Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 479093, 15 pages
doi:10.1155/2012/479093

Research Article

Matched Asymptotic Expansions to the Circular Sitnikov Problem with Long Period

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Received 3 August 2012; Revised 22 November 2012; Accepted 23 November 2012

Academic Editor: Roberto Barrio

The circular Sitnikov problem is revisited, using matched asymptotic expansions. In the case of large oscillation periods, approximate analytical expressions for the period and the orbit of the third body are found. The results are compared with those described in the literature and show that the movement of the third body can be well described by two analytical solutions, the inner and outer solutions.

1. Introduction

The Sitnikov problem [1] is a particular case of the restricted three-body problem. The configuration of the Sitnikov problem is defined by: two point-like bodies of equal masses (called primaries) orbiting around their common centre of mass due to their mutual gravitational forces, and a third body of negligible mass moving along a line, perpendicular to the orbital plane of the primaries, going through their barycentre and performs oscillations along of the straight line (Figure 1).

The circular Sitnikov problem is obtained when the two primary bodies describe circular orbits. This situation was originally discussed by MacMillan [2], demonstrating that this problem can be reduced to elliptic integrals. Later, several authors have analyzed the general and circular Sitnikov problem. The case when the two primaries bodies describe elliptical orbits was first investigated by Sitnikov [1], who proved the existence of oscillatory solutions for this problem. Subsequently, Moser [3] proved the existence the chaotic orbits. Perdios and Markellos [4] examined the stability and bifurcations of straight line motions of the third body. Dvorak [5] studied, by numerical means, motion of the planetoid limited

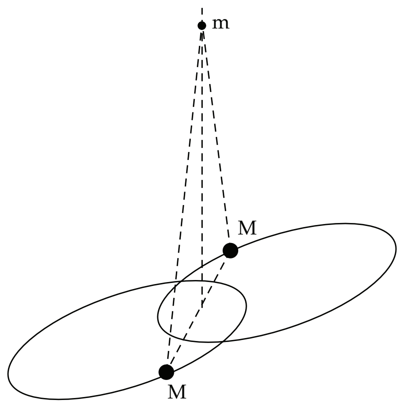

Figure 1: Sitnikov problem geometry.

to a small region around the barycentre of the primaries and found that invariant curves exist for very small oscillations centering the barycentre. For bounded orbits with moderate amplitudes, Hagel [6] showed that the equations of motion can be expressed in a polynomial form. Liu and Sun [7] derived a mapping model to investigate the problem. Wodnar [8] introduced a new formulation for the equation of motion by using the true anomaly of the primaries as independent variable. Belbruno et al. [9] derived analytical expressions for the circular case (MacMillan problem) using elliptic functions. Faruque [10] found approximate analytical solutions for small oscillations around the barycenter and moderate eccentricities. Recently, Dvorak [11] studied the complete phase space numerically. Kovács and Érdi [12] reported the extended phase space of the Sitnikov problem by using a stroboscopic map and computing escape times. The instance when the mass of the third body is nonnegligible (extended Sitnikov problem) has been studied by Dvorak and Sun [13]. Soulis et al. [14] also studied the situation in which the third body could move off the z-axis (generalized extended Sitnikov problem). In 2009, Hagel [15] derived an analytical expression for the perihelion motion of the primaries, when the third body has a finite mass.

Bountis and Papadakis [16] analyzed the problem for an arbitrary number of primary masses, Sidorenko [17] complemented the previous work on the alternation of stability and instability in the family of vertical motions. Recently, Ruzza and Lhotka [18] performed the implementation of a numerical construction of high order normal form near elliptic orbits.

In this work, we assume that the energy of the third body is close to the escape energy. In this situation, the period of the orbit is large, because the body goes to great distances from the origin and then return. The numerical calculation of the period becomes complex, mainly because computational times are very long and it becomes necessary to resort to asymptotic analysis to support the numerical calculation.

2. Equations of Motion

The equation of motion of the third body is

$$\ddot{z} = -2GM\frac{z}{\left(r^2(t)+z^2\right)^{3/2}}, \tag{2.1}$$

where $2r(t)$ is the distance between the primary bodies and $z(t)$ the distance of the third body from the center of mass and t representing time. The distance between the primary bodies is given by

$$r(t) = R(1 - e\cos E(t)), \tag{2.2}$$

where $E(t)$ is the eccentric anomaly of the primaries, e is the eccentricity of the orbits of the primaries. When the primaries move in circular orbits, $e = 0$, (2.1) is reduced to

$$\ddot{z} = -2GM\frac{z}{(R^2 + z^2)^{3/2}}. \tag{2.3}$$

Equation (2.3) represents the circular Sitnikov problem, also known as the problem of MacMillan. Integrating (2.3) with the initial conditions: $\dot{z}(t = 0) = v_0$ and $z(t = 0) = 0$ we obtain

$$\dot{z}^2 = \frac{4GM}{R}\frac{1}{\left(1 + (z/R)^2\right)^{1/2}} + v_0^2 - \frac{4GM}{R}. \tag{2.4}$$

Defining the parameter β as

$$\beta = \frac{1}{2}\frac{v_0}{\sqrt{GM/R}}. \tag{2.5}$$

Equation (2.4) is transformed into

$$\dot{z}^2 = \frac{4GM}{R}\left(\beta^2 - 1 + \frac{1}{\left(1 + (z/R)^2\right)^{1/2}}\right). \tag{2.6}$$

Making the change of variables $1 + (z/R)^2 = 1/u^2$, and then $\eta^2 = (1 - u)/\beta^2$, allows us to find the following expression to describe the position of the third body [2]:

$$\sqrt{\frac{2GM}{R^3}}t = \int_{\eta=0}^{\eta=\eta}\frac{1}{(1 - 2k^2\eta^2)^2\sqrt{(1 - \eta^2)(1 - k^2\eta^2)}}d\eta; \quad k^2 = \frac{1}{2}\beta^2. \tag{2.7}$$

The period is given by

$$\sqrt{\frac{2GM}{R^3}}\,T = 4\int_0^1\frac{1}{(1 - 2k^2\eta^2)^2\sqrt{(1 - \eta^2)(1 - k^2\eta^2)}}d\eta. \tag{2.8}$$

This integral can be performed using elliptic integrals, but it is impossible to reverse (2.7) to find an explicit expression for $z(t)$. However, it is possible to find an analytical approximation for the period of the orbit and $z(t)$ by matching asymptotic expansions when the energy of the third body is close to the energy of escape. The concept of escape velocity used in this paper is the minimum velocity at the origin to obtain an unbounded orbit. From (2.4), the following value for the escape velocity is found: $v_{0\ \mathrm{esc}} = 2\sqrt{GM/R}$. In this case, we have $\beta = 1$.

3. Asymptotic Expansions

When the third body has a speed close to the escape velocity, that is, $1 - \beta^2 = \varepsilon \ll 1$, (2.6) can be written as

$$\dot{z}^2 = \frac{4GM}{R}\left(\frac{1}{\left(1+(z/R)^2\right)^{1/2}} - \varepsilon\right). \tag{3.1}$$

The mechanical total energy (E) of the third body, can be written in terms of ε as: $E = -2GMm/R\varepsilon$, bounded orbit are obtained when $0 < \varepsilon < 1$. In the case $\varepsilon \leq 0$, an unbounded orbit is obtained, $E = 0$ is the minimum energy of the third body to escape of the gravitational field of the primary bodies.

If ε is a positive small parameter, the orbit of the third body is bounded but with a large amplitude, then the equation can be solved by matching between two asymptotic solutions.

3.1. Near Field

In the regions close to the origin, we have: $1/(1+(z/R)^2)^{1/2} \gg \varepsilon$, thus, the solution of (3.1) can be approximated to the following inner asymptotic expansion:

$$z_{in}(t) = z_0(t) + \varepsilon z_1(t) + \cdots . \tag{3.2}$$

The validity of the expansion (3.2) will be verified in the matching process (see Section 3.3). Introducing the asymptotic expansion (3.2) in (3.1), we obtain the following equation for the leading order and the first order correction:

$$\dot{z}_0^2 = \frac{4GM}{R}\left(\frac{1}{\left(1+(z_0/R)^2\right)^{1/2}}\right), \tag{3.3}$$

$$\dot{z}_1 = -\frac{2GM}{\dot{z}_0 R}\left(\frac{z_1 z_0}{R^2\left(1+(z_0/R)^2\right)^{3/2}} + 1\right). \tag{3.4}$$

The initial conditions are: $z_0(t=0) = 0$ and $z_1(t=0) = 0$.

Equation (3.3) represents the movement of a particle passing through the origin with a speed equal to the escape velocity. Integrating (3.3) gives

$$\int \left(1 + \left(\frac{z_0}{R}\right)^2\right)^{1/4} dz_0 = \sqrt{\frac{4GM}{R}}t + c_1. \tag{3.5}$$

This integral can be calculated analytically

$$\frac{2}{3}\frac{z_0}{R}\left(1 + \left(\frac{z_0}{R}\right)^2\right)^{1/4} + \frac{1}{3}\frac{z_0}{R}\,{}_2F_1\left(\frac{1}{2},\frac{3}{4};\frac{3}{2};-\left(\frac{z_0}{R}\right)^2\right) = \sqrt{\frac{4GM}{R^3}}t + c_1, \tag{3.6}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function. From the initial condition $z_0(t = 0) = 0$, we have $c_1 = 0$, (3.4) can be transformed to more useful as follows:

$$\frac{dz_1}{dz_0}\dot{z}_0^2 = -\frac{2GM}{R}\left(\frac{z_1 z_0}{R^2\left(1 + (z_0/R)^2\right)^{1/2}} + 1\right). \tag{3.7}$$

Introducing (3.3) into (3.7), we obtain the following equation for first order perturbations:

$$\frac{dz_1}{dz_0} = -\frac{1}{2}\left(\frac{z_1 z_0}{R^2\left(1 + (z_0/R)^2\right)} + \left(1 + \left(\frac{z_0}{R}\right)^2\right)^{1/2}\right). \tag{3.8}$$

The solution of (3.8) is given by the expression

$$z_1(z_0) = -\frac{1}{2}\frac{1}{\left(1 + (z_0/R)^2\right)^{1/4}}\int \left(1 + \left(\frac{z_0}{R}\right)^2\right)^{3/4} dz_0. \tag{3.9}$$

This integral can be expressed in terms of the hypergeometric function as

$$z_1(z_0) = -\frac{1}{2}\frac{1}{\left(1 + (z_0/R)^2\right)^{1/4}}\left(\frac{2}{5}z_0\left(1 + \left(\frac{z_0}{R}\right)^2\right)^{3/4} + \frac{3}{5}z_0\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{3}{2};-\left(\frac{z_0}{R}\right)^2\right)\right). \tag{3.10}$$

3.2. Far Field

In the region far from the origin, the parameter ε has an important role in (3.1), because ε defines the maximum distance reached by the third body from the origin: $z_{\max} = R/\varepsilon\sqrt{1-\varepsilon^2} \approx R/\varepsilon$. Moreover, in this region we have $z/R \gg 1$ and (3.1) can be approximated by

$$\dot{z}^2 = \frac{4GM}{R}\left(\frac{R}{z} - \varepsilon\right). \tag{3.11}$$

This differential equation represents the outer solution. Integrating this equation, the following expression is obtained:

$$\int \sqrt{\frac{z}{R-\varepsilon z}}dz = \sqrt{\frac{4GM}{R}}t + c_2. \tag{3.12}$$

This integral can also be calculated analytically, that is:

$$\frac{1}{\varepsilon^{3/2}}\frac{1}{\sqrt{R-\varepsilon z}}\left(\sqrt{\varepsilon z}\ (\varepsilon z - R) + R\sqrt{R-\varepsilon z}\ \tan^{-1}\left(\frac{\sqrt{\varepsilon z}}{\sqrt{R-\varepsilon z}}\right)\right) = \sqrt{\frac{4GM}{R}}t + c_2, \tag{3.13}$$

where the value of the constant c_2 must be calculated by matching between the inner and outer solutions.

3.3. Matching Procedure

In the region $1 \ll z/R \ll 1/\varepsilon$, the inner and outer solutions are valid, then (3.6) and (3.13) must make matching in it.

In the region $z/R \gg 1$, (3.6) has the following asymptotic representation:

$$\frac{2}{3}\left(\frac{z_0}{R}\right)^{3/2}\left(1+\frac{1}{4}\left(\frac{R}{z_0}\right)^2\right) + \frac{1}{3}\frac{\Gamma(1/4)\Gamma(3/2)}{\Gamma(3/4)\Gamma(1)} = \sqrt{\frac{4GM}{R^3}}t, \tag{3.14}$$

where we have used the following asymptotic expansion for the hypergeometric function:

$$\begin{aligned} {}_2F_1(a,b;c;z) \sim{}& \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\left(1+\frac{a(1+a-c)}{(1+a-b)\ z}+\cdots\right) \\ &+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\left(1-\frac{b(1+b-c)}{(1-a+b)z}+\cdots\right). \end{aligned} \tag{3.15}$$

In the region $z/R \ll 1/\varepsilon$, the asymptotic solution (3.13) can be expanded in power series of $\varepsilon(z/R) \ll 1$, then we obtain the equation of first order in $\varepsilon(z/R)$, namely

$$\frac{2}{3}\left(\frac{z}{R}\right)^{3/2}[1+O(\varepsilon)+\cdots] = \sqrt{\frac{4GM}{R^3}}t + \frac{c_2}{R}. \tag{3.16}$$

In the leading order, the matching between (3.14) and (3.17) delivers the following value for the constant c_2:

$$\frac{c_2}{R} = -\frac{\sqrt{2}}{12\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right). \tag{3.17}$$

Introducing (3.18) into (3.13), the following expression is obtained to $z(t)$ in the far field:

$$\frac{1}{\varepsilon^{3/2}}\left(-\sqrt{\varepsilon z}\sqrt{R-\varepsilon z}+R\tan^{-1}\left(\frac{\sqrt{\varepsilon z}}{\sqrt{R-\varepsilon z}}\right)\right)=\sqrt{\frac{4GM}{R}}t-\frac{\sqrt{2}}{12\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)R. \tag{3.18}$$

We may proceed further by combining the inner solution (3.6) and the outer solution (3.18) into a single uniform approximation $z_{\text{unif}}(t)$:

$$z_{\text{unif}}(t)=z_{\text{in}}(t)+z_{\text{out}}(t)-z_{\text{match}}(t), \tag{3.19}$$

where $z_{\text{match}}(t)$ is obtained from (3.14), given:

$$z(t)_{\text{match}}=\left(\frac{3}{2}\right)^{2/3}\left(\sqrt{\frac{4GM}{R^3}}t-\frac{\sqrt{2}}{12\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)\right)^{2/3}R. \tag{3.20}$$

The quarter-period of the orbit is obtained by evaluating (3.18) in $z=R/\varepsilon$, thus we get the following expression for the period:

$$\sqrt{\frac{4GM}{R^3}}T=\frac{2\pi}{\varepsilon^{3/2}}+\frac{\sqrt{2}}{3\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right). \tag{3.21}$$

3.4. Matching in Higher Order

To carry out the asymptotic matching for a first order in ε is more convenient to write (3.1) in the inner region as follows:

$$\int\frac{\left(1+(z/R)^2\right)^{1/4}}{\sqrt{1-\varepsilon\sqrt{1+(z/R)^2}}}dz=\sqrt{\frac{4GM}{R}}t+c_1. \tag{3.22}$$

Expanding (3.22) in powers of ε, we obtain:

$$\sum_{n=0}^{\infty}a_n\varepsilon^n\int\left(1+\left(\frac{z}{R}\right)^2\right)^{(1/4)(2n+1)}dz=\sqrt{\frac{4GM}{R}}\,t+c_1. \tag{3.23}$$

This integral can also be calculated analytically, that is:

$$\sum_{n=0}^{\infty}a_n\varepsilon^n z\,{}_2F_1\left(\frac{1}{2},-\frac{1}{4}(2n+1);\frac{3}{2};-\left(\frac{z}{R}\right)^2\right)=\sqrt{\frac{4GM}{R}}t+c_1. \tag{3.24}$$

From the initial condition $z_0(t=0)=0$, we have $c_1=0$.

In the region $z/R \gg 1$, (3.24) has this asymptotic representation

$$R\Gamma\left(\frac{3}{2}\right)\sum_{n=0}^{\infty} a_n\varepsilon^n \frac{\Gamma(-(1/4)(2n+3))}{\Gamma(-(1/4)(2n+1))} + R\,\frac{\Gamma(3/2)}{\Gamma(1/2)}\sum_{n=0}^{\infty} a_n\varepsilon^n \frac{\Gamma((1/4)(2n+3))}{\Gamma((1/4)(2n+7))}\left(\frac{z}{R}\right)^{(1/2)(2n+3)} = \sqrt{\frac{4GM}{R}}\,t. \tag{3.25}$$

To perform a more accurate estimation of the period, a more accurate estimate of the behavior of z for large z is necessary, a better approximation of (3.11) for large values of z leads to

$$\dot{z}^2 = \frac{4GM}{R}\left(\frac{1}{z/R\sqrt{1+(R/z)^2}} - \varepsilon\right). \tag{3.26}$$

Separating variables in (3.25), and after algebraic manipulation, leads to the following approximate integral:

$$\int \frac{z}{\sqrt{zR - \varepsilon(z^2 + (1/2)R^2)}}dz = \sqrt{\frac{4GM}{R}}\,t + c. \tag{3.27}$$

Finally, the following expression is found for improved outer solution:

$$\frac{1}{\varepsilon^{3/2}}\left(-\sqrt{\varepsilon\left(z - \frac{1}{2}R\varepsilon\right)}\sqrt{R\left(1 - \frac{1}{2}\varepsilon^2\right) - \varepsilon z} + R\,\tan^{-1}\left(\frac{\sqrt{\varepsilon}\;(z-(1/2)R\varepsilon)}{\sqrt{R(1-(1/2)\varepsilon^2) - \varepsilon\, z}}\right)\right) = \sqrt{\frac{4GM}{R}}\,t + c. \tag{3.28}$$

In the region $1 \ll z/R \ll 1/\varepsilon$, the asymptotic solution (3.28) can be expanded in power series of $\varepsilon(z/R) \ll 1$, then we obtain the asymptotic expansion:

$$\left(\frac{2}{3}\frac{z^{3/2}}{\sqrt{R}} + \frac{1}{5}\frac{z^{5/2}}{\left(\sqrt{R}\right)^3}\,\varepsilon + \cdots\right) = \sqrt{\frac{4GM}{R}}\,t + c. \tag{3.29}$$

The matching between (3.25) and (3.29) delivers the following value for the constant c:

$$c = -R\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(-3/4)}{\Gamma(-1/4)} - \frac{1}{2}R\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(-5/4)}{\Gamma(-3/4)}\varepsilon + O\left(\varepsilon^2\right). \tag{3.30}$$

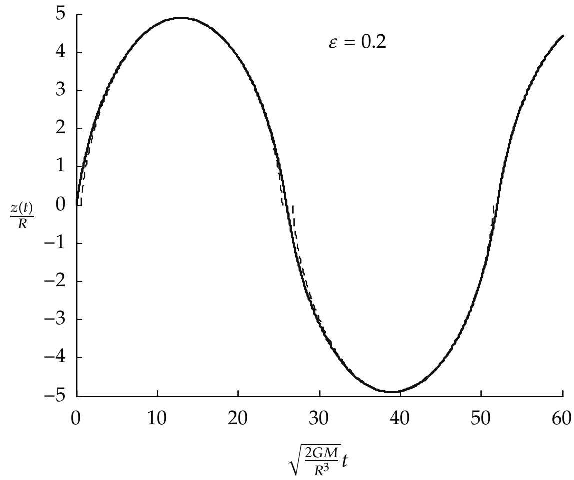

Figure 2: The function $z(t)$ with $\varepsilon = 0.2$, the solid line corresponds to numerical integration of (3.1), the dashed line represents the outer solution (3.31).

Introducing (3.30) into (3.28), the following expression is obtained to $z(t)$ in the far field

$$\begin{aligned}&\frac{1}{\varepsilon^{3/2}}\left(-\sqrt{\varepsilon\left(z-\frac{1}{2}R\varepsilon\right)}\sqrt{R\left(1-\frac{1}{2}\varepsilon^2\right)-\varepsilon z}+R\tan^{-1}\left(\frac{\sqrt{\varepsilon(z-(1/2)R\varepsilon)}}{\sqrt{R(1-(1/2)\varepsilon^2)-\varepsilon z}}\right)\right)\\&\quad=\sqrt{\frac{4GM}{R}}\,t-R\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(-3/4)}{\Gamma(-1/4)}-\frac{1}{2}R\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(-5/4)}{\Gamma(-3/4)}\varepsilon.\end{aligned}\tag{3.31}$$

The quarter-period of the orbit is obtained by evaluating (3.31) in $z = R/\varepsilon(1-(1/2)\varepsilon^2)$, thus obtaining the following expression for the period (including corrections to order ε^2):

$$\sqrt{\frac{4GM}{R^3}}\,T=\frac{2\pi}{\varepsilon^{3/2}}+\frac{\sqrt{2}}{3\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)-\frac{12\,\sqrt{2\pi^3}}{5\,\Gamma^2(1/4)}\varepsilon+\frac{5\sqrt{2}}{54\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)\varepsilon^2.\tag{3.32}$$

4. Results

In this section, the results are expressed using dimensionless variables, the dimensionless time $\tau = \sqrt{2GM/R^3}\,t$, and the dimensionless distance $\eta = z/R$. In this representation, the results are general and it is not necessary to give numerical values for the constants G, M and R.

Figure 2 presents the function $z(t)$ with $\varepsilon = 0.2$. The solid line corresponds to the solution obtained by numerical integration of (3.1), the dashed line represents the outer solution (3.31). It can be seen that, although ε is not very small, the analytical solution is quite accurate.

Figure 3 shows $z(t)$ with $\varepsilon = 0.2$, in the first quarter period of the orbit, where the outer solution (3.31) and the inner solution (3.6) can be observed (dashed line), the numerical

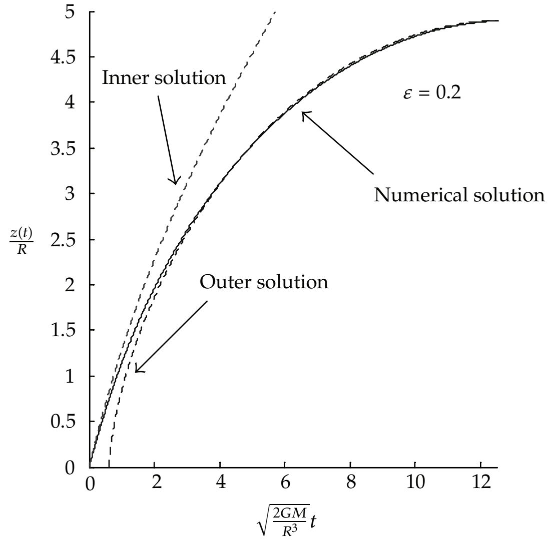

Figure 3: The function $z(t)$ in the first quarter period of the orbit, with $\varepsilon = 0.2$. Solid line corresponds to numerical integration of (3.1), the dashed line represents the outer and the inner solution.

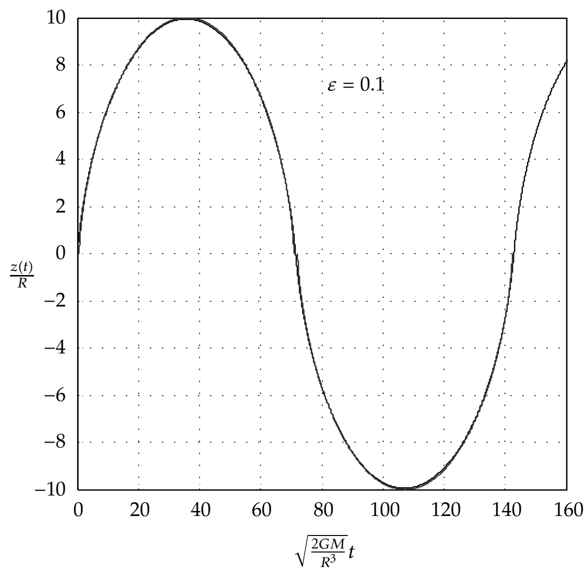

Figure 4: The function $z(t)$ with $\varepsilon = 0.1$, the solid line corresponds to numerical integration of (3.1), the dashed line represents the outer solution (3.23).

solution is represented with a solid line. Thus, in the region close to the origin of coordinates, the outer solution should be replaced by the inner solution. In Figure 3 you can see that the matching between the two solution is in a region centered around $z/R \approx 1.5$.

Figure 4 shows the function $z(t)$ con $\varepsilon = 0.1$. The solid line corresponds to the solution obtained by numerical integration of (3.1), the dashed line represents the outer

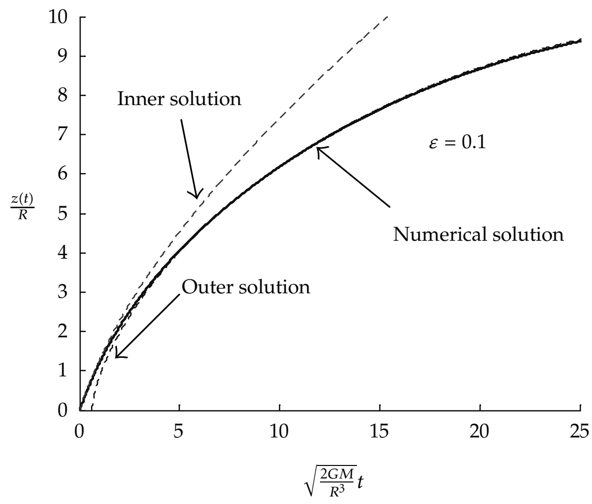

Figure 5: The function $z(t)$ with $\varepsilon = 0.1$. The solid line corresponds to numerical integration of (3.1), the dashed line represents the outer solution and the inner solution.

solution (3.31). It can be seen that the numerical and analytical solution are indistinguishable on this scale. This result indicates, that for values of $\varepsilon \ll 1$, the movement of the third body can be well described by two analytical solutions, the inner and outer solution.

Figure 5 shows $z(t)$ with $\varepsilon = 0.1$ in the region of matching between the inner solution (3.6) and the outer solution (3.31), where it can be noticed that the matching is in a region centered around $z/R \approx 2$.

As the parameter $\varepsilon \to 0$, the position of the region of matching between the two solutions will be centered approximately in a position $1/\varepsilon$, so, in the case $\varepsilon = 0$, the orbit is unbounded and the inner solution is the exact solution to the orbit.

Figure 6 shows $z(t)$ with $\varepsilon = 0.1$, in the first quarter period of the orbit, the dashed line represents the uniform approximation (3.19) and the solid line denotes the numerical solution. It can be seen that the numerical and the uniform approximation are indistinguishable on this scale.

Defining the relative error between the numerical solution and the uniform approximation as $|(z_{\text{num}}(t) - z_{\text{unif}}(t))/z_{\text{num}}(t)|$, and taking the temporal average of a quarter period, the following values for the mean relative error are obtained (Table 1).

Figure 7 shows the period of the orbit as a function of the parameter k defined as $k^2 = 1/2(1-\varepsilon)$. The solid line corresponds to the period obtained by numerical integration of Macmillan equation (2.8), the dashed line represents the asymptotic solution (3.32).

Figure 7 indicates that the asymptotic expansion (3.32) for the period of the third body matched very well with the numerical calculation of (2.8), the for the period for values of k close to $1/\sqrt{2}$.

The relative error, defined as $((T_{\text{asymp}} - T_{\text{num}})/T_{\text{num}})$, between the asymptotic solution and the numerical solution of the period of oscillations as a function of the parameter k can be seen in Figure 8. The results show that the asymptotic solution for the period of oscillation (3.32) gives good results in a wide range of values of the parameter ε. The asymptotic solution is valid in the region $\varepsilon \ll 1$, that is, when k is close to $1/\sqrt{2}$. In this region the relative error is minor than 0.005. The region of small amplitudes in the movement of the third body,

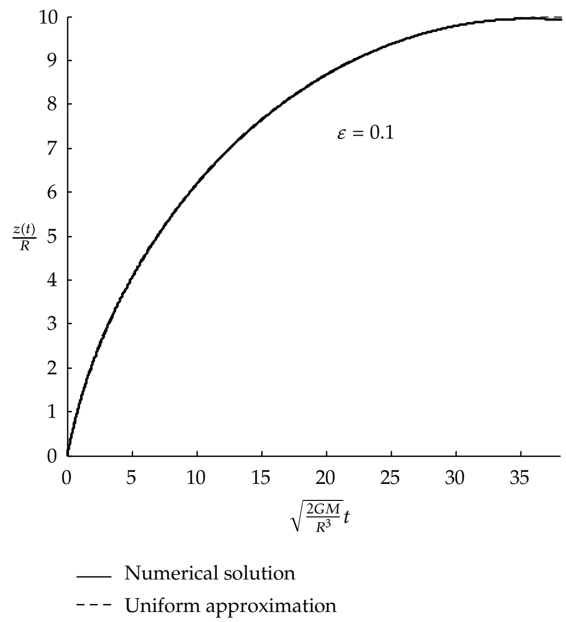

Figure 6: The function $z(t)$ in the first quarter period of the orbit, with $\varepsilon = 0.1$, solid line corresponds to numerical integration of (3.1), the dashed line represents the uniform approximation.

Table 1: Relative error between the numerical solution and the uniform approximation.

ε	Error
0.05	0.0003
0.10	0.0013
0.20	0.0058
0.40	0.0250

corresponds to $\varepsilon \approx 1$, or equivalently $k \ll 1$. In this region the relative error of the asymptotic solution is close to 0.04.

In the region of small amplitudes in the movement of the third body, MacMillan [2] finds an expression for the period of oscillations of the third body, namely:

$$\sqrt{\frac{2GM}{R^3}}T = 2\pi\left(1 + \frac{9}{4}k^2 + \frac{345}{64}k^4 + \cdots\right). \tag{4.1}$$

Figure 9 allows us to see the relative error between the solution of MacMillan (4.1) and the numerical solution of the period of oscillations as a function of the parameter k (dash line). What is also shown is the relative error between the asymptotic solution (3.32) and the numerical solution of the period of oscillations as a function of the parameter k.

Figure 9 indicates that the MacMillan solution (4.1) works well for moderate values of k, while the asymptotic solution (3.32) works well in the region where $k > 0.33$. Then,

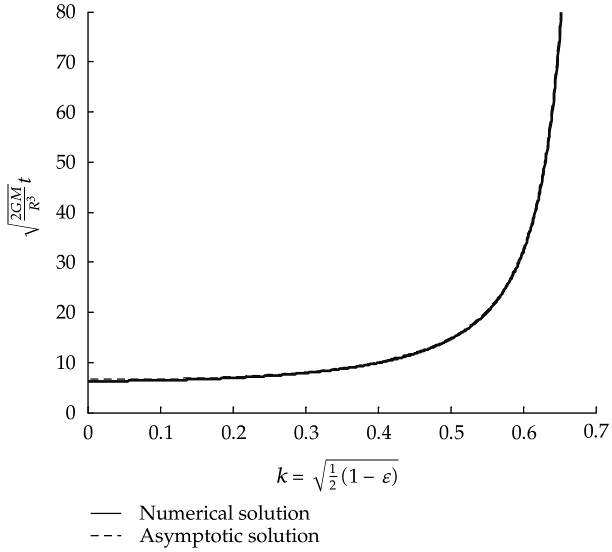

Figure 7: The period of the orbit as a function of the parameter k.

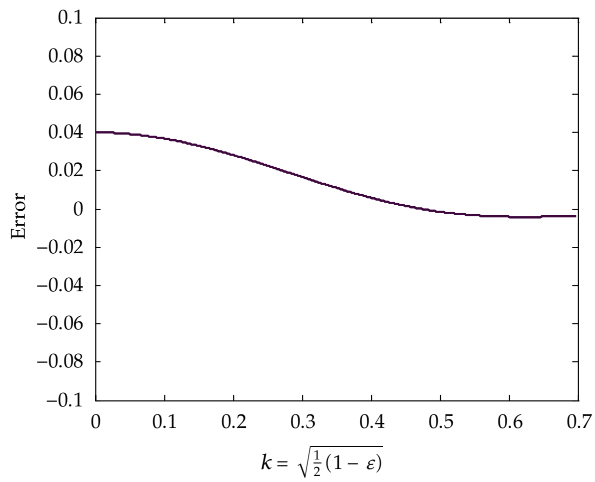

Figure 8: Relative error between the asymptotic solution and the numerical solution of the period as a function of the parameter k.

one can construct the following formula valid for the entire domain $0 < k < 1/\sqrt{2}$ to the period of the third body:

$$\sqrt{\frac{2GM}{R^3}}\,T = \begin{cases} 2\pi\left(1+\dfrac{9}{4}k^2+\dfrac{345}{64}k^4+\cdots\right) & k \le 0.33 \\ \left(\dfrac{2\pi}{(1-2k^2)^{3/2}}+\dfrac{\sqrt{2}}{3\sqrt{\pi}}\Gamma^2\left(\dfrac{1}{4}\right)-\dfrac{12\sqrt{2\pi^3}}{5\,\Gamma^2(1/4)}\left(1-2k^2\right)\right)\dfrac{1}{\sqrt{2}} & 0.33 \le k < \dfrac{1}{\sqrt{2}}. \end{cases} \tag{4.2}$$

This solution has a relative error less than 0.012 for the entire domain $0 < k < 1/\sqrt{2}$.

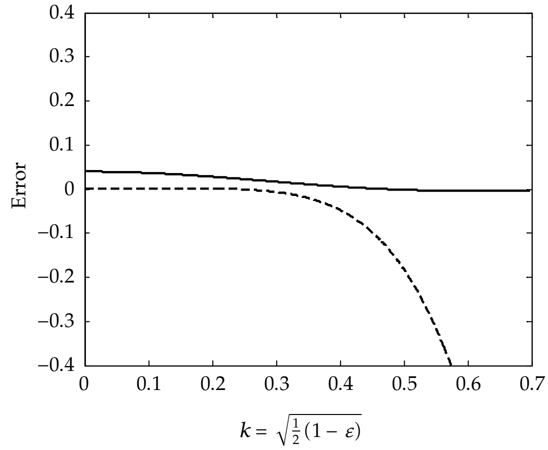

Figure 9: Relative error of the period, MacMillan ((4.1) dashed line), asymptotic solution ((3.32) solid line).

5. Conclusions

In this paper, the circular Sitnikov problem is revisited. By matching between asymptotic expansions, approximate analytical expressions for the period and the orbit of the third body in the case of large oscillation are found. The results show that for values $\varepsilon \ll 1$, the movement of the third body can be well described by two analytical solutions, the inner and outer solution.

The analytical asymptotic equation to the period of the third body matched very well with the numerical calculations. It also provides an improved expression for the period of the orbit valid for the entire domain of bounded orbits.

The method of matched asymptotic expansions employed in this paper can be extended to more complex problems. For example, the elliptic Sitnikov problem, variable mass of the third body and the relativistic Sitnikov problem. In all these problems, when the speed of the third body is very close to the escape velocity, computational times are very long and it becomes necessary to resort to asymptotic analysis.

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