Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 546784, 28 pages
doi:10.1155/2012/546784

Research Article

Approximation by Lupas-Type Operators and Szász-Mirakyan-Type Operators

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Received 28 July 2011; Accepted 5 January 2012

Academic Editor: Yuantong Gu

Lupas-type operators and Szász-Mirakyan-type operators are the modifications of Bernstein polynomials to infinite intervals. In this paper, we investigate the convergence of Lupas-type operators and Szász-Mirakyan-type operators on $[0,\infty)$.

1. Introduction and Main Results

For $f \in C([0,1])$, Bernstein operator $B_n f(x)$ is defined as follows: Let

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leqslant k \leqslant n, \tag{1.1}$$

and then we define

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right). \tag{1.2}$$

Derriennic [1] gave a modified operator of $B_n f$ such as

$$(M_n^* f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \tag{1.3}$$

and obtained the result that for $f \in C^{(2)}([0,1])$,

$$\lim_{n\to\infty}\left(\left(M_n^* f\right)(x) - f(x)\right) = (1-2x)f'(x) + x(1-x)f''(x). \tag{1.4}$$

Lupas investigated a family of linear positive operators which mapped the class of all bounded and continuous functions on $[0,\infty)$ into $C[0,\infty)$ such that

$$(L_n f)(x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \in [0,\infty). \tag{1.5}$$

Moreover, Sahai and Prasad [2] modified Lupas operators as follows: Let f be integrable on $[0,\infty)$ and let n be a positive integer. Then we define

$$(M_n[f])(x) = (n-1)\sum_{k=0}^{\infty} P_{n,k}(x)\int_0^{\infty} P_{n,k}(y) f(y)\,dy, \quad x \in [0,\infty), \tag{1.6}$$

where

$$P_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}. \tag{1.7}$$

In this paper, we assume that n is a positive integer. Then they obtained the following;

Theorem 1.1 (see [2], Theorem 1). *If f is integrable on $[0,\infty)$ and admits its $(r+1)$th and $(r+2)$th derivatives, which are bounded at a point $x \in [0,\infty)$, and $f^{(r)}(x) = O(x^\alpha)$ (α is a positive integer $\geqslant 2$) as $x \to \infty$, then*

$$\lim_{n\to\infty} n\left(\left(M_n[f]\right)^{(r)}(x) - f^{(r)}(x)\right) = (r+1)(1-2x)f^{(r+1)}(x) + x(1-x)f^{(r+2)}(x). \tag{1.8}$$

Theorem 1.1 holds only for bounded $x \leqslant K$, so it does not mean the norm convergence on $[0,\infty)$. In this paper, we improve Theorem 1.1 with respect to the norm convergence on $[0,\infty)$.

Let $0 < p \leqslant \infty$ and let w be a positive weight, that is, $w(x) \geqslant 0$ for $x \in \mathbb{R}$. For a function g on $[0,\infty)$, we define the norm by

$$\|g\|_{L_p([0,\infty))} := \begin{cases} \left(\int_{[0,\infty)} |g(t)|^p dt\right)^{1/p}, & 0 < p < \infty \\ \sup_{[0,\infty)} |g(t)|, & p = \infty. \end{cases} \tag{1.9}$$

For convenience, for nonnegative integers $n \geqslant 2$, r, and $n-r-2 \geqslant 0$, we let

$$A_{n,r} := \frac{(n-1)!(n-2)!}{(n-r-2)!(n+r-1)!}. \tag{1.10}$$

Then we have the following results:

Theorem 1.2. *Let $0 < p \leqslant \infty$. Let α and r be nonnegative integers and $n - r - 2 \geqslant 0$. Let $f \in C^{(r+1)}([0,\infty))$ satisfy*

$$\left|f^{(r)}(x)\right| \leqslant O(1)(x+1)^{\alpha}, \qquad \left|f^{(r+1)}(x)\right| \leqslant O(1)(x+1)^{\alpha+2}. \tag{1.11}$$

Then we have uniformly for f and n,

$$\left|A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x)\right| = O\left(\frac{1}{n^{1/3}}\right)(x+1)^{\alpha+2}. \tag{1.12}$$

In particular, if $\|(x+1)^{\alpha+2}w(x)\|_{L_p([0,\infty))} < \infty$, then we have uniformly for n,

$$\left\|\left(A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x)\right)w(x)\right\|_{L_p([0,\infty))} = O\left(\frac{1}{n^{1/3}}\right). \tag{1.13}$$

Remark 1.3. (a) We see that for nonnegative integers $n \geqslant 2$, r, and $n - r - 2 \geqslant 0$,

$$\frac{(n-1)!(n-2)!}{(n-r-2)!(n+r-1)!} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \tag{1.14}$$

(b) The following weight is useful.

$$w_\lambda(x) = \frac{1}{1+x^\lambda}\begin{cases} \lambda > \dfrac{1}{p} + \alpha + 2, & 0 < p < \infty, \\ \lambda \geqslant \alpha + 2, & p = \infty. \end{cases} \tag{1.15}$$

Let

$$\varphi(x) := \frac{1}{1+x}. \tag{1.16}$$

Theorem 1.4. *Let r and β be nonnegative integers and $n - r - 2 \geqslant 0$. Let $f \in C^{(r+2)}([0,\infty))$ satisfy*

$$\left\|f^{(r+1)}(x)\varphi^{2\beta+1}(x)\right\|_{L_\infty([0,\infty))} < \infty, \qquad \left\|f^{(r+2)}(x)\varphi^{2\beta}(x)\right\|_{L_\infty([0,\infty))} < \infty. \tag{1.17}$$

Then we have uniformly for f and n,

$$\begin{aligned} &\left|A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x)\right|\varphi^{2\beta+2}(x) \\ &\quad \leqslant O\left(\frac{1}{n}\right)\left(\left\|f^{(r+1)}(x)\varphi^{2\beta+1}(x)\right\|_{L_\infty([0,\infty))} + \left\|f^{(r+2)}(x)\varphi^{2\beta}(x)\right\|_{L_\infty([0,\infty))}\right). \end{aligned} \tag{1.18}$$

Let us define the weighted modulus of smoothness by

$$\omega_k(f;\eta;t) := \sup_{0\leqslant h\leqslant t}\left\|\Delta_h^k f(\cdot)\eta(\cdot)\right\|_{L_\infty([0,\infty))}, \quad t\geqslant 0,\ k=1,2, \tag{1.19}$$

where

$$\Delta_h^1 f(x) = f(x+h) - f(x), \qquad \Delta_h^2 f(x) = f(x) - 2f(x+h) + f(x+2h). \tag{1.20}$$

Theorem 1.5. *Let β and r be nonnegative integers and $n-r-2\geqslant 0$. Let $f\in C^r([0,\infty))$. Then we have uniformly for f and n,*

$$\begin{aligned}&\left\|\left(A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x)\right)\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))}\\ &\quad\leqslant C\left(\frac{1}{\sqrt{n}}\omega_1\left(f^{(r)};\varphi^{2\beta+1};\frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(r)};\varphi^{2\beta};\frac{1}{\sqrt{n}}\right)\right).\end{aligned} \tag{1.21}$$

The Szász-Mirakyan operators are also generalizations of Bernstein polynomials on infinite intervals. They are defined by:

$$S_n(f)(x) = \sum_{k=0}^{\infty} S_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.22}$$

where

$$S_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}. \tag{1.23}$$

In [3], the class of Szász-Mirakyan operators $S_{n;r,q}(f;x)$ was defined as follows:

$$S_{n;r,q}(f;x) := \frac{1}{A_r(nx)}\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n+q}\right), \quad x\in[0,\infty), \tag{1.24}$$

where $q>0$ and

$$A_r(t) = \sum_{k=0}^{\infty}\frac{t^{rk}}{(rk)!}, \quad t\in[0,\infty). \tag{1.25}$$

Theorem 1.6 (see [3]). *Let $q>0$ and $r\in\mathbb{N}$ be fixed numbers. Then there exists $M_{q,r} = \text{const.} > 0$ depending only on q and r such that, for every uniformly continuous and bounded function $f(x)e^{-qx}$ on $[0,\infty)$, the following inequalities hold;*

(a)

$$\|(S_{n;q,r}(f;x)-f(x))\varphi(x)e^{-qx}\|_{L_\infty([0,\infty))} \leqslant M_{q,r}\frac{1}{n+q}\left(\|f'(x)e^{-qx}\|_{L_\infty([0,\infty))}+\|f''(x)e^{-qx}\|_{L_\infty([0,\infty))}\right); \tag{1.26}$$

(b)

$$\|(S_{n;q,r}(f;x)-f(x))\varphi(x)e^{-qx}\|_{L_\infty([0,\infty))} \leqslant M_{q,r}\frac{1}{n+q}(\delta_{n,q}\omega_1(f;e^{-qx};\delta_{n,q})+\omega_2(f;e^{-qx};\delta_{n,q})), \tag{1.27}$$

where $\delta_{n,q} := (n+q)^{-1/2}$.

(c) *for every fixed* $x \in [0,\infty)$, *we have for every continuous* f *with* $f^{(j)}(x)e^{-qx}$, $j = 0,1,2$, *bounded on* $[0,\infty)$,

$$\lim_{n\to\infty} n(S_{n;q,r}(f;x)-f(x)) = -qxf'(x)+\frac{x}{2}f''(x). \tag{1.28}$$

Now, we modify the Szász-Mirakyan operators as follows: let f be integrable on $[0,\infty)$, then we define

$$(Q_{n,\beta}[f])(x) = (n+\beta)\sum_{k=0}^{\infty}S_{n,k}(x)\int_0^\infty S_{n+\beta,k}(y)f(y)dy, \quad x\in[0,\infty), \tag{1.29}$$

where β is a nonnegative integer. Then we have the following results:

Theorem 1.7. *Let* α, β *and* r *be nonnegative integers. Let* $f \in C^{(r+1)}([0,\infty))$ *satisfies*

$$\left|f^{(r)}(x)\right| \leqslant O(1)e^{\beta x}(x+1)^{\alpha}, \qquad \left|f^{(r+1)}(x)\right| \leqslant O(1)e^{\beta x}(x+1)^{\alpha+2}. \tag{1.30}$$

Then one has uniformly for f *and* n,

$$\left|\left(\frac{n+\beta}{n}\right)^r (Q_{n,\beta}[f])^{(r)}(x)-f^{(r)}(x)\right| = O\left(\frac{1}{n^{1/3}}\right)e^{\beta x}(x+1)^{\alpha+2}. \tag{1.31}$$

In particular, let $0 < p \leqslant \infty$. *If one supposes* $\|e^{\beta x}(x+1)^{\alpha+2}w(x)\|_{L_p([0,\infty))} < \infty$, *then one has uniformly for* f *and* n,

$$\left\|\left(\left(\frac{n+\beta}{n}\right)^r (Q_{n,\beta}[f])^{(r)}(x)-f^{(r)}(x)\right)w(x)\right\|_{L_p([0,\infty))} = O\left(\frac{1}{n^{1/3}}\right). \tag{1.32}$$

Remark 1.8. (a) We note that for nonnegative integers β and r,

$$\left(\frac{n+\beta}{n}\right)^r \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \tag{1.33}$$

(b) The following weight is useful.

$$w_{\lambda,\beta}(x) = e^{-\beta x} w_\lambda(x), \tag{1.34}$$

where $w_\lambda(x)$ is defined in Remark 1.3.

Theorem 1.9. *Let β, γ, and r be nonnegative integers. Let $f \in C^{(r+2)}([0,\infty))$ satisfies*

$$\left\| f^{(r+1)}(x) e^{-\beta x} \varphi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} < \infty, \qquad \left\| f^{(r+2)}(x) e^{-\beta x} \varphi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} < \infty. \tag{1.35}$$

Then one has uniformly for f and n,

$$\begin{aligned} &\left| \left[\left(\frac{n+\beta}{n}\right)^r \left(Q_{n,\beta}[f]\right)^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \varphi^{2\gamma+2}(x) \right| \\ &\quad \leqslant O\left(\frac{1}{n}\right) \left(\left\| f^{(r+1)}(x) e^{-\beta x} \varphi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| f^{(r+2)}(x) e^{-\beta x} \varphi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \right). \end{aligned} \tag{1.36}$$

Theorem 1.10. *Let β, γ, and r be nonnegative integers. Then one has for $f \in C^r([0,\infty))$,*

$$\begin{aligned} &\left\| \left[\left(\frac{n+\beta}{n}\right)^r \left(Q_{n,\beta}[f]\right)^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \varphi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\ &\quad \leqslant C\left(\frac{1}{\sqrt{n}} \omega_1\left(f; e^{-\beta x} \varphi^{2\gamma+1}(x); \frac{1}{\sqrt{n}}\right) + \omega_2\left(f; e^{-\beta x} \varphi^{2\gamma}(x); \frac{1}{\sqrt{n}}\right) \right). \end{aligned} \tag{1.37}$$

2. Proofs of Results

First, we will prove results for Lupas-type operators such as Theorems 1.2, 1.4, and 1.5. To prove theorems, we need some lemmas.

Lemma 2.1. *Let m and r be nonnegative integers and $n > m + r + 1$. Let*

$$T_{n,m,r}(x) := (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^\infty P_{n-r,k+r}(y)(y-x)^m dy. \tag{2.1}$$

Then

(i) $T_{n,0,r}(x) = 1$,

(ii)

$$T_{n,1,r}(x) = \frac{(r+1)(1+2x)}{(n-r+2)},$$
$$T_{n,2,r}(x) = \frac{2(n-1)x(1+x)}{(n-r-2)(n-r-3)} + \frac{(r+1)(r+2)(1+2x)^2}{(n-r-2)(n-r-3)}; \tag{2.2}$$

(iii) *for* $m \geqslant 1$,

$$(n-m-r-2)T_{n,m+1,r}(x)$$
$$= x(1+x)\left(T'_{n,m,r}(x) + 2mT_{n,m-1,r}(x)\right) + (m+r+1)(1+2x)T_{n,m,r}, \tag{2.3}$$

where $n > m+r+2$;

(iv) *for* $m \geqslant 0$,

$$T_{n,m,r}(x) = O\left(\frac{1}{n^{[(m+1)/2]}}\right)q_{n,m,r}(x), \tag{2.4}$$

where $q_{n,m,r}(x)$ *is a polynomial of degree* $\leqslant m$ *such that the coefficients are bounded independently of* n *and they are positive for* $n > m+r+1$.

Proof. (i), (ii), and (iii) have been proved in [2, Lemma 1]. So we may show only the part of (2.4). For $m = 1, 2$, (2.4) holds. Let us assume (2.4) for $m(\geqslant 2)$. We note

$$T'_{n,m,r}(x) = O\left(\frac{1}{n^{[(m+1)/2]}}\right)q'_{m,r}(x), \qquad q'_{m,r}(x) \in \mathcal{P}_{m-1}. \tag{2.5}$$

So, we have by the assumption of induction,

$$(n-m-r-2)T_{n,m+1,r}(x)$$
$$= x(1+x)\left(T'_{n,m,r}(x) + 2mT_{n,m-1,r}(x)\right) + (m+r+1)(1+2x)T_{n,m,r}$$
$$\leqslant x(1+x)\left(C\frac{1}{n^{[(m+1)/2]}}q'_{m,r}(x) + 2m\frac{1}{n^{[m/2]}}q_{m-1,r}(x)\right)$$
$$+ (m+r+1)(1+2x)\frac{1}{n^{[(m+1)/2]}}q_{m,r}(x). \tag{2.6}$$

Here, if m is even, then

$$\left[\frac{m+1}{2}\right] + 1 = \frac{m}{2} + 1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \qquad \left[\frac{m}{2}\right] + 1 = \frac{m}{2} + 1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \tag{2.7}$$

and if m is odd, then

$$\left[\frac{m+1}{2}\right]+1=\frac{m+1}{2}+1=\left[\frac{m+2}{2}\right], \qquad \left[\frac{m}{2}\right]+1=\frac{m-1}{2}+1=\frac{m+1}{2}=\left[\frac{m+2}{2}\right]. \tag{2.8}$$

Hence, we have

$$T_{n,m+1,r}(x)=O\left(\frac{1}{n^{[(m+2)/2]}}\right)q_{n,m+1,r}(x), \tag{2.9}$$

and here we see that $q_{n,m+1,r}(x)$ is a polynomial of degree $\leqslant m+1$ such that the coefficients of $q_{n,m+1,r}(x)$ are bounded independently of n. Moreover, we see from (2.6) that the coefficients of $q_{n,m+1,r}(x)$ are positive for $n>m+r+2$. □

Lemma 2.2 (see [2, Lemma 2]). *Let r be a nonnegative integer and $n-r-2\geqslant 0$. Then one has for $f\in C^r([0,\infty))$:*

$$(M_n[f])^{(r)}(x)=\frac{(n-r-1)!(n+r-1)!}{(n-1)!(n-2)!}\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)f^{(r)}(y)dy. \tag{2.10}$$

Let

$$\left(\widetilde{M}_n[f]\right)^{(r)}(x)=(n-r-1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)f^{(r)}(y)dy. \tag{2.11}$$

Then we have

$$\left(\widetilde{M}_n[f]\right)^{(r)}(x)=A_{n,r}(M_n[f])^{(r)}(x), \tag{2.12}$$

where $A_{n,r}$ is defined by (1.10).

Proof of Theorem 1.2. Let $|y-x|\leqslant 1$. By the second inequality in (1.11),

$$\begin{aligned}\left|f^{(r)}(y)-f^{(r)}(x)\right|&=|y-x|\left|f^{(r+1)}(\xi)\right|\\&\leqslant O(1)|y-x|\left|\xi^{\alpha+2}\right|\leqslant O(1)|y-x|(x+1)^{\alpha+2}.\end{aligned} \tag{2.13}$$

Let $\varepsilon := n^{-\gamma}, 0 < \gamma < 1$,

$$\begin{aligned}
&\left|\left(\left(\widetilde{M}_n[f]\right)^{(r)}(x) - f^{(r)}(x)\right)\right| \\
&\quad = \Bigg|(n-r-1)\sum_{k=0}^{\infty} P_{n+r,k}(x) \\
&\qquad \times\left(\int_{|y-x|<\varepsilon} P_{n-r,k+r}(y)\left|f^{(r)}(y) - f^{(r)}(x)\right|dy + \int_{|y-x|\geqslant\varepsilon} P_{n-r,k+r}(y)\left|f^{(r)}(y) - f^{(r)}(x)\right|dy\right)\Bigg| \\
&\quad =: A + B.
\end{aligned} \tag{2.14}$$

First, we see by (2.13) and Lemma 2.1,

$$\begin{aligned}
A &= O(1)\left|(n-r-1)\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|<\varepsilon} P_{n-r,k+r}(y)\left|y-x\right|(x+1)^{\alpha+2}dy\right| \\
&\leqslant O(1)\varepsilon|T_{n,0,r}(x)|(x+1)^{\alpha+2} = O(1)\varepsilon(x+1)^{\alpha+2}.
\end{aligned} \tag{2.15}$$

Next, we estimate B. By the first inequality in (1.11),

$$\begin{aligned}
B &\leqslant C\left|(n-r-1)\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|>\varepsilon} P_{n-r,k+r}(y)\left(\left|f^{(r)}(y)\right| + \left|f^{(r)}(x)\right|\right)dy\right| \\
&\leqslant C\left|(n-r-1)\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|>\varepsilon} P_{n-r,k+r}(y)\left((y+1)^{\alpha} + (x+1)^{\alpha}\right)dy\right|.
\end{aligned} \tag{2.16}$$

Here, using

$$(y+1)^{\alpha} = \left((y-x) + x + 1\right)^{\alpha} = \sum_{i=0}^{\alpha}\binom{\alpha}{i}(y-x)^{i}(x+1)^{\alpha-i} \tag{2.17}$$

and the notation:

$$\langle i\rangle = \begin{cases} 1, & (i : \text{odd}) \\ 0, & (i : \text{even}), \end{cases} \tag{2.18}$$

we have

$$B \leqslant C|(n-r-1)|\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_{|y-x|>\varepsilon} P_{n-r,k+r}(y) \times \left(\sum_{i=1}^{\alpha}\binom{\alpha}{i}(y-x)^{i}(x+1)^{\alpha-i} + 2(x+1)^{\alpha}\right)dy$$

$$\leqslant C|(n-r-1)|\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_{|y-x|>\varepsilon}P_{n-r,k+r}(y)\times\left(\sum_{i=1}^{\alpha}\binom{\alpha}{i}(y-x)^{i}\left|\frac{y-x}{\varepsilon}\right|^{\langle i\rangle}(x+1)^{\alpha-i}\right)dy$$
$$+C|(n-r-1)|\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_{|y-x|>\varepsilon}P_{n-r,k+r}(y)\left(\frac{y-x}{\varepsilon}\right)^{2}(x+1)^{\alpha}dy$$
$$:=B_1+B_2. \tag{2.19}$$

Then, we obtain

$$\begin{aligned}B_1&\leqslant C\left(\sum_{i=1}^{\alpha}\binom{\alpha}{i}\left|T_{n,i+\langle i\rangle,r}\right|\left(\frac{1}{\varepsilon}\right)^{\langle i\rangle}(x+1)^{\alpha-i}\right)\\&\leqslant C\sum_{i=1}^{\alpha}\binom{\alpha}{i}O\left(\frac{n^{\gamma\langle i\rangle}}{n^{[(i+\langle i\rangle+1)/2]}}\right)\left|q_{n,i+\langle i\rangle,r}(x)\right|(x+1)^{\alpha-i}\\&\leqslant O\left(\frac{1}{n^{[(i+\langle i\rangle+1)/2]-\gamma\langle i\rangle}}\right)(x+1)^{\alpha+\langle i\rangle}\leqslant O\left(\frac{1}{n^{1-\gamma}}\right)(x+1)^{\alpha+1}.\end{aligned} \tag{2.20}$$

Here, we used the following that for $i\geqslant 1$,

$$\left[\frac{i+\langle i\rangle+1}{2}\right]-\gamma\langle i\rangle\geqslant 1-\gamma, \tag{2.21}$$

because

$$\left[\frac{i+\langle i\rangle+1}{2}\right]-\gamma\langle i\rangle=\begin{cases}\dfrac{i+1}{2}-\gamma, & i:\text{odd},\\ \dfrac{i}{2}, & i:\text{even}.\end{cases} \tag{2.22}$$

And we know that

$$\begin{aligned}B_2&\leqslant C|T_{n,2,r}(x)|\left(\frac{1}{\varepsilon}\right)^{2}x^{\alpha}\leqslant O\left(\frac{1}{n^{[3/2]}}\right)\left|q_{n,2,r}(x)\right|\left(\frac{1}{\varepsilon}\right)^{2}x^{\alpha}\\&\leqslant O\left(\frac{n^{2\gamma}}{n^{[3/2]}}\right)\left|q_{n,2,r}(x)\right|x^{\alpha}\leqslant O\left(\frac{1}{n^{1-2\gamma}}\right)(x+1)^{\alpha+2}.\end{aligned} \tag{2.23}$$

Thus, we obtain

$$B\leqslant O\left(\frac{1}{n^{1-2\gamma}}\right)(x+1)^{\alpha+2}. \tag{2.24}$$

Therefore, we have uniformly on n,

$$\left|\left(\widetilde{M}_n[f]\right)^{(r)}(x)-f^{(r)}(x)\right| \leqslant O\left(\frac{1}{n^{\gamma}}\right)(x+1)^{\alpha+2}+O\left(\frac{1}{n^{1-2\gamma}}\right)(x+1)^{\alpha+2}. \quad (2.25)$$

Here, if we let $\gamma = 1/3$, then we have

$$\left|\left(\widetilde{M}_n[f]\right)^{(r)}(x)-f^{(r)}(x)\right| = O\left(\frac{1}{n^{1/3}}\right)(x+1)^{\alpha+2}, \quad (2.26)$$

that is, (1.12) is proved. So, we also have a norm convergence (1.13). □

Proof of Theorem 1.4. We know that for $f \in C^{(r+2)}([0,\infty))$,

$$f^{(r)}(t) = f^{(r)}(x)+f^{(r+1)}(x)(t-x)+\int_x^t (t-u)f^{(r+2)}(u)du, \quad (2.27)$$

$$\left|\int_x^t (t-u)f^{(r+2)}(u)du\right| \leqslant C\left\|f^{(r+2)}(x)\psi^{2\beta}(x)\right\|_{L_\infty([0,\infty))}\left((1+x)^{2\beta}+(1+t)^{2\beta}\right)(t-x)^2, \quad (2.28)$$

where $\psi(t) = 1/(1+x)$. Then we obtain from (2.10) and (2.27),

$$\begin{aligned}\left(\widetilde{M}_n[f]\right)^{(r)}(x) \\ = f^{(r)}(x)+f^{(r+1)}(x)T_{n,1,r}(x)+(n-r-1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)\int_x^y (y-u)f^{(r+2)}(u)du\,dy\end{aligned} \quad (2.29)$$

and from (2.28),

$$\begin{aligned}&\left|(n-r-1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)\int_x^y (y-u)f^{(r+2)}(u)du\,dy\right| \\ &\quad\leqslant \left\|f^{(r+2)}(x)\psi^{2\beta}(x)\right\|_{L_\infty([0,\infty))} \\ &\qquad\times\left((1+x)^{2\beta}|T_{n,2,r}(x)|+\left|(n-r-1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)(1+y)^{2\beta}(y-x)^2dy\right|\right).\end{aligned} \quad (2.30)$$

Using $(1+y)^{2\beta} \leqslant C((y-x)^{2\beta}+(1+x)^{2\beta})$, we have

$$\begin{aligned}&\left|(n-r-1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)(1+y)^{2\beta}(y-x)^2dy\right| \\ &\quad\leqslant C\left(\left|T_{n,2\beta+2,r}(x)\right|+(1+x)^{2\beta}|T_{n,2,r}(x)|\right).\end{aligned} \quad (2.31)$$

Therefore, we have

$$\begin{aligned}
&\left|\left(\widetilde{M}_n[f]\right)^{(r)}(x) - f^{(r)}(x)\right|\psi^{2\beta+2}(x) \\
&\quad\leqslant \left|f^{(r+1)}(x)\psi^{2\beta+1}(x)\right| |T_{n,1,r}(x)|\ \psi(x) \\
&\qquad + C\left\|f^{(r+2)}(x)\psi^{2\beta}(x)\right\|_{L_\infty([0,\infty))}(1+x)^{2\beta}|T_{n,2,r}(x)|\psi^{2\beta+2}(x) \\
&\qquad + C\left\|f^{(r+2)}(x)\psi^{2\beta}(x)\right\|_{L_\infty([0,\infty))}\left|T_{n,2\beta+2,r}(x)\right|\psi^{2\beta+2}(x) \\
&\quad\leqslant O\left(\frac{1}{n}\right)\left|f^{(r+1)}(x)\psi^{2\beta+1}(x)\right|\left|g_{n,1,r}(x)\right|\psi(x) \\
&\qquad + O\left(\frac{1}{n}\right)\left\|f^{(r+2)}(x)\psi^{2\beta}(x)\right\|_{L_\infty([0,\infty))}(1+x)^{2\beta}\left|g_{n,2,r}(x)\right|\psi^{2\beta+2}(x) \\
&\qquad + O\left(\frac{1}{n^{\beta+1}}\right)\left\|f^{(r+2)}(x)\psi^{2\beta}(x)\right\|_{L_\infty([0,\infty))}\left|g_{n,2\beta+2,r}(x)\right|\psi^{2\beta+2}(x).
\end{aligned} \tag{2.32}$$

Since we know that for $x \in [0,\infty)$,

$$\left|g_{n,1,r}(x)\right|\psi(x) \leqslant C, \qquad (1+x)^{2\beta}\left|g_{n,2,r}(x)\right|\psi^{2\beta+2}(x) \leqslant C, \qquad \left|g_{n,2\beta+2,r}(x)\right|\psi^{2\beta+2}(x) \leqslant C, \tag{2.33}$$

we have

$$\begin{aligned}
&\left|\left(\widetilde{M}_n[f]\right)^{(r)}(x) - f^{(r)}(x)\right|\psi^{2\beta+2}(x) \\
&\quad\leqslant O\left(\frac{1}{n}\right)\left(\left\|f^{(r+1)}(x)\psi^{2\beta+1}(x)\right\|_{L_\infty([0,\infty))} + \left\|f^{(r+2)}(x)\psi^{2\beta}(x)\right\|_{L_\infty([0,\infty))}\right).
\end{aligned} \tag{2.34}$$

□

Lemma 2.3. *Let r and β be nonnegative integers and $n - r - 2 \geqslant 0$. Let $f \in C^{(r)}([0,\infty))$ satisfies*

$$\left\|f^{(r)}\psi^{2\beta}\right\|_{L_\infty([0,\infty))} < \infty. \tag{2.35}$$

Then one has uniformly for n, f and $x \in [0,\infty)$,

$$\left|\left(\widetilde{M}_n[f]\right)^{(r)}(x)\right|\psi^{2\beta}(x) \leqslant C\left\|f^{(r)}\psi^{2\beta}\right\|_{L_\infty([0,\infty))}. \tag{2.36}$$

Proof. Using $(1+y)^{2\beta} \leqslant C((y-x)^{2\beta}+(1+x)^{2\beta})$, we have

$$\left|(n-r-1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)(1+y)^{2\beta}dy\right| \leqslant C\Big(\varphi^{-2\beta}(x)+\left|T_{n,2\beta,r}(x)\right|\Big). \tag{2.37}$$

The assumption (2.35) means

$$\left|f^{(r)}(y)\right| \leqslant C(1+y)^{2\beta}. \tag{2.38}$$

Then we can obtain by (2.10),

$$\begin{aligned}
&\left|\left(\widetilde{M}_n[f]\right)^{(r)}(x)\right| \\
&\quad\leqslant C\left|(n-r-1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(y)(1+y)^{2\beta}dy\right|\left\|f^{(r)}\varphi^{2\beta}\right\|_{L_\infty([0,\infty))} \\
&\quad\leqslant C\Big(\varphi^{-2\beta}(x)+\left|T_{n,2\beta,r}(x)\right|\Big)\left\|f^{(r)}\varphi^{2\beta}\right\|_{L_\infty([0,\infty))} \\
&\quad\leqslant C\left(\varphi^{-2\beta}(x)+O\left(\frac{1}{n^\beta}\right)\left|q_{n,2\beta,r}(x)\right|\right)\left\|f^{(r)}\varphi^{2\beta}\right\|_{L_\infty([0,\infty))}.
\end{aligned} \tag{2.39}$$

Consequently, since $|q_{n,2\beta,r}(x)|\varphi^{2\beta}(x)$ is uniformly bounded on $[0,\infty)$, we have the result. □

The Steklov function $[f]_h(x)$ for $f \in C([0,\infty))$ is defined as follows:

$$[f]_h(x) := \frac{4}{h^2}\iint_0^{h/2}\left[2f(x+s+t)-f(x+2(s+t))\right]ds\,dt, \quad x \geqslant 0, h > 0. \tag{2.40}$$

Then for the Steklov function $[f]_h(x)$ with respect to $f \in C([0,\infty))$, we have the following properties.

Lemma 2.4 (cf.[4]). *Let $f(x) \in C([0,\infty))$ and $\eta(x)$ be a positive and nonincreasing function on $[0,\infty)$. Then (i) $[f]_h(x) \in C^2([0,\infty))$;*

(ii)

$$\left\|([f]_h(x)-f(x))\eta(x)\right\|_{L_\infty([0,\infty))} \leqslant \omega_2\left(f;\eta;\frac{h}{2}\right); \tag{2.41}$$

(iii)

$$\left\|[f]_h'(x)\eta(x)\right\|_{L_\infty([0,\infty))} \leqslant \frac{4}{h}\omega_1\left(f;\eta;\frac{h}{2}\right)\frac{\eta(x)}{\eta(x+(h/2))}+\frac{1}{h}\omega_1(f;\eta;h)\frac{\eta(x)}{\eta(x+h)}; \tag{2.42}$$

(iv)

$$\left\| [f]''_h(x)\eta(x) \right\|_{L_\infty([0,\infty))} \leqslant \frac{4}{h^2}\left[2\omega_2\left(f;\eta;\frac{h}{2}\right)+\frac{1}{4}\omega_2(f;\eta;h)\right]. \tag{2.43}$$

Proof. (i) For $f \in C([0,\infty))$, we have the Steklov functions $[f]'_h(x)$ and $[f]''_h(x)$ as follows. We note

$$[f]_h(x) = \frac{4}{h^2}\int_0^{h/2}\left(\int_x^{x+h/2} 2f(u+t)du - \int_x^{x+h}\frac{1}{2}f(u+2t)du\right)dt, \quad x \geqslant 0,\ h>0. \tag{2.44}$$

Then, we can see from (2.44),

$$\begin{aligned}[f]'_h(x) &= \frac{4}{h^2}\int_0^{h/2}\left[2\left(f\left(x+\frac{h}{2}+t\right)-f(x+t)\right)-\frac{1}{2}(f(x+h+2t)-f(x+2t))\right]dt \\ &= \frac{4}{h^2}\int_0^{h/2}\left[2\Delta^1_{h/2}f(x+t)-\frac{1}{2}\Delta^1_h f(x+2t)\right]dt.\end{aligned} \tag{2.45}$$

Similarly to (2.44), we know

$$[f]'_h(x) = \frac{4}{h^2}\left[2\int_x^{x+h/2}\left(f\left(u+\frac{h}{2}\right)-f(u)\right)du - \frac{1}{4}\int_x^{x+h}(f(u+h)-f(u))du\right]. \tag{2.46}$$

Therefore, we have from (2.46),

$$\begin{aligned}[f]''_h(x) &= \frac{4}{h^2}\left[2\left(f(x+h)-2f\left(x+\frac{h}{2}\right)+f(x)\right)-\frac{1}{4}(f(x+2h)-2f(x+h)+f(x))\right] \\ &= \frac{4}{h^2}\left[2\Delta^2_{h/2}f(x)-\frac{1}{4}\Delta^2_h f(x)\right].\end{aligned} \tag{2.47}$$

Therefore, (i) is proved.

(ii) We easily see from (2.44) that

$$\begin{aligned}\left|(f(x)-[f]_h(x))\eta(x)\right| &= \left|\frac{4}{h^2}\iint_0^{h/2}\Delta^2_{s+t}f(x)\eta(x)ds\,dt\right| \\ &\leqslant \omega_2\left(f;\eta;\frac{h}{2}\right).\end{aligned} \tag{2.48}$$

(iii) From (2.46), we have

$$\begin{aligned}\left|[f]_h'(x)\eta(x)\right| &\leqslant \left|\frac{4}{h^2}\int_0^{h/2} 2\left(\Delta_{h/2}^1 f(x+t)\eta(x+t)\right)\frac{\eta(x)}{\eta(x+t)}dt\right| \\ &\quad + \left|\frac{4}{h^2}\int_0^{h/2} \frac{1}{2}\left(\Delta_h^1 f(x+2t)\eta(x+2t)\right)\frac{\eta(x)}{\eta(x+2t)}dt\right| \\ &\leqslant \frac{4}{h}\omega_1\left(f;\eta;\frac{h}{2}\right)\frac{\eta(x)}{\eta(x+h/2)} + \frac{1}{h}\omega_1(f;\eta;h)\frac{\eta(x)}{\eta(x+h)}.\end{aligned} \tag{2.49}$$

(iv) From (2.47), we have

$$\left|[f]_h''(x)\eta(x)\right| \leqslant \frac{4}{h^2}\left[2\omega_2\left(f;\eta;\frac{h}{2}\right) + \frac{1}{4}\omega_2(f;\eta;h)\right]. \tag{2.50}$$

□

Proof of Theorem 1.5. We know that for $f(x) \in C^r([0,\infty))$,

$$[f]_h^{(r)}(x) = \left[f^{(r)}\right]_h(x), \qquad [f]_h^{(r+1)}(x) = \left[f^{(r)}\right]_h'(x), \qquad [f]_h^{(r+2)}(x) = \left[f^{(r)}\right]_h''(x). \tag{2.51}$$

Then, we have

$$\begin{aligned}&\left\|\left(\left(\widetilde{M}_n[f]\right)^{(r)}(x) - f^{(r)}(x)\right)\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))} \\ &\quad\leqslant \left\|\left(\widetilde{M}_n[f-[f]_h]\right)^{(r)}(x)\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))} \\ &\qquad + \left\|\left(\left(\widetilde{M}_n[[f]_h]\right)^{(r)}(x) - [f^{(r)}]_h(x)\right)\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))} \\ &\qquad + \left\|\left[[f^{(r)}]_h(x) - f^{(r)}(x)\right]\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))}.\end{aligned} \tag{2.52}$$

From (2.51) and (2.41) of Lemma 2.4,

$$\begin{aligned}\left\|\left(\widetilde{M}_n[f-[f]_h]\right)^{(r)}(x)\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))} &\leqslant \left\|\left[f^{(r)}(x) - [f]_h^{(r)}(x)\right]\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))} \\ &= \left\|\left[f^{(r)}(x) - \left[f^{(r)}\right]_h(x)\right]\varphi^{2\beta+2}(x)\right\|_{L_\infty([0,\infty))} \\ &\leqslant \omega_2\left(f^{(r)};\varphi^{2\beta+2};h\right).\end{aligned} \tag{2.53}$$

Here, we suppose $0 < h \leqslant 1$ and then we know that

$$\frac{\varphi(x)}{\varphi(x+h)} \leqslant 2, \qquad \frac{\varphi(x)}{\varphi(x+h/2)} \leqslant 2. \tag{2.54}$$

From Theorem 1.4, (2.51), (2.42), and (2.43) of Lemma 2.4, we have

$$\begin{aligned} &\left\| \left(\left(\widetilde{M}_n [[f]_h] \right)^{(r)}(x) - [f]_h^{(r)}(x) \right) \varphi^{2\beta+2}(x) \right\|_{L_\infty([0,\infty))} \\ &\quad \leqslant O\left(\frac{1}{n}\right) \left(\left\| [f]_h^{(r+1)}(x) \varphi^{2\beta+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| [f]_h^{(r+2)}(x) \varphi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} \right) \\ &\quad \leqslant O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1\left(f^{(r)}; \varphi^{2\beta+1}; h \right) + \frac{1}{h^2} \omega_2\left(f^{(r)}; \varphi^{2\beta}; h \right) \right). \end{aligned} \tag{2.55}$$

Therefore, we have

$$\begin{aligned} &\left\| \left((\widetilde{M}_n [f])^{(r)}(x) - f^{(r)}(x) \right) \varphi^{2\beta+2}(x) \right\|_{L_\infty([0,\infty))} \\ &\quad \leqslant O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1\left(f^{(r)}; \varphi^{2\beta+1}; h \right) + \frac{1}{h^2} \omega_2\left(f^{(r)}; \varphi^{2\beta}; h \right) \right) + \omega_2\left(f^{(r)}; \varphi^{2\beta+2}; h \right). \end{aligned} \tag{2.56}$$

If we let $h = 1/\sqrt{n}$, then

$$\begin{aligned} &\left\| \left(\left(\widetilde{M}_n [f] \right)^{(r)}(x) - f^{(r)}(x) \right) \varphi^{2\beta+2}(x) \right\|_{L_\infty([0,\infty))} \\ &\quad \leqslant C\left(\frac{1}{\sqrt{n}} \omega_1\left(f^{(r)}; \varphi^{2\beta+1}; \frac{1}{\sqrt{n}} \right) + \omega_2\left(f^{(r)}; \varphi^{2\beta}; \frac{1}{\sqrt{n}} \right) \right), \end{aligned} \tag{2.57}$$

because $\omega_2(f^{(r)}; \varphi^{2\beta+2}; 1/\sqrt{n}\,) \leqslant \omega_2(f^{(r)}; \varphi^{2\beta}; 1/\sqrt{n})$. □

From now on, we will prove Theorems 1.7, 1.9, and 1.10, which are the results for the Szász-Mirakyan operators, analogously to the case of Lupas-type operators.

Lemma 2.5. *Let r be a nonnegative integer. Then one has for $f \in C^r([0,\infty))$,*

$$(Q_{n,\beta}[f])^{(r)}(x) = (n+\beta)\left(\frac{n}{n+\beta}\right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^\infty S_{n+\beta,k+r}(y) f^{(r)}(y)\,dy, \quad x \in [0,\infty). \tag{2.58}$$

Proof. We know that

$$S_{n,k}^{(r)}(x) = \sum_{i=0}^{r} \binom{r}{i} \frac{(e^{-ny})^{(r-i)}\left((ny)^k\right)^{(i)}}{k!} = \sum_{i=0}^{r} \binom{r}{i} (-1)^r (-1)^i n^r S_{n,k-i}(x),$$
$$S_{n,k}^{(r)}(x) = \sum_{i=0}^{r} \binom{r}{i} \frac{(e^{-ny})^{(i)}\left((ny)^k\right)^{(r-i)}}{k!} = \sum_{i=0}^{r} \binom{r}{i} (-1)^i n^r S_{n,k-r+i}(x). \tag{2.59}$$

Therefore, we have

$$\begin{aligned}
&(Q_{n,\beta}[f])^{(r)}(x) \\
&\quad= (n+\beta)\sum_{k=0}^{\infty} S_{n,k}^{(r)}(x) \int_0^\infty S_{n+\beta,k}(y) f(y)\,dy \\
&\quad= (n+\beta)\sum_{i=0}^{r}\sum_{k=i}^{\infty} \binom{r}{i} (-1)^r(-1)^i n^r S_{n,k-i}(x) \int_0^\infty S_{n+\beta,k}(y) f(y)\,dy \\
&\quad= (n+\beta)\left(\frac{n}{n+\beta}\right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^\infty \sum_{i=0}^{r} \binom{r}{i} (-1)^r(-1)^i (n+\beta)^r S_{n+\beta,k+i}(y) f(y)\,dy \\
&\quad= (n+\beta)\left(\frac{n}{n+\beta}\right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^\infty (-1)^r S_{n+\beta,k+r}^{(r)}(y) f(y)\,dy \\
&\quad= (n+\beta)\left(\frac{n}{n+\beta}\right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^\infty S_{n+\beta,k+r}(y) f^{(r)}(y)\,dy.
\end{aligned} \tag{2.60}$$

□

Lemma 2.6. *Let a, b, and m be nonnegative integers.*

$$R_{n,m,r}(a,b;x) := (n+b)\sum_{k=0}^{\infty} S_{n+a,k}(x) \int_0^\infty S_{n+b,k+r}(y)(y-x)^m\,dy. \tag{2.61}$$

Then one has

(i) $R_{n,0,r}(a,b;x) = 1$ *and* $R_{n,1,r}(a,b;x) = ((a-b)x + r + 1)/(n+b)$;

(ii) *For* $m \geqslant 1$

$$\begin{aligned}
&(n+b)R_{n,m+1,r}(a,b;x) \\
&\quad= xR_{n,m,r}^{(1)}(a,b;x) + ((a-b)x + m + r + 1)R_{n,m,r}(a,b;x) + 2xmR_{n,m-1,r}(a,b;x);
\end{aligned} \tag{2.62}$$

(iii)

$$R_{n,m,r}(a,b;x)=O\left(\frac{1}{n^{[(m+1)/2]}}\right)g_{n,m,r}(a,b;x), \tag{2.63}$$

where $g_{n,m,r}(a,b;x)$ *is a polynomial of degree* $\leqslant m$ *such that the coefficients of* $g_{n,m,r}(a,b;x)$ *are bounded independently of* n*.*

Proof. Let $R_{n,m,r}(x):=R_{n,m,r}(a,b;x)$. Then (i)

$$\begin{aligned}
R_{n,0,r}(x)&=(n+b)\sum_{k=0}^{\infty}S_{n+a,k}(x)\int_0^{\infty}S_{n+b,k+r}(y)dy=1,\\
R_{n,1,r}(x)&=(n+b)\sum_{k=0}^{\infty}S_{n+a,k}(x)\int_0^{\infty}S_{n+b,k+r}(y)(y-x)dy\\
&=\sum_{k=0}^{\infty}S_{n+a,k}(x)\frac{k+r+1}{n+b}-x\\
&=\sum_{k=0}^{\infty}S_{n+a,k}(x)\frac{k}{n+b}+\frac{r+1}{n+b}-x\\
&=\frac{(n+a)x}{n+b}+\frac{r+1}{n+b}-x=\frac{(a-b)x+r+1}{n+b}.
\end{aligned} \tag{2.64}$$

(ii) Using $xS^{(1)}_{n,k}(x)=(k-nx)S_{n,k}(x)$, we obtain

$$\begin{aligned}
&x\left(R^{(1)}_{n,m,r}(x)+mR_{n,m-1,r}(x)\right)\\
&\quad=(n+b)\sum_{k=0}^{\infty}xS^{(1)}_{n+a,k}(x)\int_0^{\infty}S_{n+b,k+r}(y)(y-x)^m dy\\
&\quad=(n+b)\sum_{k=0}^{\infty}S_{n,k}(x)\int_0^{\infty}(k-(n+a)x)S_{n+b,k+r}(y)(y-x)^m dy.
\end{aligned} \tag{2.65}$$

Here, we see

$$\begin{aligned}
&(k-(n+a)x)S_{n+b,k+r}(y)\\
&\quad=((k+r-(n+b)y)-(r+(a-b)x)+(n+b)(y-x))S_{n+b,k+r}(y)\\
&\quad=yS^{(1)}_{n+b,k+r}(y)-(r+(a-b)x)S_{n+b,k+r}(y)+(n+b)S_{n+b,k+r}(y)(y-x).
\end{aligned} \tag{2.66}$$

Then substituting (2.66) for (2.65), we consider the following;

$$\begin{aligned}
&x\left(R_{n,m,r}^{(1)}(x)+mR_{n,m-1,r}(x)\right)\\
&\quad=(n+b)\sum_{k=0}^{\infty}S_{n,k}(x)\int_0^{\infty}\left(yS_{n+b,k+r}^{(1)}(y)-(r+(a-b)x)S_{n+b,k+r}(y)\right.\\
&\qquad\left.+(n+b)S_{n+b,k+r}(y)(y-x)\right)(y-x)^m dy\\
&\quad:=\int_1+\int_2+\int_3 .
\end{aligned} \tag{2.67}$$

Then, we have

$$\begin{aligned}
\int_1 &= (n+b)\sum_{k=0}^{\infty}S_{n,k}(x)\int_0^{\infty}yS_{n+b,k+r}^{(1)}(y)(y-x)^m dy\\
&=(n+b)\sum_{k=0}^{\infty}S_{n+a,k}(x)\int_0^{\infty}S_{n+b,k+r}^{(1)}(y)(y-x)^{m+1}dy\\
&\quad+x(n+b)\sum_{k=0}^{\infty}S_{n+a,k}(x)\int_0^{\infty}S_{n+b,k+r}^{(1)}(y)(y-x)^m dy\\
&=-(m+1)R_{n,m,r}(x)-xmR_{n,m-1,r}(x).
\end{aligned} \tag{2.68}$$

Here the last equation follows by parts of integration. Furthermore, we have

$$\int_1+\int_2=-(r+(a-b)x)R_{n,m,r}(x)+(n+b)R_{n,m+1,r}(x). \tag{2.69}$$

Therefore, we have

$$(n+b)R_{n,m+1,r}(x)=xR_{n,m,r}^{(1)}(x)+((a-b)x+m+r+1)R_{n,m,r}(x)+2xmR_{n,m-1,r}(x). \tag{2.70}$$

(iii) It is proved by the same method as the proof of Lemma 2.1 (iv). □

Proof of Theorem 1.7. Let $|y-x|\leqslant 1$. By the second inequality in (1.30),

$$\begin{aligned}
\left|f^{(r)}(y)-f^{(r)}(x)\right| &= |y-x|\left|f^{(r+1)}(\xi)\right|\\
&\leqslant C|y-x|\left|e^{\beta\xi}\xi^{\alpha+2}\right|\leqslant C|y-x|e^{\beta x}(x+1)^{\alpha+2}.
\end{aligned} \tag{2.71}$$

Let $\varepsilon = n^{-\gamma}$, $0 < \gamma < 1$,

$$\begin{aligned}
&\left|\left(\frac{n+\beta}{n}\right)^{r}(Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x)\right| \\
&\quad= \left|(n+\beta)\sum_{k=0}^{\infty}S_{n,k}(x)\left(\int_{|y-x|<\varepsilon} S_{n+\beta,k+r}(y)\left|f^{(r)}(y) - f^{(r)}(x)\right|dy\right.\right. \\
&\qquad\qquad\left.\left.+\int_{|y-x|\geqslant\varepsilon} S_{n+\beta,k+r}(y)\left|f^{(r)}(y) - f^{(r)}(x)\right|dy\right)\right| \\
&\quad=: A + B.
\end{aligned} \tag{2.72}$$

First, we see that by (2.71) and Lemma 2.6(i),

$$\begin{aligned}
A &\leqslant C(n+\beta)\sum_{k=0}^{\infty}S_{n,k}(x)\int_{|y-x|<\varepsilon} S_{n+\beta,k+r}(y)|y-x|dy\; e^{\beta x}(x+1)^{\alpha+2} \\
&\leqslant C\varepsilon e^{\beta x}(x+1)^{\alpha+2} \leqslant O\left(\frac{1}{n^{\gamma}}\right)e^{\beta x}(x+1)^{\alpha+2}.
\end{aligned} \tag{2.73}$$

Next, to estimate B, we split it into two parts:

$$\begin{aligned}
B &= \left|(n+\beta)\sum_{k=0}^{\infty}S_{n,k}(x)\int_{|y-x|\geqslant\varepsilon} S_{n+\beta,k+r}(y)\left|f^{(r)}(y) - f^{(r)}(x)\right|dy\right| \\
&\leqslant (n+\beta)\sum_{k=0}^{\infty}S_{n,k}(x)\int_{|y-x|\geqslant\varepsilon} S_{n+\beta,k+r}(y)\left(\left|f^{(r)}(y)\right| + \left|f^{(r)}(x)\right|\right)dy \\
&\leqslant (n+\beta)\sum_{k=0}^{\infty}S_{n,k}(x)\int_{|y-x|\geqslant\varepsilon} S_{n+\beta,k+r}(y)\left(e^{\beta y}(y+1)^{\alpha} + e^{\beta x}(x+1)^{\alpha}\right)dy \\
&=: B_1 + B_2.
\end{aligned} \tag{2.74}$$

First, we estimate

$$B_1 = (n+\beta)\sum_{k=0}^{\infty}S_{n,k}(x)\int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y)e^{\beta y}(y+1)^{\alpha}dy. \tag{2.75}$$

Then, using the following facts:

$$S_{n+\beta,k+r}(y)e^{\beta y} = S_{n,k+r}(y)\left(\frac{n+\beta}{n}\right)^{k+r}, \tag{2.76}$$

$$\left(\frac{n+\beta}{n}\right)^{k+r} S_{n,k}(x) = \left(\frac{n+\beta}{n}\right)^{r} S_{n+\beta,k}(x)e^{\beta x}, \tag{2.77}$$

$$(y+1)^{\alpha} = ((y-x)+x+1)^{\alpha} = \sum_{i=0}^{\alpha}\binom{\alpha}{i}(y-x)^{i}(x+1)^{\alpha-i}, \tag{2.78}$$

we have

$$\begin{aligned}
B_1 &\leqslant Ce^{\beta x}(n+\beta)\left(\frac{n+\beta}{n}\right)^{r}\sum_{k=0}^{\infty}S_{n+\beta,k}(x)\int_{|y-x|>\varepsilon}S_{n,k+r}(y)(y+1)^{\alpha}dy \\
&= Ce^{\beta x}n\left(\frac{n+\beta}{n}\right)^{r+1}\sum_{k=0}^{\infty}S_{n+\beta,k}(x) \\
&\quad\times\int_{|y-x|>\varepsilon}S_{n,k+r}(y)\sum_{i=1}^{\alpha}\binom{\alpha}{i}(y-\mathrm{x})^{i}(x+1)^{\alpha-i}dy \\
&\quad+Ce^{\beta x}n\left(\frac{n+\beta}{n}\right)^{r}\sum_{k=0}^{\infty}S_{n+\beta,k}(x)\int_{|y-x|>\varepsilon}S_{n,k+r}(y)(x+1)^{\alpha}dy \\
&=: Ce^{\beta x}(B_{11}+B_{12}).
\end{aligned} \tag{2.79}$$

Then, using (2.18) and Lemma 2.6, we have

$$\begin{aligned}
B_{11} &= n\sum_{k=0}^{\infty}S_{n+\beta,k}(x)\int_{|y-x|>\varepsilon}S_{n,k+r}(y)\sum_{i=1}^{\alpha}\binom{\alpha}{i}(y-x)^{i}(x+1)^{\alpha-i}dy \\
&\leqslant n\sum_{k=0}^{\infty}S_{n+\beta,k}(x)\int_{|y-x|>\varepsilon}S_{n,k+r}(y)\sum_{i=1}^{\alpha}\binom{\alpha}{i}(y-x)^{i}\left|\frac{y-x}{\varepsilon}\right|^{\langle i\rangle}(x+1)^{\alpha-i}dy \\
&\leqslant n\sum_{k=0}^{\infty}S_{n+\beta,k}(x)\int_{|y-x|>\varepsilon}S_{n,k+r}(y)\sum_{i=1}^{\alpha}\binom{\alpha}{i}(y-x)^{i+\langle i\rangle}\left(\frac{1}{\varepsilon}\right)^{\langle i\rangle}(x+1)^{\alpha-i}dy \\
&\leqslant \sum_{i=1}^{\alpha}\binom{\alpha}{i}R_{n,i+\langle i\rangle r}(\beta,0;x)\left(\frac{1}{\varepsilon}\right)^{\langle i\rangle}(x+1)^{\alpha-i} \\
&= \sum_{i=1}^{\alpha}\binom{\alpha}{i}O\left(\frac{\langle i\rangle}{n^{[(i+\langle i\rangle+1)/2]}}\right)q_{n,i+\langle i\rangle,r}(\beta,0;x)(x+1)^{\alpha-i}.
\end{aligned} \tag{2.80}$$

Then by (2.21) we have

$$B_{11} \leqslant O\left(\frac{1}{n^{1-\gamma}}\right)(x+1)^{\alpha+1}. \tag{2.81}$$

For B_{12}, we have

$$\begin{aligned}
B_{12} &= n\sum_{k=0}^{\infty} S_{n+\beta,k}(x)\int_{|y-x|>\varepsilon} S_{n,k+r}(y)(x+1)^{\alpha}dy \\
&\leqslant n\sum_{k=0}^{\infty} S_{n+\beta,k}(x)\int_{|y-x|>\varepsilon} S_{n,k+r}(y)\left(\frac{y-x}{\varepsilon}\right)^{2}dy(x+1)^{\alpha} \\
&\leqslant R_{n,2,r}(\beta,0;x)\left(\frac{1}{\varepsilon}\right)^{2}(x+1)^{\alpha} = O\left(\frac{n^{2\gamma}}{n^{[3/2]}}\right)q_{n,2,r}(\beta,0;x)(x+1)^{\alpha} \\
&= O\left(\frac{1}{n^{1-2\gamma}}\right)(x+1)^{\alpha+2}.
\end{aligned} \tag{2.82}$$

From (2.81), (2.82) and (2.79), we have

$$B_1 \leqslant O\left(\frac{1}{n^{1-\gamma}}\right)e^{\beta x}(x+1)^{\alpha+2}. \tag{2.83}$$

We estimate

$$B_2 = (n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y)e^{\beta x}(x+1)^{\alpha}dy. \tag{2.84}$$

Then we can estimate B_2 by the same method as B_{12},

$$\begin{aligned}
B_2 &= (n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y)e^{\beta x}(x+1)^{\alpha}dy \\
&\leqslant (n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y)\left(\frac{y-x}{\varepsilon}\right)^{2}dy\ e^{\beta x}(x+1)^{\alpha} \\
&\leqslant R_{n,2,r}(0,\beta;x)\left(\frac{1}{\varepsilon}\right)^{2}e^{\beta x}(x+1)^{\alpha},
\end{aligned} \tag{2.85}$$

so we have

$$B_2 \leqslant O\left(\frac{n^{2\gamma}}{n^{[3/2]}}\right)q_{n,2,r}(0,\beta;x)e^{\beta x}(x+1)^{\alpha} \leqslant O\left(\frac{1}{n^{1-2\gamma}}\right)e^{\beta x}(x+1)^{\alpha+2}. \tag{2.86}$$

Consequently, we obtain from (2.83) and (2.86),

$$B \leqslant \left(O\left(\frac{1}{n^{1-\gamma}}\right)+O\left(\frac{1}{n^{1-2\gamma}}\right)\right)e^{\beta x}(x+1)^{\alpha+2} \leqslant O\left(\frac{1}{n^{1-2\gamma}}\right)e^{\beta x}(x+1)^{\alpha+2}. \tag{2.87}$$

Therefore, from (2.73) and (2.87), we have uniformly on n,

$$\left|\left(\frac{n+\beta}{n}\right)^{r}(Q_{n,\beta}[f])^{(r)}(x)-f^{(r)}(x)\right|\leqslant O\left(\frac{1}{n^{\gamma}}\right)e^{\beta x}x^{\alpha+2}+O\left(\frac{1}{n^{1-2\gamma}}\right)e^{\beta x}x^{\alpha+2}. \tag{2.88}$$

Here, if we let $\gamma=1/3$, then we have

$$\left|\left(\frac{n+\beta}{n}\right)^{r}(Q_{n,\beta}[f])^{(r)}(x)-f^{(r)}(x)\right|=O\left(\frac{1}{n^{1/3}}\right)e^{\beta x}x^{\alpha+2}, \tag{2.89}$$

that is, (1.31) is proved. So, we also have a norm convergence (1.32). □

Lemma 2.7. *Let m and b be nonnegative integers. Let*

$$U_{n,m,r}(b;x):=(n+b)\sum_{k=0}^{\infty}S_{n,k}(x)\int_{0}^{\infty}S_{n+b,k+r}(y)(y-x)^{m}e^{by}dy. \tag{2.90}$$

Then one has

$$U_{n,m,r}(b;x)=\left(\frac{n+b}{n}\right)^{r+1}e^{bx}R_{n,m,r}(b,0;x). \tag{2.91}$$

Proof. From (2.76) and (2.77) we have

$$S_{n+b,k+r}(x)e^{bx}=S_{n,k+r}(x)\left(\frac{n+b}{n}\right)^{k+r}, \tag{2.92}$$

$$\left(\frac{n+b}{n}\right)^{k+r}S_{n,k}(x)=\left(\frac{n+b}{n}\right)^{r}S_{n+b,k}(x)e^{bx}. \tag{2.93}$$

We have from (2.92), (2.93), and noting (2.61),

$$\begin{aligned}U_{n,m,r}(b;x)&=(n+b)\sum_{k=0}^{\infty}S_{n,k}(x)\int_{0}^{\infty}S_{n+b,k+r}(y)(y-x)^{m}e^{by}dy\\&=(n+b)\sum_{k=0}^{\infty}\left(\frac{n+b}{n}\right)^{k+r}S_{n,k}(x)\int_{0}^{\infty}S_{n,k+r}(y)(y-x)^{m}dy\\&=(n+b)\left(\frac{n+b}{n}\right)^{r}e^{bx}\sum_{k=0}^{\infty}S_{n+b,k}(x)\int_{0}^{\infty}S_{n,k+r}(y)(y-x)^{m}dy\\&=\left(\frac{n+b}{n}\right)^{r+1}e^{bx}R_{n,m,r}(b,0;x).\end{aligned} \tag{2.94}$$

□

Proof of Theorem 1.9. We prove this theorem, similarly to the proof of Theorem 1.4. Using (2.93) and (2.27), we have for $f \in C^{(r+2)}([0,\infty))$,

$$\begin{aligned}
&\left(\frac{n+\beta}{n}\right)^{r}\left(Q_{n,\beta}[f]\right)^{(r)}(x) \\
&\quad= (n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y) f^{(r)}(y)\,dy \\
&\quad= f^{(r)}(x) + f^{(r+1)}(x)(n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)(y-x)\,dy \\
&\qquad + (n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)\left(\int_x^y (y-u) f^{(r+2)}(u)\,du\right)dy \\
&\quad= f^{(r)}(x) + f^{(r+1)}(x) R_{n,1,r}(0,\beta;x) \\
&\qquad + (n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)\left(\int_x^y (y-u) f^{(r+2)}(u)\,du\right)dy.
\end{aligned} \tag{2.95}$$

We estimate the last term. We note the given condition:

$$\left\| f^{(r+2)}(x) e^{-\beta x} \varphi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} < \infty. \tag{2.96}$$

Using the inequality $(1+t)^{2\gamma} \leqslant C((1+x)^{2\gamma} + (t-x)^{2\gamma})$ and Lemma 2.7, we have

$$\begin{aligned}
\sum &:= (n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)\left(e^{\beta y}\varphi^{-2\gamma}(y) + e^{\beta x}\varphi^{-2\gamma}(x)\right)(y-x)^2\,dy \\
&\leqslant C(n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y) \\
&\qquad \times \left(e^{\beta y}\varphi^{-2\gamma}(x) + e^{\beta y}\left((y-x)^{2\gamma} + e^{\beta x}\varphi^{-2\gamma}(x)\right)\right)(y-x)^2\,dy \\
&\leqslant C\left(\varphi^{-2\gamma}(x) U_{n,2,r}(\beta;x) + U_{n,2\gamma+2,r}(\beta;x) + e^{\beta x}\varphi^{-2\gamma}(x) R_{n,2,r}(0,\beta;x)\right) \\
&\leqslant C\Bigg(e^{\beta x}\varphi^{-2\gamma}(x) R_{n,2,r}(\beta,0;x)\left(\frac{n+\beta}{n}\right)^{r+1} \\
&\qquad + e^{\beta x} R_{n,2\gamma+2,r}(\beta,0;x)\left(\frac{n+\beta}{n}\right)^{r+1} + e^{\beta x}\varphi^{-2\gamma}(x) R_{n,2,r}(0,\beta;x)\Bigg).
\end{aligned} \tag{2.97}$$

Then, we can estimate as follows:

$$\left|(n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)\left(\int_x^y (y-u) f^{(r+2)}(u)\,du\right)dy\right|$$

$$\begin{aligned}
&\leqslant \left\| f^{(r+2)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))} \sum \\
&\leqslant C\left\| f^{(r+2)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))} e^{\beta x} \\
&\quad \times \left(\varphi^{-2\gamma}(x) R_{n,2,r}(\beta,0;x)\left(\frac{n+\beta}{n}\right)^{r+1} + R_{n,2\gamma+2,r}(\beta,0;x)\left(\frac{n+\beta}{n}\right)^{r+1} \right. \\
&\qquad \left. + \varphi^{-2\gamma}(x) R_{n,2,r}(0,\beta;x) \right).
\end{aligned} \tag{2.98}$$

Then, we have by (iv) of Lemma 2.6,

$$\begin{aligned}
&\left| \left(\frac{n+\beta}{n}\right)^{r} (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| e^{-\beta x}\varphi^{2\gamma+2}(x) \\
&\quad \leqslant \left\| f^{(r+2)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))} \\
&\qquad \times \left(O\left(\frac{1}{n}\right)\varphi^{2}(x) g_{n,2,r}(\beta,0;x) + O\left(\frac{1}{n^{\gamma}}\right) g_{n,2\gamma+2,r}(\beta,0;x)\varphi^{2\gamma+2}(x) \right. \\
&\qquad\quad \left. + O\left(\frac{1}{n}\right)\varphi^{2}(x) g_{n,2,r}(0,\beta;x) \right) \\
&\qquad + O\left(\frac{1}{n}\right) \left| f^{(r+1)}(x)e^{-\beta x}\varphi^{2\gamma+1}(x) g_{n,1,r}(0,\beta;x)\varphi(x) \right|.
\end{aligned} \tag{2.99}$$

Consequently, we have

$$\begin{aligned}
&\left| \left[\left(\frac{n+\beta}{n}\right)^{r} (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x}\varphi^{2\gamma+2}(x) \right| \\
&\quad \leqslant O\left(\frac{1}{n}\right)\left(\left\| f^{(r+1)}(x)e^{-\beta x}\varphi^{2\gamma+1}(x)\right\|_{L_\infty([0,\infty))} + \left\| f^{(r+2)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))} \right),
\end{aligned} \tag{2.100}$$

since we know that $|g_{n,2,r}(\beta,0;x)\varphi^2(x)|$, $|g_{n,2,r}(0,\beta;x)\varphi^2(x)|$, $|g_{n,2\gamma+2,r}(\beta,0;x)\varphi^{2\gamma+2}(x)|$, and $|g_{n,1,r}(0,\beta;x)\varphi(x)|$ are uniformly bounded on $[0,\infty)$. □

Theorem 2.8. *Let β and γ be nonnegative integers and $r>0$ be a positive integer. Then one has uniformly for n, f and $x\in[0,\infty)$,*

$$\left| \left[(Q_{n,\beta}[f])^{(r)}(x) \right] e^{-\beta x}\varphi^{2\gamma}(x) \right| \leqslant C\frac{n+\beta}{n} \left\| f^{(r)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right\|_{L_\infty([0,\infty))}. \tag{2.101}$$

Proof. Using $(1+y)^{2\gamma} \leqslant C((1+x)^{2\gamma} + (y-x)^{2\gamma})$, we have

$$\begin{aligned}
&\left|(n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)e^{\beta y}(1+y)^{2\gamma}dy\right| \\
&\quad\leqslant \left|(n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)e^{\beta y}\Big((1+x)^{2\gamma}+(y-x)^{2\gamma}\Big)dy\right| \\
&\quad\leqslant C\Big(U_{n,0,r}(\beta;x)\varphi^{-2\gamma}(x) + U_{n,2\gamma,r}(\beta;x)\Big).
\end{aligned} \tag{2.102}$$

By Lemma 2.7 and (i) of Lemma 2.6, we know

$$\begin{aligned}
U_{n,0,r}(\beta;x) &= \left(\frac{n+\beta}{n}\right)^{r+1} e^{\beta x} R_{n,0,r}(\beta,0;x) = \left(\frac{n+\beta}{n}\right)^{r+1} e^{\beta x}, \\
U_{n,2\gamma,r}(\beta;x) &= \left(\frac{n+\beta}{n}\right)^{r+1} e^{\beta x} R_{n,2\gamma,r}(\beta,2\gamma;x) \\
&= O\left(\frac{1}{n^{\gamma}}\right)\left(\frac{n+\beta}{n}\right)^{r+1} e^{\beta x} g_{n,2\gamma,r}(\beta,2\gamma;x).
\end{aligned} \tag{2.103}$$

Therefore, we have

$$\begin{aligned}
&\left|(n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)e^{\beta y}(1+y)^{2\gamma}dy\right| \\
&\quad\leqslant C\left(\frac{n+\beta}{n}\right)^{r+1} e^{\beta x}\left(\varphi^{-2\gamma}(x) + O\left(\frac{1}{n^{\gamma}}\right)g_{n,2\gamma,r}(\beta,2\gamma;x)\right).
\end{aligned} \tag{2.104}$$

Since $|g_{n,2\gamma,r}(\beta,2\gamma;x)\varphi^{2\gamma}(x)|$ is uniformly bounded on $[0,\infty)$, we have

$$\begin{aligned}
&\left|\left(\frac{n+\beta}{n}\right)^{r}(Q_{n,\beta}[f])^{(r)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right| \\
&\quad\leqslant \left|(n+\beta)\sum_{k=0}^{\infty} S_{n,k}(x)\int_0^{\infty} S_{n+\beta,k+r}(y)e^{\beta y}(1+y)^{2\gamma}dy\right| e^{-\beta x}\varphi^{2\gamma}(x) \\
&\qquad\times \left\|f^{(r)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right\|_{L_{\infty}([0,\infty))} \\
&\quad\leqslant C\left(\frac{n+\beta}{n}\right)^{r+1}\left(1+O\left(\frac{1}{n^{\gamma}}\right)\right)\left\|f^{(r)}(x)e^{-\beta x}\varphi^{2\gamma}(x)\right\|_{L_{\infty}([0,\infty))}.
\end{aligned} \tag{2.105}$$

Therefore, we have the result. □

Proof of Theorem 1.10. We will prove this theorem by the same method as the proof of Theorem 1.5. First, we split it as follows:

$$\begin{aligned}
&\left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
&\quad \leqslant \left\| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f - [f]_h])^{(r)}(x) e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
&\quad + \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[[f]_h])^{(r)}(x) - [f]_h^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
&\quad + \left\| \left[[f]_h^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))}.
\end{aligned} \tag{2.106}$$

Then for the first term, we have, using Theorem 2.8 and (2.41),

$$\begin{aligned}
&\left\| (Q_{n,\beta}[f - [f]_h])^{(r)}(x) e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
&\quad \leqslant C \frac{n+\beta}{n} \left\| \left[f^{(r)}(x) - [f^{(r)}]_h(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \leqslant C\omega_2\left(f^{(r)}; e^{-\beta x} \psi^{2\gamma+2}(x); h \right).
\end{aligned} \tag{2.107}$$

For the second term, we have from Theorem 1.9,

$$\begin{aligned}
&\left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f_h])^{(r)}(x) - f_h^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
&\quad \leqslant O\left(\frac{1}{n} \right) \left(\left\| f_h^{(r+1)}(x) e^{-\beta x} \psi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| f_h^{(r+2)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \right).
\end{aligned} \tag{2.108}$$

Here, we suppose $0 < h \leqslant 1$ and then we know that $(e^{-\beta x}\psi(x))/(e^{-\beta(x+h)}\psi(x+h))$ and $e^{-\beta x}/e^{-\beta(x+h)}$ are uniformly bounded on $[0,\infty)$. Therefore, we have from (2.42) and (2.43) of Lemma 2.4,

$$\begin{aligned}
\left\| f_h^{(r+1)}(x) e^{-\beta x} \psi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} &\leqslant C \frac{1}{h} \omega_1\left(f; e^{-\beta x} \psi^{2\gamma+1}(x); h \right), \\
\left\| f_h^{(r+2)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} &\leqslant C \frac{1}{h^2} \omega_2\left(f; e^{-\beta x} \psi^{2\gamma}(x); h \right).
\end{aligned} \tag{2.109}$$

Therefore, we have

$$\begin{aligned}
&\left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f_h])^{(r)}(x) - f_h^{(r)}(x) \right] e^{-\beta x} \psi^{2}(x) \right\|_{L_\infty([0,\infty))} \\
&\quad \leqslant O\left(\frac{1}{n} \right) \left(\frac{1}{h} \omega_1\left(f; e^{-\beta x} \psi^{2+1}(x); h \right) + \frac{1}{h^2} \omega_2\left(f; e^{-\beta x} \psi^{2\gamma}(x); h \right) \right).
\end{aligned} \tag{2.110}$$

Consequently, we have

$$\begin{aligned}&\left\|\left[\left(\frac{n+\beta}{n}\right)^r (Q_{n,\beta}[f])^{(r)}(x)-f^{(r)}(x)\right]e^{-\beta x}\psi^{2\gamma+2}(x)\right\|\\&\quad\leqslant O\left(\frac{1}{n}\right)\left(\frac{1}{h}\omega_1\left(f;e^{-\beta x}\psi^{2\gamma+1}(x);h\right)+\frac{1}{h^2}\omega_2\left(f;e^{-\beta x}\psi^{2\gamma}(x);h\right)\right)\\&\qquad+C\omega_2\left(f^{(r)};e^{-\beta x}\psi^{2\gamma+2}(x)(x);h\right).\end{aligned}\tag{2.111}$$

If we let $h=1/\sqrt{n}$, then

$$\begin{aligned}&\left\|\left[\left(\frac{n+\beta}{n}\right)^r (Q_{n,\beta}[f])^{(r)}(x)-f^{(r)}(x)\right]e^{-\beta x}\psi^{2\gamma+2}(x)\right\|\\&\quad\leqslant C\left(\frac{1}{\sqrt{n}}\omega_1\left(f;e^{-\beta x}\psi^{2\gamma+1}(x);\frac{1}{\sqrt{n}}\right)+\omega_2\left(f;e^{-\beta x}\psi^{2\gamma}(x);\frac{1}{\sqrt{n}}\right)\right),\end{aligned}\tag{2.112}$$

since $\omega_2(f^{(r)};e^{-\beta x}\psi^{2\gamma+2}(x);1/\sqrt{n})\leqslant\omega_2(f;e^{-\beta x}\psi^{2\gamma}(x);1/\sqrt{n})$. □

3. Conclusion

In this paper, Lupas-type operators and Szász-Mirakyan-type operators are treated and the various weighted norm convergence on $[0,\infty)$ of these operators are investigated. Moreover, this paper proves theorems on degree of approximation of $f\in C^r([0,\infty))$ by these operators using the modulus of smoothness of f.

Acknowledgments

The authors thank the referees for many valuable suggestions and comments. H. S. Jung was supported by SEOK CHUN Research Fund, Sungkyunkwan University, 2010.

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