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t) \Big) \Big] dt$$
$$- \int_0^{kT} (\nabla F(t, u_n(t)), u_n(t) - u_0(t)) dt + k \sum_{j=1}^{m} (\nabla I_j(u_n(t_j)), u_n(t_j) - u_0(t_j)). \tag{3.20}$$

From (3.3) and (3.18), we have

$$\left| \langle \phi_k'(u_n), u_n - u_0 \rangle \right| \leq \left\| \phi_k'(u_n) \right\| \|u_n - u_0\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

By (3.8), we know that $c_1 \leq \|L\| \leq c_2$, which together with the boundedness of $\{u_n\}$ and (3.19) implies that

$$\int_0^{kT} \Big(L(t)|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t) \Big) dt \leq \|L\| \|u_n\|_{L^p}^{p-1} \|u_n - u_0\|_{L^p} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.22}$$

From the boundedness of $\{u_n\}$, the continuity of ∇I_j, and (3.18), we have

$$\sum_{j=1}^{m} (\nabla I_j(u_n(t_j)), u_n(t_j) - u_0(t_j)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.23}$$

It follows from (A), (3.18) and the boundedness of $\{u_n\}$ that

$$\int_0^{kT} (\nabla F(t, u_n(t)), u_n(t) - u_0(t)) dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.24}$$

which together with (3.20), (3.21), (3.22), and (3.23) implies that

$$\int_0^{kT} \Big(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t) \Big) dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

It is easy to see from the boundedness of $\{u_n\}$ and (3.18) that

$$\int_0^{kT} \Big(|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t) \Big) dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.26}$$

Let $f(u) = (1/p)(\int_0^{kT} |u(t)|^p dt + \int_0^{kT} |\dot{u}(t)|^p dt)$. Then, we have

$$\begin{aligned}\left\langle f'(u_n), u_n - u_0 \right\rangle &= \int_0^{kT} \left(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t) \right) dt \\ &\quad + \int_0^{kT} \left(|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t) \right) dt,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\left\langle f'(u_0), u_n - u_0 \right\rangle &= \int_0^{kT} \left(|\dot{u}_0(t)|^{p-2}\dot{u}_0(t), \dot{u}_n(t) - \dot{u}_0(t) \right) dt \\ &\quad + \int_0^{kT} \left(|u_0(t)|^{p-2}u_0(t), u_n(t) - u_0(t) \right) dt.\end{aligned} \tag{3.28}$$

It follows from (3.25) and (3.26) that

$$\left\langle f'(u_n), u_n - u_0 \right\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.29}$$

From (3.17), we get

$$\left\langle f'(u_0), u_n - u_0 \right\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

By (3.27), (3.28), and Hölder's inequality, we have

$$\begin{aligned}&\left\langle f'(u_n) - f'(u_0), u_n - u_0 \right\rangle \\ &\quad = \int_0^{kT} \left(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t) \right) dt + \int_0^{kT} \left(|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t) \right) dt \\ &\qquad - \int_0^{kT} \left(|\dot{u}_0(t)|^{p-2}\dot{u}_0(t), \dot{u}_n(t) - \dot{u}_0(t) \right) dt - \int_0^{kT} \left(|u_0(t)|^{p-2}u_0(t), u_n(t) - u_0(t) \right) dt \\ &\quad = \|u_n\|^p + \|u_0\|^p - \int_0^{kT} \left(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_0(t) \right) dt - \int_0^{kT} \left(|u_n(t)|^{p-2}u_n(t), u_0(t) \right) dt \\ &\qquad - \int_0^{kT} \left(|\dot{u}_0(t)|^{p-2}\dot{u}_0(t), \dot{u}_n(t) \right) dt - \int_0^{kT} \left(|u_0(t)|^{p-2}u_0(t), u_n(t) \right) dt \\ &\quad \geq \|u_n\|^p + \|u_0\|^p - \left(\|u_n\|_{L^p}^{p-1}\|u_0\|_{L^p} + \|\dot{u}_n\|_{L^p}^{p-1}\|\dot{u}_0\|_{L^p} \right) - \left(\|u_0\|_{L^p}^{p-1}\|u_n\|_{L^p} + \|\dot{u}_0\|_{L^p}^{p-1}\|\dot{u}_n\|_{L^p} \right) \\ &\quad \geq \|u_n\|^p + \|u_0\|^p - \left(\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p \right)^{(p-1)/p} \left(\|u_0\|_{L^p}^p + \|\dot{u}_0\|_{L^p}^p \right)^{1/p} \\ &\qquad - \left(\|u_0\|_{L^p}^p + \|\dot{u}_0\|_{L^p}^p \right)^{(p-1)/p} \left(\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p \right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&= \|u_n\|^p + \|u_0\|^p - \left(\|u_n\|^{p-1}\|u_0\| + \|u_0\|^{p-1}\|u_n\|\right) \\
&= \left(\|u_n\|^{p-1} - \|u_0\|^{p-1}\right)(\|u_n\| - \|u_0\|).
\end{aligned}
\tag{3.31}
$$

Hence, from (3.29) and (3.30), we obtain

$$
0 \leq \left(\|u_n\|^{p-1} - \|u_0\|^{p-1}\right)(\|u_n\| - \|u_0\|) \leq \langle f'(u_n) - f'(u_0), u_n - u_0 \rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.32}
$$

That is, $\|u_n\| \to \|u_0\|$ as $n \to \infty$. Since $W_{kT}^{1,p}$ has the Kadec-Klee property, we have $u_n \to u_0$ in $W_{kT}^{1,p}$. Therefore, the functional ϕ_k satisfies condition (C).

Step 2. From (H2), for any small $\varepsilon = \varepsilon(k) > 0$, there exists small enough $\delta > 0$ such that

$$
F(t,u) \leq \varepsilon |u|^p \quad \text{for } |u| \leq \delta, \text{ a.e. } t \in [0, kT]. \tag{3.33}
$$

For $u \in \widetilde{W}_{kT}^{1,p}$ and $\|u\|^p = \rho_k^p = \delta^p / (kT)^{p/q}$, it follows from (2.13) that

$$
\|u\|_\infty^p \leq (kT)^{p/q} \|\dot{u}\|_{L^p}^p \leq (kT)^{p/q} \|u\|^p = \delta^p, \tag{3.34}
$$

which implies that $|u(t)| \leq \delta$. Then from (I1), (3.8), and (3.33), we have

$$
\begin{aligned}
\varphi_k(u) &= \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt + \frac{1}{p}\int_0^{kT} \left(L(t)|u(t)|^{p-2}u(t), u(t)\right) dt \\
&\quad - \int_0^{kT} F(t,u) dt + k\sum_{j=1}^{m} I_j(u(t_j)) \\
&\geq \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt + \frac{1}{p}\int_0^{kT} c_1 |u(t)|^p dt - \int_0^{kT} \varepsilon |u(t)|^p dt \\
&\geq \min\left\{\frac{1}{p}, \frac{c_1}{p}\right\} \|u\|^p - kT\varepsilon\delta^p \\
&= C_4 \|u\|^p - kT\varepsilon\delta^p.
\end{aligned}
\tag{3.35}
$$

Let $\varepsilon = \varepsilon(k) \in (0, C_4/2(kT)^p)$; then from (3.24), we have

$$
\varphi_k(u) \geq C_4 \rho_k^p - kT\varepsilon\delta^p \geq \frac{C_4}{2}\rho_k^p \equiv \alpha > 0 \tag{3.36}
$$

for all $u \in \widetilde{W}_T^{1,p}$ and $\|u\| = \rho_k$. This implies that condition (a) of Lemma 2.2 holds.

Step 3. Let $c = \max\{c_j\}$, $j \in B$. Choose $C_7 > (c_2/p) + (mc/T)$; then from (H3), there exists $M_4 > 0$ such that

$$F(t,x) \geq C_7|x|^p, \quad |x| \geq M_4, \text{ a.e. } t \in [0,T]. \tag{3.37}$$

By assumption (A), for $|x| \leq M_4$, there exists $C_8 = \max_{|x|\leq M_4} a(|x|) > 0$ such that

$$|F(t,x)| \leq C_8 b(t), \quad \text{a.e. } t \in [0,T], \tag{3.38}$$

which together with (3.37) implies that

$$F(t,x) \geq C_7|x|^p - C_8 b(t), \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T]. \tag{3.39}$$

Thus, from (H1), (I1), (3.8), and (3.39), we have

$$\begin{aligned}
\phi_k(u) &= \frac{1}{p}\int_0^{kT}\left(L(t)|u|^{p-2}u,u\right)dt - \int_0^{kT} F(t,u)dt + k\sum_{j=1}^{m} I_j(u) \\
&= \frac{k}{p}\int_0^{T}\left(L(t)|u|^{p-2}u,u\right)dt - k\int_0^{T} F(t,u)dt + k\sum_{j=1}^{m} I_j(u) \\
&\leq \frac{c_2 k}{p}\int_0^T |u|^p dt - k\int_0^T C_7|u|^p dt + k\int_0^T C_8 b(t)dt + mc|u|^p \quad \text{for } u \in \mathbb{R}^N.
\end{aligned} \tag{3.40}$$

From (H3), we can choose C_7 suitable large such that

$$\phi_k(u) \leq 0, \quad \forall u \in \mathbb{R}^N. \tag{3.41}$$

Let $\overline{W}_{kT}^{1,p} = \operatorname{span}\{e_k\} + \mathbb{R}^N$, where $e_k = (k^{-1}\sin(k^{-1}\omega t))$, $\omega = 2\pi/T$. Since $\overline{W}_{T}^{1,p}$ is finite dimensional, there exists a constant $d > 0$ such that

$$\left(\int_0^T |x|^p dt\right)^{1/p} \geq d\left(\int_0^T |x|^2 dt\right)^{1/2}, \quad \forall x \in \overline{W}_T^{1,p}. \tag{3.42}$$

By (I1), we have

$$
\begin{aligned}
\left|\varphi(u+re_k)\right| &= \left|k\sum_{j=1}^{m} I_j\left(u+re_k\left(t_j\right)\right)\right| \\
&\le \sum_{j=1}^{m} c_j\left|u+re_k\left(t_j\right)\right|^p \\
&\le 2^p mc|u|^p + 2^p mcr^p\left|e_k\left(t_j\right)\right|^p \\
&\le 2^p mc|u|^p + 2^p mcr^p k^{-p} \\
&\le 2^p mc|u|^p + 2^p mcr^p, \quad u\in\mathbb{R}^N.
\end{aligned}
\tag{3.43}
$$

From (3.39), (3.42), and (3.43), we obtain

$$
\begin{aligned}
\phi_k(u+re_k) &= \frac{1}{p}\int_0^{kT}|r\dot{e}_k(t)|^p dt - \int_0^{kT} F(t,u+re_k(t))dt \\
&\quad + \frac{1}{p}\int_0^{kT}\left(L(t)|u+re_k(t)|^{p-2}(u+re_k(t)), u+re_k(t)\right)dt + k\sum_{j=1}^{m} I_j\left(u+re_k\left(t_j\right)\right) \\
&\le \frac{1}{p}k^{-2p}r^p\omega^p\int_0^{kT}\left|\cos\left(k^{-1}\omega t\right)\right|^p dt + \frac{c_2}{p}\int_0^{kT}|u+re_k(t)|^p dt + \int_0^{kT} C_8 b(t)dt \\
&\quad - \int_0^{kT} C_7|u+re_k(t)|^p dt + 2^p mc|u|^p + 2^p mcr^p \\
&\le \frac{1}{p}k^{-2p+1}r^p\omega^p\int_0^{T}|\cos(\omega t)|^p dt - k\int_0^{T}\left(C_7-\frac{c_2}{p}\right)|u+re_1(t)|^p dt \\
&\quad + \int_0^{T} C_8 kb(t)dt + 2^p mc|u|^p + 2^p mcr^p \\
&\le \left(\frac{T}{p}k^{-2p+1}\omega^p + 2^p mc\right)r^p - kd^p\left(C_7-\frac{c_2}{p}\right)\left(\int_0^{T}|u+re_1(t)|^2 dt\right)^{p/2} \\
&\quad + 2^p mc|u|^p + C_9 k \\
&\le \left(\frac{T}{p}k^{-2p+1}\omega^p + 2^p mc\right)r^p - kd^p\left(C_7-\frac{c_2}{p}\right)\left(\int_0^{T}\left(|u|^2+r^2|e_1(t)|^2\right)dt\right)^{p/2} \\
&\quad + 2^p mc|u|^p + C_9 k \\
&\le \left(\frac{T}{p}k^{-2p+1}\omega^p + 2^p mc\right)r^p - kd^p\left(C_7-\frac{c_2}{p}\right)\left(T|u|^2+\frac{Tr^2}{2}\right)^{p/2} \\
&\quad + 2^p mc|u|^p + C_9 k, \quad \forall r\ge 0,\ u\in\mathbb{R}^N.
\end{aligned}
\tag{3.44}
$$

From (H3), we can choose C_7 suitable such that

$$d^p\left(C_7-\frac{c_2}{p}\right)\left(\frac{T}{2}\right)^{p/2}-2^{3p}mc>0,$$
$$d^p\left(C_7-\frac{c_2}{p}\right)T^{p/2}-2^{p+1}mc>0. \tag{3.45}$$

If $k\geq 2(Tp)^{1/2p}\omega^{1/2}/[d^p(C_7-c_2/p)(T/2)^{p/2}-2^{3p}mc]\ :=C_{10}$, then we get

$$\begin{aligned}k^{-1}\phi_k(u+re_k)&\leq\left[\frac{T}{p}k^{-2p}r^p\omega^p+\frac{2^pmc}{k}-d^p\left(C_7-\frac{c_2}{p}\right)\left(\frac{T}{2}\right)^{p/2}\right]r^p+C_9\\&\leq\left[Tk^{-2p}\omega^p+2^pmc-d^p\left(C_7-\frac{c_2}{p}\right)\left(\frac{T}{2}\right)^{p/2}\right]r^p+C_9\\&\leq-\frac{1}{2}d^p\left(C_7-\frac{c_2}{p}\right)\left(\frac{T}{2}\right)^{p/2}r^p+C_9,\\k^{-1}\phi_k(u+re_k)&\leq-\frac{1}{2}d^p\left(C_7-\frac{c_2}{p}\right)T^{p/2}|u|^p+C_9.\end{aligned} \tag{3.46}$$

It follows from (3.46) that

$$\varphi_k(u+re_k)\leq 0,\quad \text{either } r\geq r_1 \text{ or } |u|\geq r_2, \tag{3.47}$$

where $r_1=\sqrt{2}(2C_9)^{1/p}/(C_7-c_2/p)^{1/p}dT^{1/2}$, $r_2=(2C_9)^{1/p}/d(C_7-c_2/p)^{1/p}T^{1/2}$. Notice that for any $u\in\mathbb{R}^N$, we have

$$\|u\|=\|u\|_{L^p}=\left(\int_0^{kT}|u|^p dt\right)^{1/p}=(kT)^{1/p}|u|\geq(C_{10}T)^{1/p}r_2:=r_3. \tag{3.48}$$

Hence, (3.47) holds for all $\|u\|\geq r_3$ whenever $u\in\mathbb{R}^N$. Set

$$Q_k=\left\{re_k\mid 0\leq r\leq r_1,\ e_k\in\widetilde{W}_{kT}^{1,p}\right\}\oplus\left\{u\in\mathbb{R}^N\mid\|u\|\leq r_3\right\}; \tag{3.49}$$

then $\partial Q_k=Q_{1k}\bigcup Q_{2k}\bigcup Q_{3k}$, where

$$\begin{aligned}Q_{1k}&=\left\{u\in\mathbb{R}^N\mid\|u\|\leq r_3\right\},\\Q_{2k}&=\left\{u+re_k\mid\|u\|=r_3,\ r\in[0,r_1],\ e_k\in\widetilde{W}_{kT}^{1,p}\right\},\\Q_{3k}&=\left\{u+re_k\mid\|u\|\leq r_3,\ r=r_1,\ e_k\in\widetilde{W}_{kT}^{1,p}\right\}.\end{aligned} \tag{3.50}$$

By (3.41) and (3.47), we have

$$\varphi(u) \le 0, \quad u \in \partial Q_k = Q_{1k} \bigcup Q_{2k} \bigcup Q_{3k}. \tag{3.51}$$

Furthermore, for all $u + re_k \in Q_k$, it follows from (H1), (3.8), and (3.43) that

$$\begin{aligned}
\phi_k(u + re_k) &= \frac{1}{p}\int_0^{kT} |r\dot{e}_k(t)|^p dt - \int_0^{kT} F(t, u + re_k(t))dt \\
&\quad + \frac{1}{p}\int_0^{kT} \Big(L(t)|u + re_k(t)|^{p-2}(u + re_k(t)), u + re_k(t)\Big)dt + k\sum_{j=1}^{m} I_j(u + re_k(t_j)) \\
&\le \frac{1}{p} r^p \int_0^{kT} |\dot{e}_k(t)|^p dt + \frac{c_2}{p}\int_0^{kT} |u + re_k(t)|^p dt + 2^p mc|u|^p + 2^p mcr^p \\
&\le \frac{1}{p} k^{-2p} r^p \omega^p \int_0^{kT} \Big|\cos\Big(k^{-1}\omega t\Big)\Big|^p dt + \frac{2^{p-1}c_2}{p}\int_0^{kT} \Big(|u|^p + r^p k^{-p}\Big|\sin\Big(k^{-1}\omega t\Big)\Big|^p\Big)dt \\
&\quad + 2^p mc|u|^p + 2^p mcr^p \\
&\le \frac{1}{p} k^{-2p+1} r^p \omega^p \int_0^{T} |\cos(\omega t)|^p dt + \frac{2^{p-1}c_2}{p}\left(\|u\|^p + r^p k^{-p+1}\int_0^{T} |\sin(\omega t)|^p dt\right) \\
&\quad + \frac{2^p mc}{T}\|u\|^p + 2^p mcr^p \\
&\le \frac{T}{p} k^{-2p+1} r^p \omega^p + \frac{2^{p-1}c_2}{p}\Big(\|u\|^p + r^p k^{-p+1} T\Big) + \frac{2^p mc}{T}\|u\|^p + 2^p mcr^p \\
&\le \frac{T}{p} r_1^p \omega^p + \frac{2^{p-1}c_2}{p}\Big(r_3^p + r_1^p T\Big) + 2^p mc\left(\frac{1}{T} r_3^p + r_1^p\right).
\end{aligned} \tag{3.52}$$

Then by Lemma 2.2, ϕ_k has at least a critical point u_k whose critical value c_k satisfies

$$0 < \alpha \le c_k = \phi_k(u_k) \le \frac{T}{p} r_1^p \omega^p + \frac{2^{p-1}c_2}{p}\Big(r_3^p + r_1^p T\Big) + 2^p mc\left(\frac{1}{T} r_3^p + r_1^p\right). \tag{3.53}$$

Similar to the proof of [28], let u_{k_1} be a k_1T-periodic solution; we can prove that there exists a positive integer $k_2 > k_1$ such that $u_{kk_1} \ne u_{k_1}$ for all $kk_1 \ge k_2$. Otherwise, $\varphi_k(u_{kk_1}) = k\varphi_k(u_{k_1}) \to \infty$ as $k \to \infty$, which contradicts to (3.53). Repeating this process, we can obtain a sequence $\{u_{k_j}\}$ of distinct periodic solutions of problem (1.1). From (3.41), we know that u_{k_j} is nonconstant. The proof is complete. □

4. Examples

In this section, we give an example to illustrate our result.

Example 4.1. Let $p = 3$, $r = 5$, $\mu = 4$, and consider the following p-Laplacian system with impulsive effects

$$\begin{gathered}\frac{d}{dt}(|\dot{u}(t)|\dot{u}(t)) - L(t)|u(t)|u(t) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \\ \Delta(|\dot{u}(t_i)|\dot{u}(t_i)) = \left|\dot{u}(t_i^+)\right|\dot{u}(t_i^+) - \left|\dot{u}(t_i^-)\right|\dot{u}(t_i^-) = \nabla I_i(u(t_i)), \quad i = 1,2,\ldots,m.\end{gathered} \tag{4.1}$$

Let

$$\begin{gathered}L(t) = \operatorname{diag}\left(1 + \exp\left(1 - \sin\left(k^{-1}\omega t\right)\right), \ldots, 1 + \exp\left(1 - \sin\left(k^{-1}\omega t\right)\right)\right), \\ I_i(x) = \frac{c_i}{k}|x|^p, \qquad F(t,x) = \frac{1+e}{3}\left(2 + \sin\left(k^{-1}\omega t\right)\right)|x|^5,\end{gathered} \tag{4.2}$$

where $c_i > 0$, $i \in B$. It is easy to check that F satisfies (A), (H1), and (H2). By a direct computation, we have

$$\begin{gathered}\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^3} = +\infty, \qquad \limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^5} \le 1 + e, \\ \liminf_{|x|\to\infty} \frac{(\nabla F(t,x), x) - 3F(t,x)}{|x|^4} \ge \frac{2(1+e)}{3},\end{gathered} \tag{4.3}$$

which show that (H3), (H4), and (H5) hold. On the other hand,

$$0 \le I_i(x) \le \frac{c}{k}|x|^p, \qquad \nabla I_i(x)x = \frac{pc_i}{k}|x|^p = pI_i(x), \tag{4.4}$$

where $c = \max\{c_i\}$, $i \in B$. It is easy to see that I_i satisfies (I1) and (I2). Hence, from Theorem 3.1, problem (4.1) has a sequence of distinct nonconstant periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.

Acknowledgments

W.-Z. Gong is supported by Guangxi Natural Science Foundation (2010GXNSFA013125) and X. H. Tang is supported by the NNSF (no. 10771215) of China.

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