

Research Article

Iterative Methods for the Sum of Two Monotone Operators

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We introduce an iterative for finding the zeros point of the sum of two monotone operators. We prove that the suggested method converges strongly to the zeros point of the sum of two monotone operators.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be a single-valued nonlinear mapping and let $B : H \rightarrow 2^H$ be a multivalued mapping. The “so-called” quasi-variational inclusion problem is to find a $u \in 2^H$ such that

$$0 \in Ax + Bx. \quad (1.1)$$

The set of solutions of (1.1) is denoted by $(A + B)^{-1}(0)$. A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions; see, for instance, [1–4]. The problem (1.1) includes many problems as special cases.

- (1) If $B = \partial\phi : H \rightarrow 2^H$, where $\phi : H \rightarrow R \cup +\infty$ is a proper convex lower semicontinuous function and $\partial\phi$ is the subdifferential of ϕ , then the variational inclusion problem (1.1) is equivalent to find $u \in H$ such that

$$\langle Au, y - u \rangle + \phi(y) - \phi(u) \geq 0, \quad \forall y \in H, \quad (1.2)$$

which is called the mixed quasi-variational inequality (see, Noor [5]).

(2) If $B = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \rightarrow [0, \infty]$ is the indicator function of C , that is,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \quad (1.3)$$

then the variational inclusion problem (1.1) is equivalent to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.4)$$

This problem is called Hartman-Stampacchia variational inequality (see, e.g., [6]).

Recently, Zhang et al. [7] introduced a new iterative scheme for finding a common element of the set of solutions to the inclusion problem, and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng et al. [8] introduced another iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping. For some related works, please see [9–27] and the references therein.

Inspired and motivated by the works in the literature, in this paper, we introduce an iterative for solving the problem (1.1). We prove that the suggested method converges strongly to the zeros point of the sum of two monotone operators $A + B$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H . Recall that a mapping $A : C \rightarrow H$ is said to be α -inverse strongly-monotone if and if only

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \quad (2.1)$$

for some $\alpha > 0$ and for all $x, y \in C$. It is known that if A is α -inverse strongly monotone, then

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\| \quad (2.2)$$

for all $x, y \in C$.

Let B be a mapping of H into 2^H . The effective domain of B is denoted by $\text{dom}(B)$, that is,

$$\text{dom}(B) = \{x \in H : Bx \neq \emptyset\}. \quad (2.3)$$

A multivalued mapping B is said to be a monotone operator on H if and if only

$$\langle x - y, u - v \rangle \geq 0 \quad (2.4)$$

for all $x, y \in \text{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if and if only its graph is not strictly contained in the graph of any other monotone operator on H . Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator:

$$J_\lambda^B = (I + \lambda B)^{-1} : H \rightarrow \text{dom}(B), \quad (2.5)$$

which is called the resolvent of B for λ . It is known that the resolvent J_λ^B is firmly nonexpansive, that is,

$$\|J_\lambda^B x - J_\lambda^B y\|^2 \leq \langle J_\lambda^B x - J_\lambda^B y, x - y \rangle \quad (2.6)$$

for all $x, y \in C$ and $B^{-1}0 = F(J_\lambda^B)$ for all $\lambda > 0$.

The following resolvent identity is well known: for $\lambda > 0$ and $\mu > 0$, there holds the following identity:

$$J_\lambda^B x = J_\mu^B \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^B x \right), \quad x \in H. \quad (2.7)$$

We use the following notation:

- (i) $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x ;
- (ii) $x_n \rightarrow x$ stands for the strong convergence of (x_n) to x .

We need the following lemmas for the next section.

Lemma 2.1 (see [28]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let the mapping $A : C \rightarrow H$ be α -inverse strongly monotone and let $\lambda > 0$ be a constant. Then, one has*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (2.8)$$

In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.2 (see [29]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1. \quad (2.9)$$

Suppose that

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n \quad (2.10)$$

for all $n \geq 0$ and

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (2.11)$$

Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.3 (see [30]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n, \quad (2.12)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Results

In this section, we will prove our main result.

Theorem 3.1. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let A be an α -inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H , such that the domain of B is included in C . Let $J_{\lambda}^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Suppose that $(A + B)^{-1}0 \neq \emptyset$. For $u \in C$ and given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{\lambda_n}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) \quad (3.1)$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0, 1)$, and $\{\beta_n\} \subset (0, 1)$ satisfy

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_n \alpha_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (iii) $a \leq \lambda_n \leq b$ where $[a, b] \subset (0, 2\alpha)$ and $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.1) converges strongly to $\tilde{x} = P_{(A+B)^{-1}0}(u)$.

Proof. First, we choose any $z \in (A + B)^{-1}0$. Note that

$$z = J_{\lambda_n}^B (z - \lambda_n(1 - \alpha_n)Az) = J_{\lambda_n}^B (\alpha_n z + (1 - \alpha_n)(z - \lambda_n Az)) \quad (3.2)$$

for all $n \geq 0$. Since J_{λ}^B is nonexpansive for all $\lambda > 0$, we have

$$\begin{aligned} & \left\| J_{\lambda_n}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) - z \right\|^2 \\ &= \left\| J_{\lambda_n}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) - J_{\lambda_n}^B (\alpha_n z + (1 - \alpha_n)(z - \lambda_n Az)) \right\|^2 \\ &\leq \left\| (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) - (\alpha_n z + (1 - \alpha_n)(z - \lambda_n Az)) \right\|^2 \\ &= \left\| (1 - \alpha_n)((x_n - \lambda_n A x_n) - (z - \lambda_n Az)) + \alpha_n(u - z) \right\|^2. \end{aligned} \quad (3.3)$$

Since A is α -inverse strongly monotone, we get

$$\begin{aligned}
& \|(1 - \alpha_n)((x_n - \lambda_n Ax_n) - (z - \lambda_n Az)) + \alpha_n(u - z)\|^2 \\
& \leq (1 - \alpha_n)\|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 + \alpha_n\|u - z\|^2 \\
& = (1 - \alpha_n)\|(x_n - z) - \lambda_n(Ax_n - Az)\|^2 + \alpha_n\|u - z\|^2 \\
& = (1 - \alpha_n)\left(\|x_n - z\|^2 - 2\lambda_n\langle Ax_n - Az, x_n - z \rangle + \lambda_n^2\|Ax_n - Az\|^2\right) + \alpha_n\|u - z\|^2 \quad (3.4) \\
& \leq (1 - \alpha_n)\left(\|x_n - z\|^2 - 2\alpha\lambda_n\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2\right) + \alpha_n\|u - z\|^2 \\
& = (1 - \alpha_n)\left(\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2\right) + \alpha_n\|u - z\|^2.
\end{aligned}$$

By (3.3) and (3.4), we obtain

$$\begin{aligned}
& \left\| J_{\lambda_n}^B(\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n Ax_n)) - z \right\|^2 \\
& \leq (1 - \alpha_n)\left(\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2\right) + \alpha_n\|u - z\|^2 \quad (3.5) \\
& \leq (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n\|u - z\|^2.
\end{aligned}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 & = \left\| \beta_n(x_n - z) + (1 - \beta_n)\left(J_{\lambda_n}^B(\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n Ax_n)) - z\right) \right\|^2 \\
& \leq \beta_n\|x_n - z\|^2 + (1 - \beta_n)\left\| J_{\lambda_n}^B(\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n Ax_n)) - z \right\|^2 \\
& \leq \beta_n\|x_n - z\|^2 + (1 - \beta_n)\left((1 - \alpha_n)\|x_n - z\|^2 + \alpha_n\|u - z\|^2\right) \quad (3.6) \\
& = [1 - (1 - \beta_n)\alpha_n]\|x_n - z\|^2 + (1 - \beta_n)\alpha_n\|u - z\|^2 \\
& \leq \max\{\|x_n - z\|^2, \|u - z\|^2\}.
\end{aligned}$$

By induction, we have

$$\|x_{n+1} - z\| \leq \max\{\|x_0 - z\|, \|u - z\|\}. \quad (3.7)$$

Therefore, $\{x_n\}$ is bounded. We deduce immediately that $\{Ax_n\}$ is also bounded. Set $u_n = \alpha_n u + (1 - \alpha_n)(x_n - \lambda_n Ax_n)$ for all n . Then $\{u_n\}$ and $\{J_{\lambda_n}^B u_n\}$ are bounded.

Next, we estimate $\|J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_n}^B u_n\|$. In fact, we have

$$\begin{aligned}
& \left\| J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_n}^B u_n \right\| \\
&= \left\| J_{\lambda_{n+1}}^B (\alpha_{n+1} u + (1 - \alpha_{n+1})(x_{n+1} - \lambda_{n+1} A x_{n+1})) - J_{\lambda_n}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) \right\| \\
&\leq \left\| J_{\lambda_{n+1}}^B (\alpha_{n+1} u + (1 - \alpha_{n+1})(x_{n+1} - \lambda_{n+1} A x_{n+1})) - J_{\lambda_{n+1}}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) \right\| \\
&\quad + \left\| J_{\lambda_{n+1}}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) - J_{\lambda_n}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) \right\| \\
&\leq \|(\alpha_{n+1} u + (1 - \alpha_{n+1})(x_{n+1} - \lambda_{n+1} A x_{n+1})) - (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n))\| \\
&\quad + \left\| J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n \right\| \\
&\leq \|(I - \lambda_{n+1} A)x_{n+1} - (I - \lambda_{n+1} A)x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\| \\
&\quad + \alpha_{n+1} (\|u\| + \|x_{n+1}\| + \lambda_{n+1} \|A x_{n+1}\|) + \alpha_n (\|u\| + \|x_n\| + \lambda_n \|A x_n\|) + \left\| J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n \right\|.
\end{aligned} \tag{3.8}$$

Since $I - \lambda_{n+1} A$ is nonexpansive for $\lambda_{n+1} \in (0, 2\alpha)$, we have $\|(I - \lambda_{n+1} A)x_{n+1} - (I - \lambda_{n+1} A)x_n\| \leq \|x_{n+1} - x_n\|$. By the resolvent identity (2.7), we have

$$J_{\lambda_{n+1}}^B u_n = J_{\lambda_n}^B \left(\frac{\lambda_n}{\lambda_{n+1}} u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \right) J_{\lambda_{n+1}}^B u_n \right). \tag{3.9}$$

It follows that

$$\begin{aligned}
\left\| J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n \right\| &= \left\| J_{\lambda_n}^B \left(\frac{\lambda_n}{\lambda_{n+1}} u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \right) J_{\lambda_{n+1}}^B u_n \right) - J_{\lambda_n}^B u_n \right\| \\
&\leq \left\| \left(\frac{\lambda_n}{\lambda_{n+1}} u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \right) J_{\lambda_{n+1}}^B u_n \right) - u_n \right\| \\
&\leq \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \left\| u_n - J_{\lambda_{n+1}}^B u_n \right\|.
\end{aligned} \tag{3.10}$$

So,

$$\begin{aligned}
\left\| J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_n}^B u_n \right\| &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\| + \alpha_{n+1} (\|u\| + \|x_{n+1}\| + \lambda_{n+1} \|A x_{n+1}\|) \\
&\quad + \alpha_n (\|u\| + \|x_n\| + \lambda_n \|A x_n\|) + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \left\| u_n - J_{\lambda_{n+1}}^B u_n \right\|.
\end{aligned} \tag{3.11}$$

Thus,

$$\limsup_{n \rightarrow \infty} \left(\left\| J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_n}^B u_n \right\| - \|x_{n+1} - x_n\| \right) \leq 0. \tag{3.12}$$

From Lemma 2.2, we get

$$\lim_{n \rightarrow \infty} \left\| J_{\lambda_n}^B u_n - x_n \right\| = 0. \quad (3.13)$$

Consequently, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \left\| J_{\lambda_n}^B u_n - x_n \right\| = 0. \quad (3.14)$$

From (3.5) and (3.6), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \left\| J_{\lambda_n}^B (\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n)) - z \right\|^2 \\ &\leq (1 - \beta_n) \left\{ (1 - \alpha_n) \left(\|x_n - z\|^2 + \lambda_n (\lambda_n - 2\alpha) \|A x_n - A z\|^2 \right) + \alpha_n \|u - z\|^2 \right\} \\ &\quad + \beta_n \|x_n - z\|^2 \\ &= [1 - (1 - \beta_n) \alpha_n] \|x_n - z\|^2 + (1 - \beta_n) \lambda_n (\lambda_n - 2\alpha) \|A x_n - A z\|^2 \\ &\quad + (1 - \beta_n) \alpha_n \|u - z\|^2 \\ &\leq \|x_n - z\|^2 + (1 - \beta_n) \lambda_n (\lambda_n - 2\alpha) \|A x_n - A z\|^2 + (1 - \beta_n) \alpha_n \|u - z\|^2. \end{aligned} \quad (3.15)$$

It follows that

$$\begin{aligned} &(1 - \beta_n) \lambda_n (2\alpha - \lambda_n) \|A x_n - A z\|^2 \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1 - \beta_n) \alpha_n \|u - z\|^2 \\ &\leq (\|x_n - z\| - \|x_{n+1} - z\|) \|x_{n+1} - x_n\| + (1 - \beta_n) \alpha_n \|u - z\|^2. \end{aligned} \quad (3.16)$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, and $\liminf_{n \rightarrow \infty} (1 - \beta_n) \lambda_n (2\alpha - \lambda_n) > 0$, we have

$$\lim_{n \rightarrow \infty} \|A x_n - A z\| = 0. \quad (3.17)$$

Put $\tilde{x} = P_{(A+B)^{-1}0}(u)$. Set $v_n = x_n - (\lambda_n / (1 - \alpha_n))(A x_n - A \tilde{x})$ for all n . Take $z = \tilde{x}$ in (3.17) to get $\|A x_n - A \tilde{x}\| \rightarrow 0$. First, we prove $\limsup_{n \rightarrow \infty} \langle u - \tilde{x}, v_n - \tilde{x} \rangle \leq 0$. We take a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - \tilde{x}, v_n - \tilde{x} \rangle = \lim_{i \rightarrow \infty} \langle u - \tilde{x}, v_{n_i} - \tilde{x} \rangle. \quad (3.18)$$

It is clear that $\{v_{n_i}\}$ is bounded due to the boundedness of $\{x_n\}$ and $\|A x_n - A \tilde{x}\| \rightarrow 0$. Then, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to some point $w \in C$.

Hence, $\{x_{n_i}\}$ also converges weakly to w because of $\|v_{n_i} - x_{n_i}\| \rightarrow 0$. By the similar argument as that in [31], we can show that $w \in (A + B)^{-1}0$. This implies that

$$\limsup_{n \rightarrow \infty} \langle u - \tilde{x}, v_n - \tilde{x} \rangle = \lim_{j \rightarrow \infty} \langle u - \tilde{x}, v_{n_j} - \tilde{x} \rangle = \langle u - \tilde{x}, w - \tilde{x} \rangle. \quad (3.19)$$

Note that $\tilde{x} = P_{(A+B)^{-1}0}(u)$. Then, $\langle u - \tilde{x}, w - \tilde{x} \rangle \leq 0$, $w \in (A + B)^{-1}0$. Therefore,

$$\limsup_{n \rightarrow \infty} \langle u - \tilde{x}, v_n - \tilde{x} \rangle \leq 0. \quad (3.20)$$

Finally, we prove that $x_n \rightarrow \tilde{x}$. From (3.1), we have

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &\leq \beta_n \|x_n - \tilde{x}\|^2 + (1 - \beta_n) \left\| J_{\lambda_n}^B u_n - \tilde{x} \right\|^2 \\ &= \beta_n \|x_n - \tilde{x}\|^2 + (1 - \beta_n) \left\| J_{\lambda_n}^B u_n - J_{\lambda_n}^B (\tilde{x} - (1 - \alpha_n) \lambda_n A \tilde{x}) \right\|^2 \\ &\leq \beta_n \|x_n - \tilde{x}\|^2 + (1 - \beta_n) \|u_n - (\tilde{x} - (1 - \alpha_n) \lambda_n A \tilde{x})\|^2 \\ &= \beta_n \|x_n - \tilde{x}\|^2 + (1 - \beta_n) \|\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n A x_n) - (\tilde{x} - (1 - \alpha_n) \lambda_n A \tilde{x})\|^2 \\ &= \beta_n \|x_n - \tilde{x}\|^2 + (1 - \beta_n) \|(1 - \alpha_n)((x_n - \lambda_n A x_n) - (\tilde{x} - \lambda_n A \tilde{x})) + \alpha_n(u - \tilde{x})\|^2 \\ &= \beta_n \|x_n - \tilde{x}\|^2 + (1 - \beta_n) \\ &\quad \times \left((1 - \alpha_n)^2 \|(x_n - \lambda_n A x_n) - (\tilde{x} - \lambda_n A \tilde{x})\|^2 \right. \\ &\quad \left. + 2\alpha_n(1 - \alpha_n) \langle u - \tilde{x}, (x_n - \lambda_n A x_n) - (\tilde{x} - \lambda_n A \tilde{x}) \rangle + \alpha_n^2 \|u - \tilde{x}\|^2 \right) \\ &\leq \beta_n \|x_n - \tilde{x}\|^2 + (1 - \beta_n) \\ &\quad \times \left((1 - \alpha_n) \|x_n - \tilde{x}\|^2 + 2\alpha_n(1 - \alpha_n) \langle u - \tilde{x}, x_n - \lambda_n(Ax_n - A\tilde{x}) - \tilde{x} \rangle \right. \\ &\quad \left. + \alpha_n^2 \|u - \tilde{x}\|^2 \right) \\ &\leq [1 - (1 - \beta_n)\alpha_n] \|x_n - \tilde{x}\|^2 + (1 - \beta_n)\alpha_n \left\{ 2(1 - \alpha_n) \langle u - \tilde{x}, v_n - \tilde{x} \rangle + \alpha_n \|u - \tilde{x}\|^2 \right\}. \end{aligned} \quad (3.21)$$

It is clear that $\sum_n (1 - \beta_n)\alpha_n = \infty$ and $\limsup_{n \rightarrow \infty} (2(1 - \alpha_n) \langle u - \tilde{x}, v_n - \tilde{x} \rangle + \alpha_n \|u - \tilde{x}\|^2) \leq 0$. We can therefore apply Lemma 2.3 to conclude that $x_n \rightarrow \tilde{x}$. This completes the proof. \square

4. Applications

Next, we consider the problem for finding the minimum norm solution of a mathematical model related to equilibrium problems. Let C be a nonempty, closed, and convex subset of a Hilbert space and let $G : C \times C \rightarrow R$ be a bifunction satisfying the following conditions:

- (E1) $G(x, x) = 0$ for all $x \in C$;
- (E2) G is monotone, that is, $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
- (E3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;
- (E4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

Then, the mathematical model related to equilibrium problems (with respect to C) is to find $\tilde{x} \in C$ such that

$$G(\tilde{x}, y) \geq 0 \quad (4.1)$$

for all $y \in C$. The set of such solutions \tilde{x} is denoted by $EP(G)$. The following lemma appears implicitly in Blum and Oettli [32].

Lemma 4.1. *Let C be a nonempty, closed, and convex subset of H and let G be a bifunction of $C \times C$ into R satisfying (E1)–(E4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \quad (4.2)$$

The following lemma was given by Combettes and Hirstoaga [33].

Lemma 4.2. *Assume that $G : C \times C \rightarrow R$ satisfies (E1)–(E4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \quad (4.3)$$

for all $x \in H$. Then, the following holds:

- (1) T_r is single valued;
- (2) T_r is a firmly nonexpansive mapping, that is, for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \quad (4.4)$$

- (3) $F(T_r) = EP(G)$;
- (4) $EP(G)$ is closed and convex.

We call such T_r the resolvent of G for $r > 0$. Using Lemmas 4.1 and 4.2, we have the following lemma. See [34] for a more general result.

Lemma 4.3. *Let H be a Hilbert space and let C be a nonempty, closed, and convex subset of H . Let $G : C \times C \rightarrow R$ satisfy (E1)–(E4). Let A_G be a multivalued mapping of H into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z \rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \quad (4.5)$$

Then, $EP(G) = A_G^{-1}(0)$ and A_G is a maximal monotone operator with $\text{dom}(A_G) \subset C$. Further, for any $x \in H$ and $r > 0$, the resolvent T_r of G coincides with the resolvent of A_G ; that is,

$$T_r x = (I + rA_G)^{-1}x. \quad (4.6)$$

Form Lemma 4.3 and Theorems 3.1, we have the following result.

Theorem 4.4. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let G be a bifunction from $C \times C \rightarrow \mathbb{R}$ satisfying (E1)–(E4) and let T_λ be the resolvent of G for $\lambda > 0$. Suppose $EP(G) \neq \emptyset$. For $u \in C$ and given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T_{\lambda_n}(\alpha_n u + (1 - \alpha_n)x_n) \quad (4.7)$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, and $\{\beta_n\} \subset (0, 1)$ satisfy

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_n \alpha_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (iii) $a \leq \lambda_n \leq b$ where $[a, b] \subset (0, \infty)$ and $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $\tilde{x} = P_{EP(G)}(u)$.

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