Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 437391, 9 pages
doi:10.1155/2012/437391

Research Article

Implicit Methods for Equilibrium Problems on Hadamard Manifolds

Muhammad Aslam Noor,[1] Saira Zainab,[2] and Yonghong Yao[3]

[1] *Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan*
[2] *Mathematics Department, HITEC University, Taxila Cantt, Pakistan*
[3] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China*

Correspondence should be addressed to Muhammad Aslam Noor, noormaslam@hotmail.com

Received 26 March 2012; Accepted 3 April 2012

Academic Editor: Rudong Chen

We use the auxiliary principle technique to suggest and analyze an implicit method for solving the equilibrium problems on Hadamard manifolds. The convergence of this new implicit method requires only the speudomonotonicity, which is a weaker condition than monotonicity. Some special cases are also considered.

1. Introduction

Recently, much attention has been given to study the variational inequalities, equilibrium and related optimization problems on the Riemannian manifold and Hadamard manifold. This framework is a useful for the developments of various fields. Several ideas and techniques form the Euclidean space have been extended and generalized to this nonlinear framework. Hadamard manifolds are examples of hyperbolic spaces and geodesics, see [1–8] and the references therein. Németh [9], Tang et al. [7], M. A. Noor and K. I. Noor [5], and Colao et al. [2] have considered the variational inequalities and equilibrium problems on Hadamard manifolds. They have studied the existence of a solution of the equilibrium problems under some suitable conditions. To the best of our knowledge, no one has considered the auxiliary principle technique for solving the equilibrium problems on Hadamard manifolds. In this paper, we use the auxiliary principle technique to suggest and analyze an implicit method for solving the equilibrium problems on Hadamard manifold. As a special case, our result includes the recent result of Noor and Oettli [10] for variational inequalities on Hadamard manifold. This shows that the results obtained in this paper continue to hold for variational inequalities on Hadamard manifold, which are due to M. A. Noor and K. I. Noor [5], Tang et al. [7], and Németh [9]. We hope that the technique and idea of this paper may stimulate further research in this area.

2. Preliminaries

We now recall some fundamental and basic concepts needed for reading of this paper. These results and concepts can be found in the books on Riemannian geometry [1–3, 6].

Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \cup_{x\in M} T_xM$, which is naturally a manifold. A vector field A on M is a mapping of M into TM which associates to each point $x \in M$, a vector $A(x) \in T_xM$. We always assume that M can be endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle , \cdot, \rangle$ the scalar product on T_xM with the associated norm $\| \cdot \|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a, b] \to M$ joining x to y (i.e., $\gamma(a) = x$ and $\gamma(b) = y$,) by using the metric, we can define the length of γ as $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then, for any $x, y \in M$, the Riemannian distance $d(x, y)$, which includes the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y.

Let Δ be the Levi-Civita connection with $(M, \langle\cdot,\cdot\rangle)$. Let γ be a smooth curve in M. A vector field A is said to be parallel along γ if $\Delta_{\gamma'} A = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal if its length equals $d(x, y)$.

A Riemannian manifold is complete, if for any $x \in M$ all geodesics emanating from x are defined for all $t \in R$. By the Hopf-Rinow Theorem, we know that if M is complete, then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.

Let M be complete. Then the exponential map $\exp_x : T_xM \to M$ at x is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(.) = \gamma_v(., x)$ is the geodesic starting at x with velocity v (i.e., $\gamma(0) = x$ and $\gamma'(0) = v$). Then $\exp_x tv = \gamma_v(t, x)$ for each real number t.

A complete simply-connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. Throughout the remainder of this paper, we always assume that M is an m-manifold Hadamard manifold.

We also recall the following well-known results, which are essential for our work.

Lemma 2.1 (See [6]). *Let $x \in M$. Then $\exp_x : T_xM \to M$ is a diffeomorphism, and, for any two points $x, y \in M$, there exists a unique normalized geodesic joining x to y, $\gamma_{x,y}$, which is minimal.*

So from now on, when referring to the geodesic joining two points, we mean the unique minimal normalized one. Lemma 2.1 says that M is diffeomorphic to the Euclidean space R^m. Thus M has the same topology and differential structure as R^m. It is also known that Hadamard manifolds and euclidean spaces have similar geometrical properties. Recall that a geodesic triangle $\triangle(x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3 and three minimal geodesics joining these points.

Lemma 2.2 (See [2, 3, 6] (comparison theorem for triangles)). *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 (\mathrm{mod} 3)$, by $\gamma_i : [0, l_i] \to M$, the geodesic joining x_i to x_{i+1}, and*

$\alpha_i ;= L(\gamma_i'(0), -\gamma_l'(i-1)(li-1))$, the angle between the vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(l_{i-1})$, and $l_i ;= L(\gamma_i)$. Then,

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi, \tag{2.1}$$

$$l_l^2 + l_{i+1}^2 - 2L_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \tag{2.2}$$

In terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle \leq d^2(x_{i-1}, x_i), \tag{2.3}$$

since

$$\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle = d(x_i, x_{i+1}) d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}. \tag{2.4}$$

Lemma 2.3 (See [23]). *Let $\triangle(x, y, z)$ be a geodesic triangle in a Hadamard manifold M. Then, there exist $x', y', z' \in R^2$ such that*

$$d(x, y) = \|x' - y'\|, \qquad d(y, z) = \|y' - z'\|, \qquad d(z, x) = \|z' - x'\|. \tag{2.5}$$

The triangle $\triangle(x', y', z')$ is called the comparison triangle of the geodesic triangle $\triangle(x, y, z)$, which is unique up to isometry of M.

From the law of cosines in inequality (2.3), we have the following inequality, which is a general characteristic of the spaces with nonpositive curvature [6]:

$$\left\langle \exp_x^{-1} y, \exp_x^{-1} z \right\rangle + \left\langle \exp_y^{-1} x, \exp_y^{-1} z \right\rangle \geq d^2(x, y). \tag{2.6}$$

From the properties of the exponential map, we have the following known result.

Lemma 2.4 (See [6]). *Let $x_0 \in M$ and $\{x_n\} \subset M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$,*

$$\exp_{x_n}^{-1} y \longrightarrow \exp_{x_0}^{-1} y, \qquad \exp_y^{-1} x_n \longrightarrow \exp_y^{-1} x_0. \tag{2.7}$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n} M$ and $v_n \to v_0$, then $v_0 \in T_{x_0} M$.*

(iii) *Given the sequences $\{u_n\}$ and $\{v_n\}$ satisfying $u_n, v_n \in T_{x_n} M$, if $u_n \to u_0$ and $v_n \to v_0$, with $u_0, v_0 \in T_{x_0} M$, then*

$$\langle u_n, v_n \rangle \longrightarrow \langle u_0, v_0 \rangle. \tag{2.8}$$

A subset $K \subseteq M$ is said to be convex if for any two points $x, y \in K$, the geodesic joining x and y is contained in K, K that is, if $\gamma : [a,b] \to M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in K$, for all $t \in [0,1]$. From now on $K \subseteq M$ will denote a nonempty, closed, and convex set, unless explicitly stated otherwise.

A real-valued function f defined on K is said to be convex, if, for any geodesic γ of M, the composition function $f \circ \gamma : R \to R$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b), \quad \forall a, b \in R,\ t \in [0,1]. \tag{2.9}$$

The subdifferential of a function $f : M \to R$ is the set-valued mapping $\partial f : M \to 2^{TM}$ defined as

$$\partial f(x) = \left\{ u \in T_x M : \left\langle u, \exp_x^{-1} y \right\rangle \leq f(y) - f(x),\ \forall y \in M \right\}, \quad \forall x \in M, \tag{2.10}$$

and its elements are called subgradients. The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed and convex (possibly empty) set. Let $D(\partial f)$ denote the domain of ∂f defined by

$$D(\partial f) = \{ x \in M : \partial f(x) \neq \emptyset \}. \tag{2.11}$$

The existence of subgradients for convex functions is guaranteed by the following proposition, see [8].

Lemma 2.5 (See [6, 8]). *Let M be a Hadamard manifold and let $f : M \to R$ be convex. Then, for any $x \in M$, the subdifferential $\partial f(x)$ of f at x is nonempty. That is, $D(\partial f) = M$.*

For a given bifunction $F(\cdot,\cdot) : K \times K \to R$, we consider the problem of finding $u \in K$ such that

$$F(u, v) \geq 0, \quad \forall v \in K, \tag{2.12}$$

which is called the equilibrium problem on Hadamard manifolds. This problem was considered by Colao et al. [2]. They proved the existence of a solution of the problem (2.12) using the KKM maps. Colao et al. [2] have given an example of the equilibrium problem defined in a Euclidean space whose set K is not a convex set, so it cannot be solved using the technique of Blum and Oettli [11]. However, if one can reformulate the equilibrium problem on a Riemannian manifold, then it can be solved. This shows the importance of considering these problems on Hadamard manifolds. For the applications, formulation, and other aspects of the equilibrium problems in the linear setting, see [4, 9–22].

If $F(u,v) = \langle Tu, \exp_u^{-1} v \rangle$, where T is a single valued vector filed $T : K \to TM$, then problem (2.12) is equivalent to finding $u \in K$ such that

$$\left\langle Tu, \exp_x^{-1} v \right\rangle \geq 0, \quad \forall v \in K, \tag{2.13}$$

which is called the variational inequality on Hadamard manifolds. Németh [9], Colao et al. [2], Noor and Oettli [10], and M. A. Noor and K. I. Noor [5] studied variational inequalities on Hadamard manifold from different point of views. In the linear setting, variational inequalities have been studied extensively, see [5, 10, 11, 13–28] and the references therein.

Definition 2.6. A bifunction $F(\cdot,\cdot)$ is said to be speudomonotone, if and only if

$$F(u,v) \geq 0, \Longrightarrow F(v,u) \leq 0, \quad \forall u,v \in K. \tag{2.14}$$

3. Main Results

We now use the auxiliary principle technique of Glowinski et al. [23] to suggest and analyze an implicit iterative method for solving the equilibrium problems (2.12).

For a given $u \in K$ satisfying (2.12), consider the problem of finding $w \in K$ such that

$$\rho F(w,v) + \left\langle \exp_u^{-1} w, \exp_w^{-1} v \right\rangle \geq 0, \quad \forall v \in K, \tag{3.1}$$

which is called the auxiliary equilibrium problem on Hadamard manifolds. We note that, if $w = u$, then w is a solution of (2.12). This observation enables to suggest and analyze the following implicit method for solving the equilibrium problems (2.12). This is the main motivation of this paper.

Algorithm 3.1. For a given u_0, compute the approximate solution by the iterative scheme

$$\rho F(u_{n+1},v) + \left\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v \right\rangle \geq 0, \quad \forall v \in K. \tag{3.2}$$

Algorithm 3.1 is called the implicit (proximal point) iterative method for solving the equilibrium problem on the Hadamard manifold. Algorithm 3.1 can be written in the following equivalent form.

Algorithm 3.2. For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative scheme:

$$\begin{aligned} \rho F(u_n,v) + \left\langle \exp_{u_n}^{-1} y_n, \exp_{y_n}^{-1} v \right\rangle &\geq 0, \quad \forall v \in K, \\ \rho F(y_n,v) + \left\langle \exp_{y_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v \right\rangle &\geq 0, \quad \forall v \in K. \end{aligned} \tag{3.3}$$

Algorithm 3.2 is a two-step iterative method for solving the equilibrium problems on Hadamard manifolds. This method can be viewed as the extragradient method for solving the equilibrium problems.

If K is a convex set in R^n, then Algorithm 3.1 collapses to the following.

Algorithm 3.3. For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative scheme:

$$\rho F(u_{n+1},v) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K, \tag{3.4}$$

which is known as the implicit method for solving the equilibrium problem. For the convergence analysis of Algorithm 3.2, see [16, 19, 20].

If $F(u,v) = \langle Tu, \exp_u^{-1} v\rangle$, where T is a single valued vector filed $T : K \to TM$, then Algorithm 3.1 reduces to the following implicit method for solving the variational inequalities.

Algorithm 3.4. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\left\langle \rho T u_{n+1} + \left(\exp_{u_n}^{-1} u_{n+1}\right), \exp_{u_{n+1}}^{-1} v \right\rangle \geq 0, \quad \forall v \in K. \tag{3.5}$$

Algorithm 3.4 is due according to Tang et al. [7] and M. A. Noor and K. I. Noor [5]. We can also rewrite Algorithm 3.4 in the following equivalent form.

Algorithm 3.5. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{gathered} \left\langle \rho T u_n + \exp_{u_n}^{-1} y_n, \exp_{y_n}^{-1} v \right\rangle \geq 0, \quad \forall v \in K, \\ \left\langle \rho T y_n + \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} v \right\rangle \geq 0, \quad \forall v \in K, \end{gathered} \tag{3.6}$$

which is the extragradient method for solving the variational inequalities on Hadamard manifolds and appears to be a new one.

In a similar way, one can obtain several iterative methods for solving the variational inequalities on the Hadamard manifold.

We now consider the convergence analysis of Algorithm 3.1 and this is the motivation of our next result.

Theorem 3.6. *Let $F(\cdot,\cdot)$ be a speudomonotone bifunction. Let u_n be the approximate solution of the equilibrium problem* (2.12) *obtained from Algorithm 3.1, then*

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) \leq d^2(u_n, u), \tag{3.7}$$

where $u \in K$ is a solution of the equilibrium problem (2.12).

Proof. Let $u \in K$ be a solution of the equilibrium problem (2.12). Then, by using the speudomonotonicity of the bifunction $F(\cdot,\cdot)$, we have

$$F(v,u) \leq 0, \quad \forall v \in K. \tag{3.8}$$

Taking $v = u_{n+1}$ in (3.9), we have

$$F(u_{n+1}, u) \leq 0. \tag{3.9}$$

Taking $v = u$ in (3.2), we have

$$\rho F(u_{n+1}, u) + \left\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} u \right\rangle \geq 0. \tag{3.10}$$

From (3.10) and (3.9), we have

$$\left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \leq 0. \tag{3.11}$$

For the geodesic triangle $\triangle(u_n, u_{n+1}, u)$, the inequality (2.3) can be written as

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) - \left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \leq d^2(u_n, u). \tag{3.12}$$

Thus, from (3.12) and (3.11), we obtained the inequality (3.8), the required result. □

Theorem 3.7. *Let $u \in K$ be solution of* (2.12) *and let u_{n+1} be the approximate solution obtained from Algorithm 3.1, then* $\lim_{n \to} u_{n+1} = u$.

Proof. Let $\widehat{u}$ be a solution of (2.12). Then, from (3.8), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} d^2(u_{n+1}, u_n) \leq d^2(u_0, u), \tag{3.13}$$

then it follows that

$$\lim_{n \to \infty} d(u_{n+1}, u_n) = 0. \tag{3.14}$$

Let $\widehat{u}$ be a cluster point of $\{u_n\}$. Then there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{u_i}\}$ converges to $\widehat{u}$. Replacing u_{n+1} by u_{n_i} in (3.2), taking the limit, and using (3.14), we have

$$\langle F(\widehat{u}, v) \geq 0, \quad \forall v \in K. \tag{3.15}$$

This shows that $\widehat{u} \in K$ solves (2.12) and

$$d^2(u_{n+1}, \widehat{u}) \leq d^2(u_n, \widehat{u}) \tag{3.16}$$

which implies that the sequence $\{u_n\}$ has unique cluster point and $\lim_{n \to \infty} u_n = \widehat{u}$ is a solution of (2.12), the required result. □

4. Conclusion

In this paper, we have suggested and analyzed an implicit iterative method for solving the equilibrium problems on Hadamard manifold. It is shown that the convergence analysis of this methods requires only the speudomonotonicity, which is a weaker condition than monotonicity. Some special cases are also discussed. Results proved in this paper may stimulate research in this area.

Acknowledgments

The authors would like to thank Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Islamabad, Pakistan, for providing excellent research facilities. Professor Y. Yao was supported in part by Colleges and Universities Science and Technology Development Foundation (20091003) of Tianjin, NSFC 11071279, and NSFC 71161001-G0105.

References

[1] D. Azagra, J. Ferrera, and F. López-Mesas, "Nonsmooth analysis and Hamilton-Jacobi equations on Riemannian manifolds," *Journal of Functional Analysis*, vol. 220, no. 2, pp. 304–361, 2005.
[2] V. Colao, G. Lopez, G. Marino, and V. Martin-Marquez, "Equilibrium problems in Hadamard manifolds," *Journal of Mathematical Analysis and Applications*, vol. 388, pp. 61–77, 2012.
[3] M. P. DoCarmo, *Riemannian Geometry*, Mathematics: Theory & Applications, Birkhäuser, Boston, Mass, USA, 1992.
[4] O. P. Ferreira and P. R. Oliveira, "Proximal point algorithm on Riemannian manifolds," *Optimization*, vol. 51, no. 2, pp. 257–270, 2002.
[5] M. A. Noor and K. I. Noor, "Proximal point methods for solving mixed variational inequalities on Hadamard manifolds," *Journal of Applied Mathematics*. In press.
[6] T. Sakai, *Riemannian Geometry*, vol. 149 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1996.
[7] G. Tang, L. W. Zhou, and N. J. Huang, "The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds," *Optimization Letters*. In press.
[8] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, vol. 297 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.
[9] S. Z. Németh, "Variational inequalities on Hadamard manifolds," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 52, no. 5, pp. 1491–1498, 2003.
[10] M. A. Noor and W. Oettli, "On general nonlinear complementarity problems and quasi-equilibria," *Le Matematiche*, vol. 49, no. 2, pp. 313–331, 1994.
[11] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
[12] C. Li, G. López, V. Martín-Márquez, and J.-H. Wang, "Resolvents of set-valued monotone vector fields in Hadamard manifolds," *Set-Valued and Variational Analysis*, vol. 19, no. 3, pp. 361–383, 2011.
[13] M. A. Noor, "General variational inequalities," *Applied Mathematics Letters*, vol. 1, no. 2, pp. 119–122, 1988.
[14] M. A. Noor, "New approximation schemes for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.
[15] M. A. Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.
[16] M. A. Noor, "Fundamentals of mixed quasi variational inequalities," *International Journal of Pure and Applied Mathematics*, vol. 15, no. 2, pp. 137–258, 2004.
[17] M. A. Noor, "Auxiliary principle technique for equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 122, no. 2, pp. 371–386, 2004.
[18] M. A. Noor, "Fundamentals of equilibrium problems," *Mathematical Inequalities & Applications*, vol. 9, no. 3, pp. 529–566, 2006.
[19] M. A. Noor, "Extended general variational inequalities," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 182–186, 2009.
[20] M. A. Noor, "On an implicit method for nonconvex variational inequalities," *Journal of Optimization Theory and Applications*, vol. 147, no. 2, pp. 411–417, 2010.
[21] M. A. Noor, "Auxiliary principle technique for solving general mixed variational inequalities," *Journal of Advanced Mathematical Studies*, vol. 3, no. 2, pp. 89–96, 2010.
[22] M. A. Noor, "Some aspects of extended general variational inequalities," *Abstract and Applied Analysis*, vol. 2012, Article ID 303569, 16 pages, 2012.
[23] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, vol. 8 of *Studies in Mathematics and Its Applications*, North-Holland Publishing, Amsterdam, The Netherlands, 1981.

[24] M. A. Noor and K. I. Noor, "On equilibrium problems," *Applied Mathematics E-Notes*, vol. 4, pp. 125–132, 2004.

[25] M. A. Noor, K. I. Noor, and T. M. Rassias, "Some aspects of variational inequalities," *Journal of Computational and Applied Mathematics*, vol. 47, no. 3, pp. 285–312, 1993.

[26] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," vol. 258, pp. 4413–4416, 1964.

[27] Y. Yao, M. A. Noor, and Y. C. Liou, "Strong convergence of a modified extra-gradient method to the minimum-norm solution of variational inequalities," *Abstract and Applied Analysis*, vol. 2012, Article ID 817436, 9.

[28] Y. Yao, M. A. Noor, Y. C. Liou, and S. M. Kang, "Iterative algorithms for general multi-valued variational inequalities," *Abstract and Applied Analysis*, vol. 2012, Article ID 768272, 10 pages, 2012.