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Research Article

Existence and Duality of Generalized ε-Vector Equilibrium Problems

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We consider a generalized ε-vector equilibrium problem which contain vector equilibrium problems and vector variational inequalities as special cases. By using the KKM theorem, we obtain some existence theorems for the generalized ε-vector equilibrium problem. We also investigate the duality of this generalized ε-vector equilibrium problem and discuss the equivalence relation between solutions of primal and dual problems.

1. Introduction

It is known that vector equilibrium problems provide a unified model for several different problems appearing in the fields of vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems. Many important results for various kinds of vector equilibrium problems and their extensions have been extensively investigated, see [1–8] and the references therein. Ansari et al. [9] introduced an implicit vector variational problem which contain vector equilibrium problems and vector variational inequalities as special cases. They also established an equivalent relationship between the solution sets for an implicit variational problem and its dual problem. Li and Zhao [10] introduced a dual scheme for a mixed vector equilibrium problem by using the method of Fenchel conjugate functions and established a relationship between the solution sets of the primal and dual problems. Sun and Li [11] introduced a dual scheme for an extended Ky Fan inequality. By using the obtained duality assertions, they also obtained some Farkas-type results which characterize the optimal value of this extended Ky Fan inequality.

From the computational viewpoint, the algorithms proposed in the literature for solving nonlinear optimization problems, in general, can only obtain approximate solutions (ε-optimal solutions) of such problems. In this regard, many researchers have studied optimality conditions for ε-solutions for scalar and vector optimization problems, see [12–16] and the references cited therein.

However, there are very little results for optimality conditions for ε-solution (approximate solution) of vector equilibrium problems. Moreover, the study of approximate vector equilibrium problems is very important since many approximate optimization problems may be considered as their special cases, see [17–19] and the references cited therein. Sach et al. [20] introduced new versions of ε-dual problems of a vector quasi-equilibrium problem with set-valued maps and obtained an ε-duality result between approximate solutions of the primal and dual problems. X. B. Li and S. J. Li [21] considered parametric scalar and vector equilibrium problems and obtained sufficient conditions for Hausdorff continuity and Berge continuity of approximate solution mappings for parametric scalar and vector equilibrium problems. Sun and Li [22] considered a generalized multivalued ε-vector variational inequality, formulated its dual problem, and proved duality results between the primal and dual problems.

To the best of our knowledge, there is no paper to deal with the generalized ε-vector equilibrium problems. Motivated by the work reported in [3–5, 9, 22], in this paper, we first introduce a generalized ε-vector equilibrium problem $(\mathrm{GVEP})_\varepsilon$ and establish an existence theorem for $(\mathrm{GVEP})_\varepsilon$ by using the known KKM Theorem. Then, we discuss duality results between $(\mathrm{GVEP})_\varepsilon$ and its dual problem. We also show that our results on existence contain known results in the literature as special cases.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions and introduce a generalized ε-vector equilibrium problem $(\mathrm{GVEP})_\varepsilon$. In Section 3, by using the known KKM Theorem, we establish an existence theorem for the $(\mathrm{GVEP})_\varepsilon$. As a special case, we also derive some existence results for a generalized ε-vector variational inequality introduced and studied by Sun and Li [22]. In Section 4, we give a dual ε-vector variational inequality $(\mathrm{DVEP})_\varepsilon$ for $(\mathrm{GVEP})_\varepsilon$ and prove an equivalence relation between $(\mathrm{GVEP})_\varepsilon$ and $(\mathrm{DVEP})_\varepsilon$.

2. Mathematical Preliminaries

Throughout this paper, let X and Y be two Banach spaces, and let $\mathcal{L}(X,Y)$ be the set of all linear continuous operators from X to Y. Let $K \subseteq Y$ be a convex cone with a nonempty interior int K. Given $x,\ y \in Y$, we define the following ordering relations:

$$\begin{aligned} y <_K x &\Longleftrightarrow y - x \in -\operatorname{int} K, & y \nless_K x &\Longleftrightarrow y - x \notin -\operatorname{int} K, \\ y \leq_K x &\Longleftrightarrow y - x \in -K, & y \nleq_K x &\Longleftrightarrow y - x \notin -K. \end{aligned} \tag{2.1}$$

Given two sets $A,\ B \subseteq Y$, we also consider the following set ordering relationships:

$$\begin{aligned} A <_K B &\Longleftrightarrow y <_K x, \quad \forall y \in A,\ x \in B, \\ A \nless_K B &\Longleftrightarrow y \nless_K x, \quad \forall y \in A,\ x \in B, \\ A \leq_K B &\Longleftrightarrow y \leq_K x, \quad \forall y \in A,\ x \in B, \\ A \nleq_K B &\Longleftrightarrow y \nleq_K x, \quad \forall y \in A,\ x \in B. \end{aligned} \tag{2.2}$$

Let $\varphi : X \times X \to Y$ and $g : X \to Y$ be two vector-valued mappings satisfying $\varphi(x,x) = 0$, for any $x \in X$. Consider the following generalized ε-vector equilibrium problem $(\mathrm{GVEP})_\varepsilon$:

$$\text{Find } x_0 \in X \text{ such that } \varphi(x_0,x) + g(x) - g(x_0) + \varepsilon\|x - x_0\| \nless_K 0, \text{ for any } x \in X, \qquad (\mathrm{GVEP})_\varepsilon$$

where $\varepsilon \in K$. We say that x_0 is an ε-solution of $(\mathrm{GVEP})_\varepsilon$ if and only if

$$\varphi(x_0,x) + g(x) - g(x_0) + \varepsilon\|x - x_0\| \nless_K 0, \quad \text{for any } x \in X. \tag{2.3}$$

If $\varepsilon = 0$, then $(\mathrm{GVEP})_\varepsilon$ collapses to the following generalized vector equilibrium problem (GVEP), introduced and studied by Li and Zhao [10]:

$$\text{Find } x_0 \in X \text{ such that } \varphi(x_0,x) + g(x) - g(x_0) \nless_K 0, \text{ for any } x \in X. \qquad (\mathrm{GVEP})$$

If $g = 0$ and $\varepsilon = 0$, then $(\mathrm{GVEP})_\varepsilon$ becomes the following vector equilibrium problem (VEP)

$$\text{Find } x_0 \in X \text{ such that } \varphi(x_0,x) \nless_K 0, \text{ for any } x \in X. \qquad (\mathrm{VEP})$$

Let $T : X \to \mathcal{L}(X,Y)$ be a vector-valued mapping. If $\varphi(x_0,x) = \langle T(x_0), x - x_0\rangle$, for any $x \in X$, $(\mathrm{GVEP})_\varepsilon$ collapses to the following generalized ε-vector variational inequality $(\mathrm{GVVI})_\varepsilon$, introduced and studied by Sun and Li [22]:

$$\text{Find } x_0 \in X \text{ such that } \langle T(x_0), x - x_0\rangle + g(x) - g(x_0) + \varepsilon\|x - x_0\| \nless_K 0, \text{ for any } x \in X, \qquad (\mathrm{GVVI})_\varepsilon$$

where $\varepsilon \in K$ and $\langle T(x_0), x - x_0\rangle$ is the evaluation of $T(x_0)$ at $x - x_0$.

In this paper, we consider the generalized ε-vector equilibrium problem $(\mathrm{GVEP})_\varepsilon$ and establish the existence theorem for solutions of $(\mathrm{GVEP})_\varepsilon$. Then, we give a dual ε-vector equilibrium problem $(\mathrm{DVEP})_\varepsilon$ for $(\mathrm{GVEP})_\varepsilon$ and prove an equivalence relation between $(\mathrm{GVEP})_\varepsilon$ and $(\mathrm{DVEP})_\varepsilon$.

At first, let us recall some important definitions.

Definition 2.1 (see [23]). Let Q be a nonempty subset of Y. A point $\hat{y} \in Q$ is said to be a weak maximal point of Q, if there is no $y' \in Q$ such that $\hat{y} <_K y'$. The set of all maximal points of Q is called the weak maximum of Q and is denoted by $\mathrm{WMax}_K Q$. The weak minimum of Q, $\mathrm{WMin}_K Q$, is defined analogously.

Definition 2.2 (see [23]). Let $g : X \to Y$ be a vector-valued mapping. g is said to be K-convex if for any $x_1, x_2 \in X$ and $\alpha \in [0,1]$,

$$g(\alpha x_1 + (1-\alpha)x_2) \leq_K \alpha g(x_1) + (1-\alpha)g(x_2). \tag{2.4}$$

Furthermore, g is said to be K-concave, if $-g$ is K-convex.

Definition 2.3 (see [24]). A vector mapping $g : X \to Y$ is said to be ε-convex with $\varepsilon \in K$ if for any $x_1, x_2 \in X$ and $\alpha \in [0,1]$,

$$g(\alpha x_1 + (1-\alpha)x_2) \leq_K \alpha g(x_1) + (1-\alpha)g(x_2) + \varepsilon\alpha(1-\alpha)\|x_1 - x_2\|. \tag{2.5}$$

Definition 2.4 (see [22]). Let $g : X \to Y$ be an ε-convex mapping with $\varepsilon \in K$.

(i) A set-valued mapping $g^*_{x_0,\varepsilon} : \mathcal{L}(X,Y) \rightrightarrows Y$ defined by

$$g^*_{x_0,\varepsilon}(\Lambda) = \mathrm{WMax}_K\{\langle\Lambda, x\rangle - g(x) - \varepsilon\|x - x_0\| : x \in X\}, \quad \text{for any } \Lambda \in \mathcal{L}(X,Y), \tag{2.6}$$

is called the ε-conjugate mapping of g at $x_0 \in X$.

(ii) A set-valued mapping $g^{**}_{x_0,\varepsilon} : X \rightrightarrows Y$ defined by

$$g^{**}_{x_0,\varepsilon}(x) = \mathrm{WMax}_K\{\langle\Lambda, x\rangle - g^*_{x_0,\varepsilon}(\Lambda) : \Lambda \in \mathcal{L}(X,Y)\}, \quad \text{for any } x \in X, \tag{2.7}$$

is called the ε-biconjugate mapping of g at $x_0 \in X$.

Definition 2.5 (see [25]). Let $g : X \to Y$ be a given mapping. A subdifferential of g at $x_0 \in X$ is defined as

$$\partial g(x_0) = \{\Lambda \in \mathcal{L}(X,Y) : g(x) - g(x_0) \not<_K \langle\Lambda, x - x_0\rangle, \forall x \in X\}. \tag{2.8}$$

Definition 2.6 (see [26]). Let $g : X \to Y$ be a given mapping with $\varepsilon \in K$. An ε-subdifferential of g at $x_0 \in X$ is defined as

$$\partial_\varepsilon g(x_0) = \{\Lambda \in \mathcal{L}(X,Y) : g(x) - g(x_0) \not<_K \langle\Lambda, x - x_0\rangle - \varepsilon\|x - x_0\|,\ \forall x \in X\}. \tag{2.9}$$

Remark 2.7. (i) It is easy to note that, when $\varepsilon = 0$, $\partial_\varepsilon g(x_0) = \partial g(x_0)$ and $\partial_\varepsilon g(x_0) = \partial(g + \varepsilon\|\cdot - x_0\|)(x_0)$.

(ii) Note that Li and Guo [26, Section 3] have given some existence theorems of ε-subdifferential for a vector-valued mapping under the condition that the cone K is connected (i.e., $K \cup (-K) = Y$).

Next, we give KKM theorem needed for the proof of the existence results.

Definition 2.8 (see [27]). A set-valued mapping $G : X \rightrightarrows Y$ is called the KKM mapping if, for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X,

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i), \tag{2.10}$$

where $\mathrm{co}\{x_1, x_2, \ldots, x_n\}$ is the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Theorem 2.9 (see [27, KKM Theorem]). *Let $G : X \rightrightarrows Y$ be a KKM mapping. If for each $x \in X$, $G(x)$ is closed and is compact for at least one $x \in X$, then, $\bigcap_{x \in X} G(x) \neq \emptyset$.*

3. Existence Theorems for $(\mathbf{GVEP})_\varepsilon$

In this section, we prove some existence results for the generalized ε-vector equilibrium problem $(\mathrm{GVEP})_\varepsilon$ by using the known KKM theorem. As a special case, we derive some existence results for the generalized ε-vector variational inequality $(\mathrm{GVVI})_\varepsilon$.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(i) $\varphi : X \times X \to Y$ *and* $g : X \to Y$ *are two continuous mappings;*

(ii) *for any* $y \in X$, $B_y := \{x \in X : \varphi(x, y) + g(y) - g(x) + \varepsilon\|x - y\| <_K 0\}$ *is convex;*

(iii) *there exists a nonempty compact subset* C *and* $x' \in C$ *such that for any* $y \in X \setminus C$, *one has*

$$\varphi(x', y) + g(y) - g(x') + \varepsilon\|x' - y\| <_K 0. \tag{3.1}$$

Then, the generalized ε*-vector equilibrium problem* $(GVEP)_\varepsilon$ *is solvable.*

Proof. Define a set-valued mapping $G : X \rightrightarrows X$ by: for any $x \in X$,

$$G(x) := \{y \in X : \varphi(x, y) + g(y) - g(x) + \varepsilon\|x - y\| \not<_K 0\}. \tag{3.2}$$

We first prove that G is a KKM mapping. In fact, suppose that $G(x)$ is not a KKM mapping. Then, there exists a finite subset $\{x'_1, x'_2, \dots, x'_n\}$ of X such that

$$\mathrm{co}\{x'_1, x'_2, \dots, x'_n\} \not\subset \bigcup_{i=1}^{n} G(x'_i). \tag{3.3}$$

Let $y' \in \mathrm{co}\{x'_1, x'_2, \dots, x'_n\}$. Then, $y' = \sum_{i=1}^{n} \alpha_i x'_i$ for some $\alpha_i \in [0, 1]$, $i = 1, 2, \dots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$ and

$$y' \notin \bigcup_{i=1}^{n} G(x'_i). \tag{3.4}$$

So, for any $i \in \{1, 2, \dots, n\}$, we have

$$\varphi(x'_i, y') + g(y') - g(x'_i) + \varepsilon\|x'_i - y'\| <_K 0. \tag{3.5}$$

Hence,

$$\{x'_1, x'_2, \dots, x'_n\} \subset B_{y'}. \tag{3.6}$$

Since $B_{y'}$ is convex, we have

$$\mathrm{co}\{x'_1, x'_2, \dots, x'_n\} \subset B_{y'}. \tag{3.7}$$

By $y' \in \operatorname{co}\{x'_1, x'_2, \ldots, x'_n\}$, we have

$$y' \in B_{y'}. \tag{3.8}$$

Thus, we have

$$\varphi(y', y') + g(y') - g(y') + \varepsilon\|y' - y'\| <_K 0, \tag{3.9}$$

which is a contradiction. Therefore, G is a KKM mapping.

Next, we prove that, for any $x \in X$, $G(x)$ is closed. Indeed, let any sequence $\{y_n\}$ with $y_n \in G(x)$ and $y_n \to y_0$. Then,

$$\varphi(x, y_n) + g(y_n) - g(x) + \varepsilon\|x - y_n\| \nless_K 0. \tag{3.10}$$

Taking limit in (3.10) with φ and g being two continuous mappings, we have

$$\varphi(x, y_0) + g(y_0) - g(x) + \varepsilon\|x - y_0\| \nless_K 0. \tag{3.11}$$

Thus, $y_0 \in G(x)$ and $G(x)$ is closed.

Assumption (iii) implies that $G(x') \subset C$. Hence, $G(x')$ is compact. Therefore, the assumptions of Theorem 2.9 hold. By Theorem 2.9, we have

$$\bigcap_{x \in X} G(x) \neq \emptyset. \tag{3.12}$$

Hence, there exist $x_0 \in X$ such that

$$\varphi(x, x_0) + g(x_0) - g(x) + \varepsilon\|x - x_0\| \nless_K 0, \tag{3.13}$$

for any $x \in X$. This completes the proof. □

Now, we give an example to illustrate Theorem 3.1.

Example 3.2. Let $X = R$, $Y = R^2$, $C = [-1, 1]$, $K = R^2_+$, and $\varepsilon = (0, 1)$. Suppose that $\varphi : R \times R \to R^2$ and $g : R \to R^2$ are defined by

$$\varphi(x, y) = \left(-2x^2 + y^2, -2x^2 + y^2 - |x - y|\right), \tag{3.14}$$

$$g(x) = \left(-2x^2, -2x^2\right), \tag{3.15}$$

respectively. Then, assumption (i) in Theorem 3.1 is clearly satisfied. It can be checked that, for any $y \in R$,

$$\begin{aligned} B_y &= \Big\{x \in X : \left(-2x^2 + y^2, -2x^2 + y^2 - |x-y|\right) + \left(-2y^2, -2y^2\right) - \left(-2x^2, -2x^2\right) \\ &\qquad +(0, |x-y|) <_{R_+^2} 0\Big\} \\ &= \Big\{x \in X : \left(-y^2, -y^2\right) <_{R_+^2} 0\Big\}. \end{aligned} \tag{3.16}$$

Obviously, B_y is a convex set. Thus, assumption (ii) of Theorem 3.1 holds.

Obviously, $[-1,1]$ is a nonempty compact set. Let $x' = 1 \in [-1,1]$. For any $y \in R \setminus [-1,1]$, we have

$$\begin{aligned} \varphi(x', y) + g(y) - g(x') + \varepsilon\|x' - y\| &= \left(-2 + y^2, -2 + y^2 - |1-y|\right) + \left(-2y^2, -2y^2\right) \\ &\quad - (-2,-2) + (0, |1-y|) \\ &= \left(-y^2, -y^2\right) \\ &<_{R_+^2} 0. \end{aligned} \tag{3.17}$$

Thus, assumption (iii) of Theorem 3.1 holds. Obviously, $x_0 = 0$ is a solution of $(\mathrm{GVEP})_\varepsilon$.

Remark 3.3. When the function g is K-concave and $\varphi(\cdot, y)$ is K-convex, we have that condition (ii) of Theorem 3.1 holds, that is, B_y is convex.

Indeed, let $x_1, x_2 \in B_y$ and $\alpha \in [0,1]$. Then, we have

$$\begin{aligned} \varphi(x_1, y) + g(y) - g(x_1) + \varepsilon\|x_1 - y\| \in -\operatorname{int} K, \\ \varphi(x_2, y) + g(y) - g(x_2) + \varepsilon\|x_2 - y\| \in -\operatorname{int} K. \end{aligned} \tag{3.18}$$

Since g is K-concave and $\varphi(\cdot, y)$ is K-convex, we have

$$\begin{aligned} &\varphi(x_1 + (1-\alpha)x_2, y) + g(y) - g(\alpha x_1 + (1-\alpha)x_2) + \varepsilon\|\alpha x_1 + (1-\alpha)x_2 - y\| \\ &\quad \in \alpha\left[\varphi(x_1, y) + g(y) - g(x_1) + \varepsilon\|x_1 - y\|\right] \\ &\qquad + (1-\alpha)\left[\varphi(x_2, y) + g(y) - g(x_2) + \varepsilon\|x_2 - y\|\right] - K - K - K \\ &\quad \subseteq -\operatorname{int} K - \operatorname{int} K - K - K - K \\ &\quad = -\operatorname{int} K. \end{aligned} \tag{3.19}$$

So, B_y is convex and we have the following corollary.

Corollary 3.4. *Suppose that the following conditions are satisfied:*

(i) $\varphi(\cdot, y) : X \to Y$ *is a continuous* K*-convex mapping and* $g : X \to Y$ *is a continuous* K*-concave mapping;*

(ii) *there exists a nonempty compact subset C and $x' \in C$ such that for any $y \in X \setminus C$, one has*

$$\varphi(x', y) + g(y) - g(x') + \varepsilon\|x' - y\| <_K 0. \tag{3.20}$$

Then, the generalized ε-vector equilibrium problem $(GVEP)_\varepsilon$ is solvable.

By Theorem 3.1 and Corollary 3.4, we can derive the following existence results for the generalized ε-vector variational inequality $(\mathrm{GVVI})_\varepsilon$.

Theorem 3.5. *Suppose that the following conditions are satisfied:*

(i) *$T : X \to \mathcal{L}(X, Y)$ and $g : X \to Y$ are two continuous mappings;*

(ii) *for any $y \in X$, $B_y := \{x \in X : \langle T(y), x - y\rangle + g(x) - g(y) + \varepsilon\|x - y\| <_K 0\}$ is convex;*

(iii) *there exists a nonempty compact subset C and $x' \in C$ such that for any $y \in X \setminus C$, one has*

$$\langle T(y), x' - y\rangle + g(x') - g(y) + \varepsilon\|x' - y\| <_K 0. \tag{3.21}$$

Then, the generalized ε-vector variational inequality $(GVVI)_\varepsilon$ is solvable.

Remark 3.6. If the function g *is* K-convex, we can prove that condition (ii) of Theorem 3.5 holds by the similar method of Remark 3.3.

Corollary 3.7. *Suppose that the following conditions are satisfied:*

(i) *$T : X \rightrightarrows \mathcal{L}(X, Y)$ is a continuous mapping, and $g : X \to Y$ is a continuous K-convex mapping;*

(ii) *there exists a nonempty compact subset C and $x' \in C$ such that for any $y \in X \setminus C$, one has*

$$\langle T(y), x' - y\rangle + g(x') - g(y) + \varepsilon\|x' - y\| <_K 0. \tag{3.22}$$

Then, the generalized ε-vector variational inequality $(GVVI)_\varepsilon$ is solvable.

Remark 3.8. (i) In case when T is a set-valued mapping, similar results of Theorem 3.5 and Corollary 3.7 have been studied by Sun and Li [22].

(ii) Theorem 3.5 and Corollary 3.7 can be viewed as an extension of Theorem 1 in Yang [3].

4. Dual Results for $(\mathbf{GVEP})_\varepsilon$

In this section, we first introduce the dual generalized ε-vector equilibrium problems for $(\mathrm{GVEP})_\varepsilon$. Then, we establish an equivalence between primal and dual problems.

We first recall that g is said to be externally stable at $x \in X$ if $g(x) \in g^{**}_{x_0,\varepsilon}(x)$. The external stability was introduced in [25] when the vector conjugate function is defined via the set of efficient points.

The following result plays an important role in our study.

Lemma 4.1. *Let $g : X \to Y$ and $x_0 \in X$. Then,*

$$\Lambda \in \partial_\varepsilon g(x_0) \Longleftrightarrow \langle \Lambda, x_0 \rangle - g(x_0) \in g^*_{x_0,\varepsilon}(\Lambda). \tag{4.1}$$

Proof. It follows from the definitions of the ε-conjugate function and the ε-subdifferentials that

$$\Lambda \in \partial_\varepsilon g(x_0) \tag{4.2}$$

if and only if

$$g(x) - g(x_0) \nless_K \langle \Lambda, x - x_0 \rangle - \varepsilon \|x - x_0\|, \quad \text{for any } x \in X, \tag{4.3}$$

equivalently,

$$\langle \Lambda, x \rangle - g(x) - \varepsilon \|x - x_0\| \ngtr_K \langle \Lambda, x_0 \rangle - g(x_0), \quad \text{for any } x \in X, \tag{4.4}$$

which means that

$$\langle \Lambda, x_0 \rangle - g(x_0) \in g^*_{x_0,\varepsilon}(\Lambda), \tag{4.5}$$

and the proof is completed. □

It is shown in [28] that, if $\varphi(x_0, \cdot)$ is convex, then $\partial\varphi_{x_0}(x) \neq \emptyset$, where $\partial\varphi_{x_0}(x)$ denotes the subdifferential of φ with respect to its second component. Now, we define the dual generalized ε-vector equilibrium problem $(\text{DVEP})_\varepsilon$ of $(\text{GVEP})_\varepsilon$ as follows:

$$\text{Find } x_0 \in X,\ -\Gamma_0 \in \partial\varphi_{x_0}(x_0),\ \text{and } y_0 \in g^*_{x_0,\varepsilon}(\Gamma_0) \text{ such that } y_0 - \langle \Gamma_0, x_0 \rangle \ngtr_K g^*_{x_0,\varepsilon}(\Gamma) - \langle \Gamma, x_0 \rangle, \quad \text{for any } \Gamma \in \mathcal{L}(X, Y). \tag{$(\text{DVEP})_\varepsilon$}$$

Moreover, (x_0, Γ_0) is called a solution of $(\text{DVEP})_\varepsilon$.

Remark 4.2. In [8], Sach et al. gave some examples to prove that the dual problems of [3, 5] are not suitable for the duality property of vector variational inequalities, and hence, all possible applications of them cannot be seen to be justified. This fact shows that, when dealing with duality in vector variational inequality problems which are generalizations of those considered in [3, 5], we must use dual problems different from those of [3, 5]. So, in this section we consider this problem $(\text{DVEP})_\varepsilon$ called the dual problem of $(\text{GVEP})_\varepsilon$.

In the following, we will discuss the relationships between the solutions of $(\text{GVEP})_\varepsilon$ and $(\text{DVEP})_\varepsilon$.

Theorem 4.3. *Suppose that g is externally stable at $x_0 \in X$. If x_0 is a solution of $(GVEP)_\varepsilon$ and $\partial_\varepsilon(\varphi_{x_0} + g)(x_0) = \partial\varphi_{x_0}(x_0) + \partial_\varepsilon g(x_0)$, then, there exists $\Gamma_0 \in \mathcal{L}(X, Y)$ such that (x_0, Γ_0) is a solution of $(DVEP)_\varepsilon$.*

Proof. If x_0 is a solution of $(\mathrm{GVEP})_\varepsilon$, then,

$$\varphi(x_0, x) + g(x) - g(x_0) + \varepsilon\|x - x_0\| \nless_K 0, \quad \text{for any } x \in X. \tag{4.6}$$

So, it is easy to see that

$$\left(\varphi(x_0, x) + g(x)\right) - \left(\varphi(x_0, x_0) + g(x_0)\right) \nless_K \langle 0, x - x_0\rangle - \varepsilon\|x - x_0\|, \quad \text{for any } x \in X, \tag{4.7}$$

which means that

$$0 \in \partial_\varepsilon\left(\varphi_{x_0} + g\right)(x_0). \tag{4.8}$$

Hence,

$$0 \in \partial\varphi_{x_0}(x_0) + \partial_\varepsilon g(x_0), \tag{4.9}$$

or, equivalently, there exists $\Gamma_0 \in \mathcal{L}(X, Y)$ such that

$$\Gamma_0 \in -\partial\varphi_{x_0}(x_0) \cap \partial_\varepsilon g(x_0). \tag{4.10}$$

Then, from (4.1), we get

$$\langle \Gamma_0, x_0\rangle - g(x_0) \in g^*_{x_0,\varepsilon}(\Gamma_0). \tag{4.11}$$

Since g is externally stable at $x_0 \in X$, we have

$$g(x_0) \in g^{**}_{x_0,\varepsilon}(x_0) = \mathrm{WMax}_K\left\{\langle \Gamma, x_0\rangle - g^*_{x_0,\varepsilon}(\Gamma) : \Gamma \in \mathcal{L}(X, Y)\right\}. \tag{4.12}$$

Thus,

$$g(x_0) \nless_K \langle \Gamma, x_0\rangle - g^*_{x_0,\varepsilon}(\Gamma), \quad \text{for any } \Gamma \in \mathcal{L}(X, Y). \tag{4.13}$$

By (4.11), there exists $y_0 \in g^*_{x_0,\varepsilon}(\Gamma_0)$ such that

$$y_0 = \langle \Gamma_0, x_0\rangle - g(x_0). \tag{4.14}$$

By (4.13) and (4.14), we get

$$y_0 - \langle \Gamma_0, x_0\rangle \ngtr_K g^*_{x_0,\varepsilon}(\Gamma) - \langle \Gamma, x_0\rangle, \quad \text{for any } \Gamma \in \mathcal{L}(X, Y), \tag{4.15}$$

and the proof is completed. □

Theorem 4.4. *If (x_0, Γ_0) is a solution of $(DVEP)_\varepsilon$ and $\partial_\varepsilon(\varphi_{x_0} + g)(x_0) = \partial\varphi_{x_0}(x_0) + \partial_\varepsilon g(x_0)$, then, x_0 is a solution of $(GVEP)_\varepsilon$.*

Proof. This is obtained by inverting the reasoning in the proof of Theorem 4.3 step by step. □

Remark 4.5. Although the equality of the approximate subdifferentials used in Theorems 4.3 and 4.4 is difficult to be verified, Li and Guo [26, Section 5] have given some sufficient conditions for the validity of the equality under the condition that the cone K is connected (i.e., $K \cup (-K) = Y$).

Now, we give an example to illustrate Theorem 4.4.

Example 4.6. Let $X = R$, $Y = R^2$, $K = R^2_+$, *and* $\varepsilon = (1,1)$. Suppose that $\varphi : R \times R \to R^2$ and $g : R \to R^2$ are defined by

$$\varphi(x_0, x) = (|x_0|, |x_0|), \tag{4.16}$$

$$g(x) = \left(x - |x|, x^2 - |x|\right), \tag{4.17}$$

respectively. Then, for $x_0 = 0$, we have

$$\begin{gathered} \partial_\varepsilon(\varphi_{x_0} + g)(x_0) = \partial_\varepsilon g(x_0) = \{1, 0\}, \\ \partial\varphi_{x_0}(x_0) = \{0, 0\}. \end{gathered} \tag{4.18}$$

Obviously,

$$\partial_\varepsilon(\varphi_{x_0} + g)(x_0) = \partial\varphi_{x_0}(x_0) + \partial_\varepsilon g(x_0). \tag{4.19}$$

Moreover, for any $\Gamma = (\Gamma_1, \Gamma_2) \in R^2$,

$$g^*_{x_0,\varepsilon}(\Gamma) = \begin{cases} \left\{(y_1, y_2) \in R^2 : y_2 \le \dfrac{\Gamma_2^2}{4}, y_1 = 0\right\}, & \text{if } \Gamma_1 = 1, \\ \left\{(y_1, y_2) \in R^2 : y_2 = -\dfrac{1}{(\Gamma_1 - 1)^2} y_1^2 + \dfrac{\Gamma_2}{\Gamma_1 - 1} y_1, y_1 \ge \dfrac{\Gamma_2(\Gamma_1 - 1)}{2}\right\}, & \text{if } \Gamma_1 \ne 1. \end{cases} \tag{4.20}$$

Then, for $\Gamma_0 = (0,0) \in \partial\varphi_{x_0}(x_0)$, we obtain that

$$\begin{gathered} g^*_{x_0,\varepsilon}(\Gamma_0) = \left\{(y_1, y_2) \in R^2 : y_2 = -y_1^2,\ y_1 \ge 0\right\}, \\ y_0 = \langle \Gamma_0, x_0 \rangle - g(x_0) = (0,0) \in g^*_{x_0,\varepsilon}(\Gamma_0). \end{gathered} \tag{4.21}$$

Obviously, (x_0, Γ_0) is a solution of $(\mathrm{DVEP})_\varepsilon$ and x_0 is a solution of $(\mathrm{GVEP})_{2\varepsilon}$.

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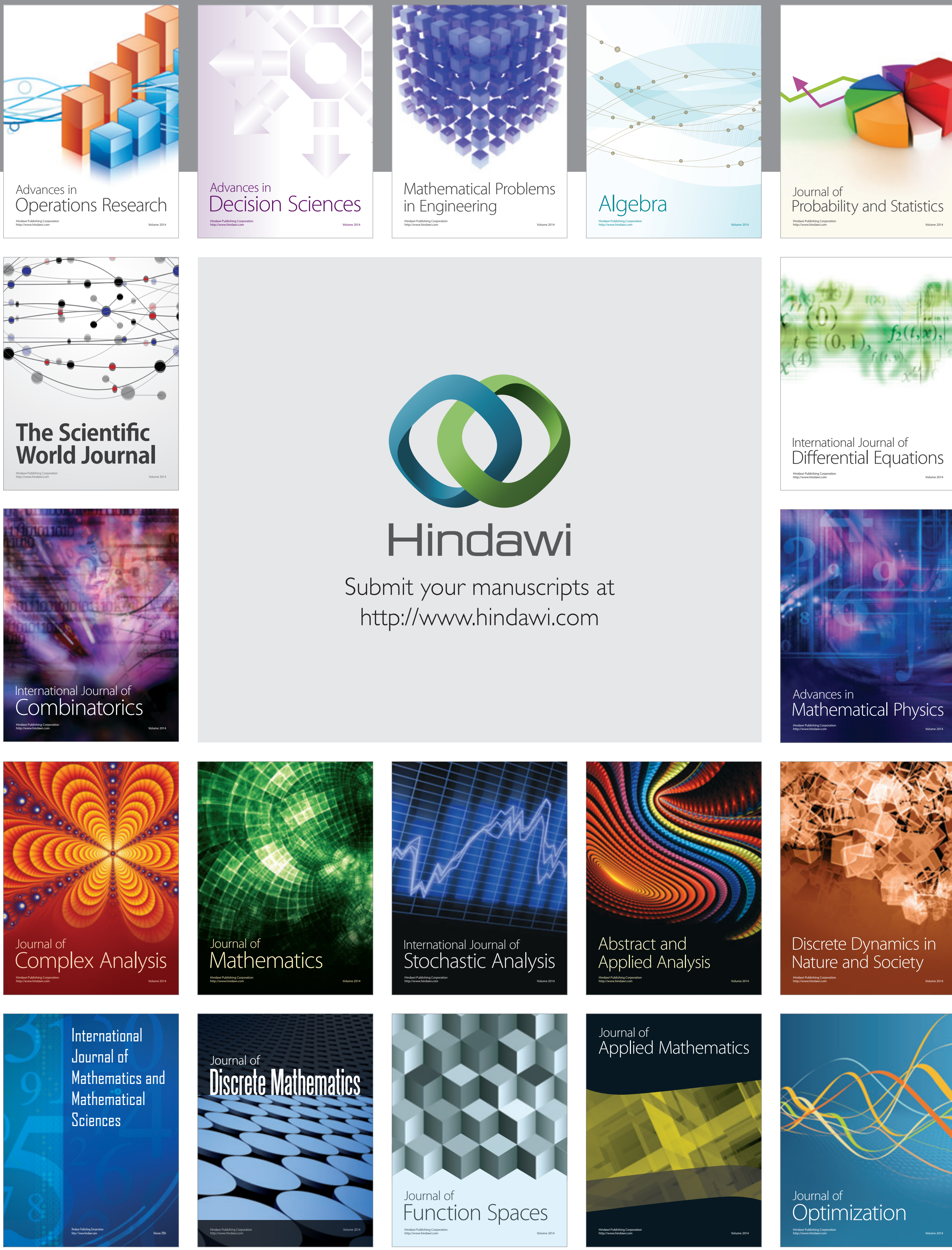
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