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Research Article

Stabilities of Cubic Mappings in Various Normed Spaces: Direct and Fixed Point Methods

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In 1940 and 1964, Ulam proposed the general problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?". In 1941, Hyers solved this stability problem for linear mappings. According to Gruber (1978) this kind of stability problems are of the particular interest in probability theory and in the case of functional equations of different types. In 1981, Skof was the first author to solve the Ulam problem for quadratic mappings. In 1982–2011, J. M. Rassias solved the above Ulam problem for linear and nonlinear mappings and established analogous stability problems even on restricted domains. The purpose of this paper is the generalized Hyers-Ulam stability for the following cubic functional equation: $f(mx+y)+f(mx-y)=mf(x+y)+mf(x-y)+2(m^3-m)f(x), m\geq 2$ in various normed spaces.

1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?"

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1964.

In the next year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.

In 1978, Th. M. Rassias [3] proved a generalization of Hyers' theorem for additive mappings.

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then, the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.3}$$

for all $x \in E$. If $p < 0$, then inequality (1.1) *holds for $x, y \neq 0$ and* (1.3) *for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.*

The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [4] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.4}$$

is called a *quadratic functional equation*. In particular, every solution of the above quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [7] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2–4, 8–48]).

On the other hand, J. M. Rassias [38] considered the Cauchy difference controlled by a product of different powers of norm.

Theorem 1.2 (J. M. Rassias). *Let $f : E \to E'$ be a mapping from a real normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\|x\|^p\|y\|^q \tag{1.5}$$

for all $x, y \in E$, *where* ϵ *and* $r = p + q$ *are constants with* $\epsilon > 0$ *and* $r \neq 1$. *Then,* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^r} \|x\|^r \tag{1.6}$$

for all $x \in E$.

However, there was a singular case, for this singularity a counterexample was given by Găvruţa [19]. This stability phenomenon is called the Ulam-Gavruta-Rassias product stability (see also [13–17, 49]). In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function. This stability is called JMRassias mixed product-sum stability (see also [44, 50–53]).

Jun and Kim [22] introduced the functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{1.7}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.7) in Banach spaces.

Park and Jung [35] introduced the functional equation

$$f(3x + y) + f(3x - y) = 3f(x + y) + 3f(x - y) + 48f(x), \tag{1.8}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.8) in Banach spaces.

It is easy to see that the function $f(x) = x^3$ is a solution of the functional equations (1.7) and (1.8). Thus, it is natural that functional equations (1.7) and (1.8) are called cubic functional equations and every solution of these cubic functional equations is said to be a cubic mapping.

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation:

$$f(mx + y) + f(mx - y) = mf(x + y) + mf(x - y) + 2\left(m^3 - m\right)f(x). \tag{1.9}$$

where m is a positive integer greater than 2, in various normed spaces.

2. Preliminaries

In the sequel, we will adopt the usual terminology, notions, and conventions of the theory of random normed spaces as in [54]. Throughout this paper, the space of all probability distribution functions is denoted by Δ^+. Elements of Δ^+ are functions $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$, such that F is left continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. It is clear that the subset $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$, where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of Δ^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, for all $t \in R$, $F \leq G$ if and only if $F(t) \leq G(t)$. For every $a \geq 0$, $H_a(t)$ is the element of D^+ defined by

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases} \tag{2.1}$$

One can easily show that the maximal element for Δ^+ in this order is the distribution function $H_0(t)$.

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ is a continuous triangular norm (briefly a t-norm) if Tsatisfies the following conditions:

(i) T is commutative and associative;

(ii) T is continuous;

(iii) $T(x,1) = x$ for all $x \in [0,1]$;

(iv) $T(x,y) \leq T(z,w)$ whenever $x \leq z$ and $y \leq w$ for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are $T(x,y) = xy, T(x,y) = \max\{a+b-1,0\}$, and $T(x,y) = \min(a,b)$. Recall that, if T is a t-norm and $\{x_n\}$ is a given group of numbers in $[0,1]$, $T_{i=1}^{n} x_i$ is defined recursively by $T_{i=1}^{1} x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$.

Definition 2.2. A random normed space (briefly RN-space) is a triple (X, μ', T), where X is a vector space, T is a continuous t-norm and $\mu' : X \to D^+$ is a mapping such that the following conditions hold:

(i) $\mu'_x(t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;

(ii) $\mu'_{\alpha x}(t) = \mu'_x(t/|\alpha|)$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $x \in X$ and $t \geq 0$;

(iii) $\mu'_{x+y}(t+s) \geq T(\mu'_x(t), \mu'_y(s))$, for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ', T_M) where, for every $t > 0$,

$$\mu'_u(t) = \frac{t}{t + \|u\|} \tag{2.2}$$

and T_M is the minimum t-norm. This space is called the induced random normed space.

If the t-norm T is such that $\sup_{0<a<1} T(a,a) = 1$, then every RN-space (X, μ', T) is a metrizable linear topological space with the topology τ (called the μ'-topology or the (ϵ, δ)-topology) induced by the base of neighborhoods of θ, $\{U(\epsilon, \lambda) \mid \epsilon > 0, \lambda \in (0,1)\}$, where

$$U(\epsilon, \lambda) = \{x \in X \mid \mu'_x(\epsilon) > 1 - \lambda\}. \tag{2.3}$$

Definition 2.3. Let (X, μ', T) be an RN-space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ in X if, for all $t > 0$, $\lim_{n\to\infty} \mu'_{x_n - x}(t) = 1$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy sequence in X if, for all $t > 0$, $\lim_{n\to\infty}\mu'_{x_n-x_m}(t) = 1$.

(iii) The RN-space (X,μ',T) is said to be complete if every Cauchy sequence in X is convergent.

Theorem 2.4. *If (X,μ',T) is RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\mu'_{x_n}(t) = \mu'_x(t)$.*

A valuation is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r||s|$, and the triangle inequality holds, that is,

$$|r+s| \le \max\{|r|,|s|\}. \tag{2.4}$$

A field $\mathbb{K}$ is called a *valued field* if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\} \tag{2.5}$$

for all $r,s \in \mathbb{K}$, then the function $|\cdot|$ is called a *non-Archimedean valuation* and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \ge 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 2.5. Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a non-Archimedean norm if the following conditions hold:

(a) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;

(b) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(c) the strong triangle inequality holds:

$$\|x+y\| \le \max\{\|x\|,\|y\|\} \tag{2.6}$$

for all $x,y \in X$. Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space.

Definition 2.6. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

(a) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a *Cauchy sequence* if and only if, the sequence $\{x_{n+1}-x_n\}_{n=1}^{\infty}$ converges to zero.

(b) The sequence $\{x_n\}$ is said to be convergent if, for any $\varepsilon > 0$, there are a positive integer N and $x \in X$ such that

$$\|x_n - x\| \le \varepsilon \tag{2.7}$$

for all $n \geq N$. Then, the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.

(c) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

Definition 2.7. Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(c) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 2.8. *Let (X,d) be a complete generalized metric space and $J : X \to X$ a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{2.8}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(J^n x, J^{n+1} x) < \infty$ *for all $n_0 \geq n_0$;*

(b) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(c) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*

(d) $d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ *for all $y \in Y$.*

3. Random Stability of Functional Equation (1.9): A Direct Method

In this section, using direct method, we prove the generalized Hyers-Ulam stability of cubic functional equation (1.9) in random normed spaces.

Lemma 3.1. *Let E_1 and E_2 be real vector spaces. A function $f : E_1 \to E_2$ satisfies the functional equation* (1.7) *if and only if $f : E_1 \to E_2$ satisfies the functional equation* (1.9)*. Therefore, every solution of functional equation* (1.9) *is also cubic function.*

Proof. Let $f : E_1 \to E_2$ satisfy the equation (1.7). Putting $x = y = 0$ in (1.7), we get $f(0) = 0$. Set $y = 0$ in (1.7) to get $f(-y) = -f(y)$. By induction, we lead to $f(kx) = k^3 f(x)$ for all positive integer k. Replacing y by $x + y$ in (1.7), we have

$$f(3x+y) + f(x-y) = 2f(2x+y) - 2f(-y) + 12f(x), \tag{3.1}$$

for all $x, y \in E_1$. Once again replacing y by $y - x$ in (1.7), we have

$$f(x+y) + f(3x-y) = 2f(y) + 2f(2x-y) + 12f(x), \tag{3.2}$$

for all $x, y \in E_1$. Adding (3.1) to (3.2) and using (1.7), we obtain

$$f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x), \tag{3.3}$$

for all $x, y \in E_1$. By using the previous method, by induction, we infer that

$$f(mx+y) + f(mx-y) = mf(x+y) + mf(x-y) + 2\left(m^3 - m\right)f(x), \tag{3.4}$$

for all $x, y \in E_1$ and each positive integer $m \geq 3$.

Let $f : E_1 \to E_2$ satisfy the functional equation (1.9) with the positive integer $m \geq 3$. Putting $x = y = 0$ in (1.9), we get $f(0) = 0$. Setting $x = 0$, we get $f(-y) = -f(y)$. Let k be a positive integer. Replacing y by $kx + y$ in (1.9), we have

$$\begin{aligned} f((m+k)x+y) + f((m-k)x-y) = mf&((k+1)x+y) \\ &- mf((k-1)x+y) \\ &+ 2\left(m^3 - m\right)f(x), \end{aligned} \tag{3.5}$$

for all $x, y \in E_1$. Replacing y by $y - kx$ in (1.9), we have

$$\begin{aligned} f((m-k)x+y) + f((m+k)x-y) = mf&((k+1)x-y) \\ &- mf((k-1)x-y) \\ &+ 2\left(m^3 - m\right)f(x), \end{aligned} \tag{3.6}$$

for all $x, y \in E_1$. Adding (3.5) to (3.6), we obtain

$$\begin{aligned} f&((m+k)x+y) + f((m-k)x-y) + f((m-k)x+y) + f((m+k)x-y) \\ &= mf((k+1)x+y) + mf((k+1)x-y) \\ &\quad - m(f((k-1)x+y) + f((k-1)x-y)) + 2\left(m^3 - m\right)f(x), \end{aligned} \tag{3.7}$$

for all $x, y \in E_1$ and for all integer $k \geq 1$. Let $\psi_m(x, y) = f(mx+y) + f(mx-y)$ for each integer $m \geq 0$. Then, (3.7) means that

$$\psi_{m+k}(x,y) + \psi_{m-k}(x,y) = m\psi_{k+1}(x,y) - m\psi_{k-1}(x,y) + 4\left(m^3 - m\right)f(x), \tag{3.8}$$

for all $x, y \in E_1$ and for all integer $k \geq 1$. For $k = 1$ and $k = m$ in (3.8), we obtain

$$\begin{aligned} \psi_{m+1}(x,y) + \psi_{m-1}(x,y) &= m\psi_2(x,y) + 4\left(m^3 - m\right)f(x), \\ \psi_{2m}(x,y) &= m\psi_{m+1}(x,y) - m\psi_{m-1}(x,y) + 4\left(m^3 - m\right)f(x), \end{aligned} \tag{3.9}$$

for all $x, y \in E_1$. By the proof of the first part, since $f : E_1 \to E_2$ satisfies the functional equation (1.9) with the positive integer $m \geq 3$, then f satisfies the functional equation (1.9) with the positive integer $k \geq m$. It follows from (3.9) that f satisfies the functional equation (1.7) and

$$\begin{aligned} f((m-1)x+y)+f((m-1)x-y) &= (m-1)f(x+y) \\ &\quad+(m-1)f(x-y) \\ &\quad+2\Big((m-1)^3-(m-1)\Big)f(x). \end{aligned} \tag{3.10}$$

□

Theorem 3.2. *Let X be a real linear space, $(Z,\mu',\min)$ an RN-space, and $\varphi : X^2 \to Z$ a function such that, for some $0<\alpha<m^3$,*

$$\mu'_{\varphi(mx,0)}(t) \geq \mu'_{\alpha\varphi(x,0)}(t) \quad \forall x \in X,\ t>0, \tag{3.11}$$

and, for all $x,y \in X$ and $t>0$, $\lim_{n\to\infty}\mu'_{\varphi(m^n x,m^n y)/m^{3n}}(t)=1$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f : X \to Y$ is a mapping with $f(0)=0$ such that for all $x \in X$ and $t>0$

$$\mu_{f(mx\pm y)-mf(x\pm y)-2(m^3-m)f(x)}(t) \geq \mu'_{\varphi(x,y)}(t), \tag{3.12}$$

then the limit $C(x)=\lim_{n\to\infty}(f(m^n x)/m^{3n})$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\varphi(x,0)/2(m^3-\alpha)}(t). \tag{3.13}$$

Proof. Putting $y=0$ in (3.12) we see that, for all $x \in X$,

$$\mu_{f(mx)/m^3-f(x)}(t) \geq \mu'_{\varphi(x,0)/2m^3}(t). \tag{3.14}$$

Replacing x by $m^n x$ in (3.14) and using (3.11), we obtain

$$\mu_{f(m^{n+1}x)/m^{3n+3}-f(m^n x)/m^{3n}}(t) \geq \mu'_{\varphi(m^n x,0)/2m^{3n+3}}(t) \geq \mu'_{\alpha^n\varphi(x,0)/2m^{3n+3}}(t). \tag{3.15}$$

So

$$\begin{aligned} \mu_{(f(m^n x)/m^{3n})-f(x)/(\sum_{k=0}^{n-1}(\alpha^k/2m^{3k+3}))}(t) &= \mu_{((\sum_{k=0}^{n-1}(f(m^{k+1}x)/m^{3k+3}))-(f(m^k x)/m^{3k}))/(\sum_{k=0}^{n-1}(\alpha^k/2m^{3k+3}))}(t) \\ &\geq T_{k=0}^{n-1}\Big(\mu_{((f(m^{k+1}x)/m^{3k+3})-(f(m^k x)/m^{3k}))/(\alpha^k/2m^{3k+3})}(t)\Big) \\ &= T_{k=0}^{n-1}\Big(\mu'_{\varphi(x,0)}(t)\Big) \\ &= \mu'_{\varphi(x,0)}(t). \end{aligned} \tag{3.16}$$

This implies that

$$\mu_{f(m^n x)/m^{3n}-f(x)}(t) \geq \mu'_{\sum_{k=0}^{n-1}(\alpha^k \varphi(x,0)/2m^{3k+3})}(t). \tag{3.17}$$

Replacing x by m_x^p in (3.17), we obtain

$$\begin{aligned}\mu_{f(m^{n+p}x)/m^{3(n+p)}-f(m^p x)/m^{3p}}(t) &\geq \mu'_{\sum_{k=0}^{n-1}(\alpha^k \varphi(m^p x,0)/2m^{3(k+p)+3})}(t) \geq \mu'_{\sum_{k=0}^{n-1}(\alpha^{k+p} \varphi(x,0)/2m^{3(k+p)+3})}(t) \\ &= \mu'_{\sum_{k=p}^{n+p-1}(\alpha^k \varphi(x,0)/2m^{3k+3})}(t).\end{aligned} \tag{3.18}$$

As

$$\lim_{p,n\to\infty} \mu'_{\sum_{k=p}^{n+p-1}(\alpha^k \varphi(x,0)/2m^{3k+3})}(t) = 1, \tag{3.19}$$

$\{f(m^n x)/m^{3n}\}$ is a Cauchy sequence in complete RN-space $(Y, \mu, \min)$, so there exists some point $C(x) \in Y$ such that $\lim_{n\to\infty}(f(m^n x)/m^{3n}) = C(x)$. Fix $x \in X$ and put $p = 0$ in (3.18). Then, we obtain

$$\mu_{f(m^n x)/m^{3n}-f(x)}(t) \geq \mu'_{\sum_{k=0}^{n-1}(\alpha^k \varphi(x,0)/2m^{3k+3})}(t), \tag{3.20}$$

and so, for every $\epsilon > 0$, we have

$$\begin{aligned}\mu_{C(x)-f(x)}(t+\epsilon) &\geq T\left(\mu_{C(x)-f(m^n x)/m^{3n}}(\epsilon), \mu_{f(m^n x)/m^{3n}-f(x)}(t)\right) \\ &\geq T\left(\mu_{C(x)-f(m^n x)/m^{3n}}(\epsilon), \mu'_{\sum_{k=0}^{n-1}(\alpha^k \varphi(x,0)/2m^{3k+3})}(t)\right).\end{aligned} \tag{3.21}$$

Taking the limit as $n \to \infty$ and using (3.21), we get

$$\mu_{C(x)-f(x)}(t+\epsilon) \geq \mu'_{\varphi(x,0)/2(m^3-\alpha)}(t). \tag{3.22}$$

Since ϵ was arbitrary by taking $\epsilon \to 0$ in (3.22), we get

$$\mu_{C(x)-f(x)}(t) \geq \mu'_{\varphi(x,0)/2(m^3-\alpha)}(t). \tag{3.23}$$

Replacing x and y by $m^n x$ and $m^n y$ in (3.12), respectively, we get, for all $x, y \in X$ and for all $t > 0$,

$$\mu_{(f(m^{n+1}x \pm m^n y)-mf(m^n x \pm m^n y)-2(m^3-m)f(m^n x))/m^{3n}}(t) \geq \mu'_{\varphi(m^n x, m^n y)/m^{3n}}(t). \tag{3.24}$$

Since $\lim_{n\to\infty}\mu'_{\varphi(m^n x, m^n y)/m^{3n}}(t) = 1$, we conclude that $C(mx \pm y) = C(x \pm y) + 2(m^3 - m)C(x)$. To prove the uniqueness of the cubic mapping C, assume that there exists another cubic mapping $L : X \to Y$ which satisfies (3.13).

By induction one can easily see that, since f is a cubic functional equation, so, for all $n \in \mathbb{N}$ and every $x \in X$, $C(m^n x) = m^{3n} C(x)$, and $L(m^n x) = m^{3n} L(x)$, we have

$$\mu_{C(x)-L(x)}(t) = \lim_{n\to\infty} \mu_{(C(m^n x)/m^{3n})-(L(m^n x)/m^{3n})}(t), \tag{3.25}$$

so

$$\begin{aligned}
\mu_{C(m^n x)/m^{3n}-L(m^{3n}x)/m^{3n}}(t) &\geq \min\left\{\mu_{C(m^n x)/m^{3n}-f(m^n x)/m^{3n}}\left(\frac{t}{2}\right), \mu_{L(m^n x)/m^{3n}-f(m^n x)/m^{3n}}\left(\frac{t}{2}\right)\right\} \\
&\geq \mu'_{\varphi(m^n x,0)/m^{3n}(m^3-\alpha)}(t) \\
&\geq \mu'_{\alpha^n \varphi(x,0)/m^{3n}(m^3-\alpha)}(t).
\end{aligned} \tag{3.26}$$

Since $\lim_{n\to\infty} \mu'_{\alpha^n \varphi(x,0)/m^{3n}(m^3-\alpha)}(t) = 1$, it follows that, for all $t > 0$, $\mu_{C(x)-L(x)}(t) = 1$ and so $C(x) = L(x)$. This completes the proof. □

Corollary 3.3. *Let X be a real linear space, $(Z, \mu', \min)$ an RN-space, and $(Y, \mu, \min)$ a complete RN-space. Let $0 < r < 1$ and $z_0 \in Z$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying*

$$\mu_{f(mx\pm y)-mf(x\pm y)-2(m^3-m)f(x)}(t) \geq \mu'_{(\|x\|^r+\|y\|^r)z_0}(t), \tag{3.27}$$

for all $x, y \in X$ and $t > 0$. Then, the limit $C(x) = \lim_{n\to\infty}(f(m^n x)/m^{3n})$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\|x\|^r z_0/2(m^3-m^{3r})}(t) \tag{3.28}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = m^{3r}$, and let $\varphi : X^2 \to Z$ be defined as $\varphi(x, y) = (\|x\|^r + \|y\|^r)z_0$. □

Remark 3.4. In Corollary 3.3, if we assume that $\varphi(x, y) = (\|x\|^r.\|y\|^r)z_0$ or $\varphi(x, y) = (\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s)z_0$, then we get Ulam-Gavruta-Rassias product stability and JMRassias mixed product-sum stability, respectively. But, since we put $y = 0$ in this functional equation, the Ulam-Gavruta-Rassias product stability and JMRassias mixed product-sum stability corollaries will be obvious. Meanwhile, the JMRassias mixed product-sum stability when $r + s = 3$ is an open question.

Corollary 3.5. *Let X be a real linear space, $(Z, \mu', \min)$ an RN-space, and $(Y, \mu, \min)$ a complete RN-space. Let $z_0 \in Z$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying*

$$\mu_{f(mx\pm y)-mf(x\pm y)-2(m^3-m)f(x)}(t) \geq \mu'_{\delta z_0}(t), \tag{3.29}$$

for all $x, y \in X$ and $t > 0$. Then, the limit $C(x) = \lim_{n\to\infty}(f(m^n x)/m^{3n})$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\delta z_0/2(m^3-1)}(t) \tag{3.30}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 1$, and let $\varphi : X^2 \to Z$ be defined by $\varphi(x, y) = \delta z_0$. □

Theorem 3.6. *Let X be a real linear space, $(Z, \mu', \min)$ an RN-space, and $\varphi : X^2 \to Z$ a function such that for some $0 < \alpha < 1/m^3$*

$$\mu'_{\varphi(x/m,0)}(t) \geq \mu'_{\alpha\varphi(x,0)}(t) \quad \forall x \in X,\ t > 0, \tag{3.31}$$

and, for all $x, y \in X$ and $t > 0$,

$$\lim_{n\to\infty} \mu'_{m^{3n}\varphi(x/m^n, y/m^n)}(t) = 1. \tag{3.32}$$

Let $(Y, \mu, \min)$ be a complete RN-space. If $f : X \to Y$ is a mapping with $f(0) = 0$ and satisfying (3.12), then the limit $C(x) = \lim_{n\to\infty} m^{3n} f(x/m^n)$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\varphi(x,0)/2(1-m^3\alpha)}(t). \tag{3.33}$$

Proof. Putting $y = 0$ in (3.12) and replacing x by x/m, we obtain that for all $x \in X$

$$\mu_{f(x)-m^3 f(x/m)}(t) \geq \mu'_{\varphi(x,0)/2}(t). \tag{3.34}$$

Replacing x by x/m^n in (3.34) and using (3.31), we obtain

$$\mu_{m^{3n} f(x/m^n) - m^{3n+3} f(x/m^{n+1})}(t) \geq \mu'_{m^{3n}\varphi(x/m^n,0)/2}(t) \geq \mu'_{\alpha^n m^{3n}\varphi(x,0)/2}(t). \tag{3.35}$$

So

$$\begin{aligned}
\mu_{(m^{3n} f(x/m^n) - f(x))/\sum_{k=0}^{n-1}(\alpha^k m^{3k}/2)}(t) &= \mu_{(\sum_{k=0}^{n-1} m^{3k+3} f(x/m^{k+1}) - m^{3k} f(x/m^k))/\sum_{k=0}^{n-1}(\alpha^k m^{3k}/2)}(t) \\
&\geq T_{k=0}^{n-1}\left(\mu'_{\varphi(x,0)}(t)\right) \\
&= \mu'_{\varphi(x,0)}(t).
\end{aligned} \tag{3.36}$$

This implies that

$$\mu_{m^{3n} f(x/m^n) - f(x)}(t) \geq \mu'_{\sum_{k=0}^{n-1} \alpha^k m^{3k}\varphi(x,0)/2}(t). \tag{3.37}$$

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.7. *Let X be a real linear space, $(Z, \mu', \min)$ an RN-space, and $(Y, \mu, \min)$ a complete RN-space. Let $r > 1$ and $z_0 \in Z$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (3.27). Then, the limit $C(x) = \lim_{n\to\infty} m^{3n} f(x/m^n)$ exists for all $x \in X$ and defines a unique cubic* *mapping $C : X \to Y$ such that*

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{m^{3r}\|x\|^r z_0/2(m^{3r}-m^3)}(t) \tag{3.38}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = m^{-3r}$, and let $\psi : X^2 \to Z$ be defined as $\psi(x, y) = (\|x\|^r + \|y\|^r)z_0$. □

Corollary 3.8. *Let X be a real linear space, $(Z, \mu', \min)$ be an RN-space, and $(Y, \mu, \min)$ a complete RN-space. Let $z_0 \in Z$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (3.29). Then, the limit $C(x) = \lim_{n\to\infty} m^{3n} f(x/m^n)$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that*

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{m\delta z_0/2(m-1)}(t) \tag{3.39}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\alpha = 1/m^4$, and let $\psi : X^2 \to Z$ be defined by $\psi(x, y) = \delta z_0$. □

4. Random Stability of the Functional Equation (1.9): A Fixed Point Approach

In this section, using the fixed point alternative approach, we prove the generalized Hyers-Ulam stability of functional equation (1.9) in random normed spaces.

Theorem 4.1. *Let X be a linear space, (Y, μ, T_M) a complete RN-space, and Φ a mapping from X^2 to D^+($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that there exists $0 < \alpha < 1/m^3$ such that*

$$\Phi_{x,y}(t) \leq \Phi_{x/m,y/m}(\alpha t) \tag{4.1}$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying

$$\mu_{f(mx\pm y)-mf(x\pm y)-2(m^3-m)f(x)}(t) \geq \Phi_{x,y}(t) \tag{4.2}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$

$$C(x) := \lim_{n\to\infty} m^{3n} f\left(\frac{x}{m^n}\right) \tag{4.3}$$

exists and $C : X \to Y$ is a unique cubic mapping such that

$$\mu_{f(x)-C(x)}(t) \geq \Phi_{x,0}\left(\frac{(2-2m^3\alpha)t}{\alpha}\right) \tag{4.4}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ in (4.2) and replacing x by x/m, we have

$$\mu_{f(x)-m^3 f(x/m)}(t) \geq \Phi_{x/m,0}(2t) \geq \Phi_{x,0}\left(\frac{2t}{\alpha}\right) \tag{4.5}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S := \{g : X \longrightarrow Y; g(0) = 0\} \tag{4.6}$$

and the generalized metric d in S defined by

$$d(f, g) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,0}(t),\ \forall x \in X,\ t > 0\}, \tag{4.7}$$

where $\inf\ \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [26], Lemma 2.1). Now, we consider a linear mapping $J : S \longrightarrow S$ such that

$$Jh(x) := m^3 h\left(\frac{x}{m}\right) \tag{4.8}$$

for all $x \in X$. First, we prove that J is a strictly contractive mapping with the Lipschitz constant $m^3\alpha$.

In fact, let $g, h \in S$ be such that $d(g, h) < \epsilon$. Then, we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,0}(t) \tag{4.9}$$

for all $x \in X$ and $t > 0$, and so

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}\left(m^3\alpha\epsilon t\right) = \mu_{m^3 g(x/m)-m^3 h(x/m)}\left(m^3\alpha\epsilon t\right) &= \mu_{g(x/m)-h(x/m)}(\alpha\epsilon t) \\ &\geq \Phi_{x/m,0}(\alpha t) \\ &\geq \Phi_{x,0}(t) \end{aligned} \tag{4.10}$$

for all $x \in X$ and $t > 0$. Thus, $d(g, h) < \epsilon$ implies that

$$d(Jg, Jh) = d\left(m^3 g\left(\frac{x}{m}\right), m^3 h\left(\frac{x}{m}\right)\right) < m^3\alpha\epsilon. \tag{4.11}$$

This means that

$$d(Jg, Jh) = d\left(m^3 g\left(\frac{x}{m}\right), m^3 h\left(\frac{x}{m}\right)\right) \leq m^3\alpha d(g, h) \tag{4.12}$$

for all $g, h \in S$. It follows from (4.5) that

$$d(f, Jf) = d\left(f, m^3 f\left(\frac{x}{m}\right)\right) \leq \frac{\alpha}{2}. \tag{4.13}$$

By Theorem 2.8, there exists a mapping $C : X \to Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C\left(\frac{x}{m}\right) = \frac{1}{m^3} C(x) \tag{4.14}$$

for all $x \in X$.

The mapping C is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}. \tag{4.15}$$

This implies that C is a unique mapping satisfying (4.14) such that there exists $u \in (0, \infty)$ satisfying

$$\mu_{f(x)-C(x)}(ut) \geq \Phi_{x,0}(t) \tag{4.16}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} m^{3n} f\left(\frac{x}{m^n}\right) = C(x) \tag{4.17}$$

for all $x \in X$.

(3) $d(f, C) \leq d(f, Jf)/(1 - m^3\alpha)$ with $f \in \Omega$, which implies the inequality $d(f, C) \leq \alpha/(2 - 2m^3\alpha)$ and so

$$\mu_{f(x)-C(x)}\left(\frac{\alpha t}{2 - 2m^3\alpha}\right) \geq \Phi_{x,0}(t) \tag{4.18}$$

for all $x \in X$ and $t > 0$. This implies that inequality (4.4) holds. Now, we have

$$\mu_{m^{3n} f((mx \pm y)/m^n) - m^{3n+1} f((x \pm y)/m^n) - 2m^{3n}(m^3 - m) f(x/m^n)}(t) \geq \Phi_{x/m^n, y/m^n}\left(\frac{t}{m^{3n}}\right) \tag{4.19}$$

for all $x, y \in X$, $t > 0$, and $n \geq 1$, and so, from (4.1), it follows that

$$\Phi_{x/m^n, y/m^n}\left(\frac{t}{m^{3n}}\right) \geq \Phi_{x,y}\left(\frac{t}{m^{3n}\alpha^n}\right), \tag{4.20}$$

Since

$$\lim_{n\to\infty} \Phi_{x,y}\left(\frac{t}{m^{3n}\alpha^n}\right) = 1, \tag{4.21}$$

for all $x, y \in X$ and $t > 0$, we have

$$\mu_{C(mx\pm y)-mC(x\pm y)-2(m^3-m)C(x)}(t) = 1 \tag{4.22}$$

for all $x, y \in X$ and $t > 0$. Thus, the mapping $C : X \to Y$ is cubic. This completes the proof. □

Corollary 4.2. *Let X be a real normed space, $\theta \geq 0$, and p a real number with $p \in (1, +\infty)$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying*

$$\mu_{f(mx\pm y)-mf(x\pm y)-2(m^3-m)f(x)}(t) \geq \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{4.23}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$, the limit $C(x) = \lim_{n\to\infty} m^{3n} f(x/m^n)$ exists and $C : X \to Y$ is a unique cubic mapping such that

$$\mu_{f(x)-C(x)}(t) \geq \frac{2\left(m^{3p} - m^3\right)t}{2\left(m^{3p} - m^3\right)t + \theta\|x\|^p} \tag{4.24}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 4.1 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{4.25}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = m^{-3p}$, then we get the desired result. □

Theorem 4.3. *Let X be a linear space, (Y, μ, T_M) a complete RN-space, and Φ a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that for some $0 < \alpha < m^3$*

$$\Phi_{x/m,y/m}(t) \leq \Phi_{x,y}(\alpha t) \tag{4.26}$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (4.2). Then, for all $x \in X$, the limit $C(x) := \lim_{n\to\infty} f(m^n x)/m^{3n}$ exists and $C : X \to Y$ is a unique cubic mapping such that

$$\mu_{f(x)-C(x)}(t) \geq \Phi_{x,0}\left(\left(2m^3 - 2\alpha\right)t\right) \tag{4.27}$$

for all $x \in X$ and $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 4.1. Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{m^3}h(mx) \tag{4.28}$$

for all $x \in X$.

Let $g, h \in S$ be such that $d(g, h) < \epsilon$. Then, we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,0}(t) \tag{4.29}$$

for all $x \in X$ and $t > 0$ and so

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}\left(\frac{\alpha\epsilon t}{m^3}\right) = \mu_{(1/m^3)g(mx)-(1/m^3)h(mx)}\left(\frac{\alpha\epsilon t}{m^3}\right) &= \mu_{g(mx)-h(mx)}(\alpha\epsilon t) \\ &\geq \Phi_{mx,0}(\alpha t) \\ &\geq \Phi_{x,0}(t)\end{aligned} \tag{4.30}$$

for all $x \in X$ and $t > 0$. Thus, $d(g, h) < \epsilon$ implies that

$$d(Jg, Jh) = d\left(\frac{g(mx)}{m^3}, \frac{h(mx)}{m^3}\right) < \frac{\alpha\epsilon}{m^3}. \tag{4.31}$$

This means that

$$d(Jg, Jh) = d\left(\frac{g(mx)}{m^3}, \frac{h(mx)}{m^3}\right) \leq \frac{\alpha}{m^3}d(g, h) \tag{4.32}$$

for all $g, h \in S$.

Putting $y = 0$ in (4.2), we see that, for all $x \in X$,

$$\mu_{f(mx)/m^3-f(x)}\left(\frac{t}{2m^3}\right) \geq \Phi_{x,0}(t). \tag{4.33}$$

It follows from (4.33) that

$$d(f, Jf) = d\left(f, \frac{f(mx)}{m^3}\right) \leq \frac{1}{2m^3}. \tag{4.34}$$

By Theorem 2.8, there exists a mapping $C : X \to Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C(mx) = m^3C(x) \tag{4.35}$$

for all $x \in X$.

The mapping C is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g,h) < \infty\}. \tag{4.36}$$

This implies that C is a unique mapping satisfying (4.35) such that there exists $u \in (0,\infty)$ satisfying

$$\mu_{f(x)-C(x)}(ut) \geq \Phi_{x,0}(t) \tag{4.37}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(m^n x)}{m^{3n}} = C(x) \tag{4.38}$$

for all $x \in X$.

(3) $d(f,C) \leq d(f,Jf)/(1-\alpha/m^3)$ with $f \in \Omega$, which implies the inequality $d(f,C) \leq 1/(2m^3 - 2\alpha)$, and so

$$\mu_{f(x)-C(x)}\left(\frac{t}{2m^3 - 2\alpha}\right) \geq \Phi_{x,0}(t) \tag{4.39}$$

for all $x \in X$ and $t > 0$. The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let X be a real normed space, $\theta \geq 0$, and p a real number with $p \in (0,1)$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (4.23). Then, for all $x \in X$, the limit $C(x) = \lim_{n\to\infty} f(m^n x)/m^{3n}$ exists and $C : X \to Y$ is a unique cubic mapping such that*

$$\mu_{f(x)-C(x)}(t) \geq \frac{2(m^3 - m^{3p})t}{2(m^3 - m^{3p})t + \theta\|x\|^p} \tag{4.40}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 6.3 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{4.41}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = m^{3p}$, then we get the desired result. □

Remark 4.5. In Corollaries 4.2 and 4.4, if we assume that $\Phi_{x,y}(t) = t/(t + \theta(\|x\|^p \cdot \|y\|^p))$ *or* $\Phi_{x,y}(t) = t/(t + \theta(\|x\|^{p+q} + \|y\|^{p+q} + \|x\|^p\|y\|^q))$, then we get Ulam-Gavruta-Rassias product stability and JMRassias mixed product-sum stability, respectively. But, since we put $y = 0$ in this functional equation, the Ulam-Gavruta-Rassias product stability and JMRassias mixed product-sum stability corollaries will be obvious. Meanwhile, the JMRassias mixed product-sum stability when $p + q = 3$ is an open question.

5. Non-Archimedean Stability of Functional Equation (1.9): A Direct Method

In this section, using direct method, we prove the generalized Hyers-Ulam stability of cubic functional equation (1.9) in non-Archimedean normed spaces. Throughout this section, we assume that G is an additive semigroup and X is a complete non-Archimedean space.

Theorem 5.1. *Let $\zeta : G^2 \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |m|^{3n} \zeta\left(\frac{x}{m^n}, \frac{y}{m^n}\right) = 0 \tag{5.1}$$

for all $x, y \in G$ and let for each $x \in G$ the limit

$$\Theta(x) = \lim_{n\to\infty} \max\left\{ |m|^{3(k+1)} \zeta\left(\frac{x}{m^{k+1}}, 0\right); 0 \le k < n \right\} \tag{5.2}$$

exist. Suppose that $f : G \to X$ a mapping with $f(0) = 0$ and satisfying the following inequality:

$$\left\| f(mx \pm y) - mf(x \pm y) - 2\left(m^3 - m\right) f(x) \right\| \le \zeta(x, y, z). \tag{5.3}$$

Then, the limit $C(x) := \lim_{n\to\infty} m^{3n} Q(x/m^n)$ exists for all $x \in G$ and defines a cubic mapping $C : G \to X$ such that

$$\| f(x) - C(x) \| \le \frac{\Theta(x)}{|2m^3|}. \tag{5.4}$$

Moreover, if

$$\lim_{j\to\infty} \lim_{n\to\infty} \max\left\{ |m|^{3(k+1)} \zeta\left(\frac{x}{m^{k+1}}, 0\right); j \le k < n + j \right\} = 0, \tag{5.5}$$

then C is the unique cubic mapping satisfying (5.4).

Proof. Putting $y = 0$ in (5.3), we get

$$\left\| f(mx) - m^3 f(x) \right\| \le \frac{\zeta(x, 0)}{|2|} \tag{5.6}$$

for all $x \in G$. Replacing x by x/m^{n+1} in (5.6), we obtain

$$\left\| m^{3n+3} f\left(\frac{x}{m^{n+1}}\right) - m^{3n} f\left(\frac{x}{m^n}\right) \right\| \le \frac{|m|^{3n}}{|2|} \zeta\left(\frac{x}{m^{n+1}}, 0\right). \tag{5.7}$$

It follows from (5.1) and (5.7) that the sequence $\{m^{3n} f(x/m^n)\}_{n\ge1}$ is a Cauchy sequence. Since X is complete, $\{m^{3n} f(x/m^n)\}_{n\ge1}$ is convergent. Set $C(x) := \lim_{n\to\infty} m^{3n} f(x/m^n)$.

Using induction, one can show that

$$\left\| m^{3n} f\left(\frac{x}{m^n}\right) - f(x) \right\| \le \frac{1}{|2m^3|} \max\left\{ |m|^{3(k+1)} \zeta\left(\frac{x}{m^{k+1}}, 0\right); 0 \le k < n \right\} \tag{5.8}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (5.8) and using (5.2), one obtains (5.4). By (5.1) and (5.3), we get

$$\begin{aligned}
&\left\| C(mx \pm y) - mC(x \pm y) - 2\left(m^3 - m\right) C(x) \right\| \\
&\quad = \lim_{n\to\infty} \left\| m^{3n} f\left(\frac{mx \pm y}{m^n}\right) - m^{3n+1} f\left(\frac{x \pm y}{m^n}\right) - 2m^{3n}\left(m^3 - m\right) f\left(\frac{x}{m^n}\right) \right\| \\
&\quad \le \lim_{n\to\infty} |m|^{3n} \zeta\left(\frac{x}{m^n}, \frac{y}{m^n}\right) \\
&\quad = 0
\end{aligned} \tag{5.9}$$

for all $x, y \in G$. Therefore, the function $C : G \to X$ satisfies (1.9). To prove the uniqueness property of C, let $L : G \to X$ be another function satisfying (5.4). Then,

$$\begin{aligned}
\|C(x) - L(x)\| &= \lim_{j\to\infty} |m|^{3j} \left\| C\left(\frac{x}{m^j}\right) - L\left(\frac{x}{m^j}\right) \right\| \\
&\le \lim_{j\to\infty} |m|^{3j} \max\left\{ \left\| C\left(\frac{x}{m^j}\right) - f\left(\frac{x}{m^j}\right) \right\|, \left\| f\left(\frac{x}{m^j}\right) - L\left(\frac{x}{m^j}\right) \right\| \right\} \\
&\le \lim_{j\to\infty} \lim_{n\to\infty} \max\left\{ |m|^{3(k+1)} \zeta\left(\frac{x}{m^{k+1}}, 0\right); j \le k < n + j \right\} \\
&= 0
\end{aligned} \tag{5.10}$$

for all $x \in G$. Therefore, $C = L$, and the proof is complete. □

Corollary 5.2. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi\left(\frac{t}{|m|}\right) \le \xi\left(\frac{1}{|m|}\right) \xi(t) \quad (t \ge 0) \quad \xi\left(\frac{1}{|m|}\right) < |m|^{-3}. \tag{5.11}$$

Let $\kappa > 0$, and let $f : G \to X$ be a mapping with $f(0) = 0$ and satisfying the following inequality:

$$\left\| f(mx \pm y) - mf(x \pm y) - 2\left(m^3 - m\right) f(x) \right\| \le \kappa(\xi(|x|) + \xi(|y|)) \tag{5.12}$$

for all $x, y \in G$. Then there exists a unique cubic mapping $C : G \to X$ such that

$$\|f(x) - C(x)\| \le \frac{\kappa \xi(|x|)}{|2m^3|}. \tag{5.13}$$

Proof. Defining $\zeta : G^2 \to [0,\infty)$ by $\zeta(x,y) := \kappa(\xi(|x|) + \xi(|y|))$, we have

$$\lim_{n\to\infty} |m|^{3n} \zeta\left(\frac{x}{m^n}, \frac{y}{m^n}\right) \le \lim_{n\to\infty} \left(|m|^3 \xi\left(\frac{1}{|m|}\right)\right)^n \zeta(x,y) = 0 \tag{5.14}$$

for all $x, y \in G$. The last equality comes from the fact that $|m|^3\xi(1/|m|) < 1$. On the other hand,

$$\Theta(x) = \lim_{n\to\infty} \max\left\{ |m|^{3k+3} \zeta\left(\frac{x}{m^{k+1}}, 0\right); 0 \le k < n \right\} = |m|^3 \zeta\left(\frac{x}{m}, 0\right) = \kappa\xi(|x|), \tag{5.15}$$

for all $x \in G$, exists. Also,

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{ |m|^{3k+3} \zeta\left(\frac{x}{m^{k+1}}, 0\right); j \le k < n+j \right\} = \lim_{j\to\infty} |m|^{3j+3} \zeta\left(\frac{x}{m^{j+1}}, 0\right) = 0. \tag{5.16}$$

Applying Theorem 5.1, we get the desired result. □

Theorem 5.3. *Let $\zeta : G^3 \to [0,+\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\zeta(m^n x, m^n y)}{|m|^{3n}} = 0 \tag{5.17}$$

for all $x, y \in G$, and let for each $x \in G$ the limit

$$\Theta(x) = \lim_{n\to\infty} \max\left\{ \frac{\zeta(m^k x, 0)}{|m|^{3k+3}}; 0 \le k < n \right\} \tag{5.18}$$

exist. Suppose that $f : G \to X$ a mapping with $f(0) = 0$ and satisfying (5.3)*. Then, the limit $C(x) := \lim_{n\to\infty} f(m^n x)/m^{3n}$ exists for all $x \in G$ and defines a cubic mapping $C : G \to X$ such that*

$$\|f(x) - C(x)\| \le \frac{1}{|2|}\Theta(x). \tag{5.19}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{ \frac{\zeta(m^k x, 0)}{|m|^{3k+3}}; j \le k < n+j \right\} = 0, \tag{5.20}$$

then C is the unique cubic mapping satisfying (5.19)*.*

Proof. Putting $y = 0$ in (5.3), we get

$$\left\| f(x) - \frac{f(mx)}{m^3} \right\| \le \frac{\zeta(x,0)}{|2m^3|} \tag{5.21}$$

for all $x \in G$. Replacing x by $m^n x$ in (5.21), we obtain

$$\left\| \frac{f(m^n x)}{m^{3n}} - \frac{f(m^{n+1}x)}{m^{3n+3}} \right\| \leq \frac{\zeta(m^n x, 0)}{|2||m|^{3n+3}}. \tag{5.22}$$

It follows from (5.17) and (5.22) that the sequence $\{f(m^n x)/m^{3n}\}_{n\geq 1}$ is convergent. Set $C(x) := \lim_{n\to\infty} f(m^n x)/m^{3n}$. On the other hand, it follows from (5.22) that

$$\begin{aligned}
\left\| \frac{f(m^p x)}{m^{3p}} - \frac{f(m^q x)}{m^{3q}} \right\| &= \left\| \sum_{k=p}^{q-1} \frac{f(m^{k+1}x)}{m^{3k+3}} - \frac{f(m^k x)}{m^{3k}} \right\| \\
&\leq \max\left\{ \left\| \frac{f(m^{k+1}x)}{m^{3k+3}} - \frac{f(m^k x)}{m^{3k}} \right\| ; p \leq k < q-1 \right\} \\
&\leq \frac{1}{|2|} \max\left\{ \frac{\zeta(m^k x, 0)}{|m|^{3k+3}} ; p \leq k < q \right\}
\end{aligned} \tag{5.23}$$

for all $x \in G$ and all nonnegative integers p, q with $q > p \geq 0$. Letting $p = 0$, passing the limit $q \to \infty$ in the last inequality, and using (5.18), we obtain (5.19). The rest of the proof is similar to the proof of Theorem 5.1. $\square$

6. Non-Archimedean Stability of Functional Equation (1.9): A Fixed Point Method

In this section, using the fixed point alternative approach, we prove the generalized Hyers-Ulam stability of cubic functional equation (1.9) in non-Archimedean normed spaces. Throughout this section, let X be a non-Archimedean normed space that Y a complete non-Archimedean normed space. Also, $|2m^3| \neq 1$.

Theorem 6.1. *Let $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\zeta\left(\frac{x}{m}, \frac{y}{m}\right) \leq \frac{L\zeta(x,y)}{|m^3|} \tag{6.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying the following inequality:

$$\left\| f(mx \pm y) - mf(x \pm y) - 2\left(m^3 - m\right) f(x) \right\| \leq \zeta(x,y) \tag{6.2}$$

for all $x, y \in X$. Then, there is a unique cubic mapping $C : X \to Y$ such that

$$\| f(x) - C(x) \| \leq \frac{L\zeta(x,0)}{|2m^3| - |2m^3|L} \tag{6.3}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (6.2) and replacing x by x/m, we have

$$\left\| m^3 f\left(\frac{x}{m}\right) - f(x) \right\| \le \frac{1}{|2|}\zeta\left(\frac{x}{m}, 0\right) \tag{6.4}$$

for all $x \in X$. Consider the set

$$S := \{g : X \longrightarrow Y; g(0) = 0\} \tag{6.5}$$

and the generalized metric d in S defined by

$$d(f, g) = \inf_{\mu \in \mathbb{R}^+} \{\|g(x) - h(x)\| \le \mu\zeta(x, 0),\ \forall x \in X\}, \tag{6.6}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [26], Lemma 2.1).

Now, we consider a linear mapping $J : S \longrightarrow S$ such that

$$Jh(x) := m^3 h\left(\frac{x}{m}\right) \tag{6.7}$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then,

$$\|g(x) - h(x)\| \le \epsilon\zeta(x, 0) \tag{6.8}$$

for all $x \in X$, and so

$$\begin{aligned} \|Jg(x) - Jh(x)\| = \left\| m^3 g\left(\frac{x}{m}\right) - m^3 h\left(\frac{x}{m}\right) \right\| &\le \left|m^3\right| \epsilon\zeta\left(\frac{x}{m}, 0\right) \\ &\le \left|m^3\right| \frac{L\epsilon}{|m^3|}\zeta(x, 0) \end{aligned} \tag{6.9}$$

for all $x \in X$. Thus, $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \le L\epsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h) \tag{6.10}$$

for all $g, h \in S$. It follows from (6.4) that

$$d(f, Jf) \le \frac{L}{|2m^3|}. \tag{6.11}$$

By Theorem 2.8, there exists a mapping $C : X \longrightarrow Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C\left(\frac{x}{m}\right) = \frac{1}{m^3}C(x) \tag{6.12}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}. \tag{6.13}$$

This implies that C is a unique mapping satisfying (6.12) such that there exists $\mu \in (0, \infty)$ satisfying

$$\|f(x) - C(x)\| \le \mu\zeta(x, 0) \tag{6.14}$$

for all $x \in X$.

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} m^{3n} f\left(\frac{x}{m^n}\right) = C(x) \tag{6.15}$$

for all $x \in X$.

(3) $d(f, C) \le d(f, Jf)/(1 - L)$ with $f \in \Omega$, which implies the inequality

$$d(f, C) \le \frac{L}{|2m^3| - |2m^3|L}. \tag{6.16}$$

This implies that inequality (6.3) holds.

By (6.1) and (6.2), we obtain

$$\begin{aligned}\left\|C(mx \pm y) - mC(x \pm y) - 2\left(m^3 - m\right)C(x)\right\| &\le \lim_{n\to\infty} |m|^{3n} \zeta\left(\frac{x}{m^n}, \frac{y}{m^n}\right) \\ &\le \lim_{n\to\infty} |m|^{3n} \cdot \frac{L^n}{|m|^{3n}} \zeta(x, y)\end{aligned} \tag{6.17}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So,

$$C(mx \pm y) = mC(x \pm y) + 2\left(m^3 - m\right)C(x) \tag{6.18}$$

for all $x, y \in X$. Thus, the mapping $C : X \to Y$ is cubic, as desired. □

Corollary 6.2. *Let $\theta \ge 0$, and let r be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying inequality*

$$\left\|f(mx \pm y) - mf(x \pm y) - 2\left(m^3 - m\right)f(x)\right\| \le \theta(\|x\|^r + \|y\|^r) \tag{6.19}$$

for all $x, y \in X$. Then, the limit $C(x) = \lim_{n\to\infty} m^{3n} f(x/m^n)$ exists for all $x \in X$ and $C : X \to Y$ is a unique cubic mapping such that

$$\|f(x) - C(x)\| \leq \frac{\left|2m^3\right|\theta\|x\|^r}{|2m^3|^{r+1} - |2m^3|^2}. \tag{6.20}$$

for all $x \in X$.

Proof. The proof follows from Theorem 6.1 by taking

$$\zeta(x, y) = \theta\left(\|x\|^r + \|y\|^r\right) \tag{6.21}$$

for all $x, y \in X$. In fact, if we choose $L = |2m^3|^{1-r}$, then we get the desired result. □

Theorem 6.3. *Let $\zeta : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\zeta(x, y) \leq \left|m^3\right| L\zeta\left(\frac{x}{m}, \frac{y}{m}\right) \tag{6.22}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying the inequality (6.2). Then, there is a unique cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\| \leq \frac{\zeta(x, 0)}{|2m^3| - |2m^3|L}. \tag{6.23}$$

Proof. By (5.21), we know that

$$\left\|f(x) - \frac{f(mx)}{m^3}\right\| \leq \frac{\zeta(x, 0)}{|2m^3|} \tag{6.24}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 6.1. Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{m^3} f(mx) \tag{6.25}$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then, $\|g(x) - h(x)\| \leq \epsilon\zeta(x, 0)$ for all $x \in X$, and so

$$\|Jg(x) - Jh(x)\| = \left\|\frac{g(mx)}{m^3} - \frac{h(mx)}{m^3}\right\| \leq \frac{1}{|m^3|}\epsilon\zeta(mx, 0) \leq \frac{1}{|m^3|}\left|m^3\right| L\zeta(x, 0) \tag{6.26}$$

for all $x \in X$. Thus, $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h) \tag{6.27}$$

for all $g, h \in S$. It follows from (6.24) that

$$d(f, Jf) \leq \frac{1}{|2m^3|}. \tag{6.28}$$

By Theorem 2.8, there exists a mapping $C : X \to Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C(mx) = m^3 C(x) \tag{6.29}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}. \tag{6.30}$$

This implies that C is a unique mapping satisfying (6.29) such that there exists $\mu \in (0, \infty)$ satisfying

$$\|f(x) - C(x)\| \leq \mu\zeta(x, 0) \tag{6.31}$$

for all $x \in X$.

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(mx)}{m^3} = C(x) \tag{6.32}$$

for all $x \in X$.

(3) $d(f, C) \leq d(f, Jf)/(1 - L)$ with $f \in \Omega$, which implies the inequality

$$d(f, C) \leq \frac{1}{|2m^3| - |2m^3|L}. \tag{6.33}$$

This implies that inequality (6.23) holds.

The rest of the proof is similar to the proof of Theorem 6.1. □

Corollary 6.4. *Let $\theta \geq 0$, and let r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying* (6.19). *Then, the limit $C(x) = \lim_{n\to\infty}(f(m^n x)/m^{3n})$ exists for all $x \in X$ and $C : X \to Y$ is a unique cubic mapping such that*

$$\|f(x) - C(x)\| \leq \frac{\theta\|x\|^r}{|2m^3| - |2m^3|^r} \tag{6.34}$$

for all $x \in X$.

Proof. The proof follows from Theorem 6.3 by taking

$$\zeta(x, y) = \theta(\|x\|^r + \|y\|^r) \tag{6.35}$$

for all $x, y \in X$. In fact, if we choose $L = |2m^3|^{r-1}$, then we get the desired result. □

Remark 6.5. In Corollaries 6.2 and 6.4, if we assume that $\zeta(x, y) = \theta(\|x\|^r \cdot \|y\|^r)$ or $\zeta(x, y) = \theta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s)$, then we get Ulam-Gavruta-Rassias product stability and JMRassias mixed product-sum stability, respectively. But, since we put $y = 0$ in this functional equation, the Ulam-Gavruta-Rassias product stability and JMRassias mixed product-sum stability corollaries will be obvious. Meanwhile, the JMRassias mixed product-sum stability when $r + s = 3$ is an open question.

7. Conclusion

We linked here four different disciplines, namely, the random normed spaces, non-Archimedean normed spaces, functional equations, and fixed point theory. We established the generalized Hyers-Ulam stability of the functional equation (1.9) in random and non-Archimedean normed spaces.

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