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Research Article

Fixed Point and Asymptotic Analysis of Cellular Neural Networks

Xianghong Lai[1] and Yutian Zhang[2]

[1] *School of Economics & Management, Nanjing University of Information Science & Technology, Nanjing 210044, China*
[2] *School of Mathematics & Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China*

Correspondence should be addressed to Yutian Zhang, ytzhang81@163.com

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We firstly employ the fixed point theory to study the stability of cellular neural networks without delays and with time-varying delays. Some novel and concise sufficient conditions are given to ensure the existence and uniqueness of solution and the asymptotic stability of trivial equilibrium at the same time. Moreover, these conditions are easily checked and do not require the differentiability of delays.

1. Introduction

Cellular neural networks (CNNs) were firstly proposed by Chua and Yang in 1988 [1, 2] and have become a research focus for their numerous successful applications in various fields such as optimization, linear, and nonlinear programming, associative memory, pattern recognition, and computer vision. Owing to the finite switching speed of neurons and amplifiers in the implementation of neural networks, it turns out that the time delays are inevitable and therefore the model of delayed cellular neural networks (DCNNs) is of greater realistic significance. Research on the dynamic behaviors of CNNs and DCNNs has received much attention, and nowadays there have been a large number of achievements reported [3–5].

In fact, besides delay effects, stochastic and impulsive as well as diffusing effects are also likely to exist in the neural networks. As a result, they have formed complex CNNs including impulsive delayed reaction-diffusion CNNs, stochastic delayed reaction-diffusion CNNs, and so forth. One can refer to [6–11] for the relevant researches. Synthesizing the existing publications about complex CNNs, we find that Lyapunov method is the primary technique. However, we also notice that there exist lots of difficulties in the applications of corresponding results to practical problems and so it does seem that new methods are needed to address those difficulties.

Encouragingly, Burton and other authors have recently applied the fixed point theory to investigate the stability of deterministic systems and obtained more applicable results, for example, see the monograph [12] and the papers [13–24]. Furthermore, there has been found that the fixed point theory is also effective to the stability analysis of stochastic (delayed) differential equations, see [25–31]. Particularly, in [26–28], Luo used the fixed point theory to study the exponential stability of mild solutions of stochastic partial differential equations with bounded delays and with infinite delays. In [29, 30], Sakthivel and Luo used the fixed point theory to investigate the asymptotic stability in pth moment of mild solutions to nonlinear impulsive stochastic partial differential equations with bounded delays and with infinite delays. In [31], Luo used the fixed point theory to study the exponential stability of stochastic Volterra-Levin equations. With these motivations, we wonder if we can use the fixed point theory to study the stability of complex neural networks, thus obtaining more applicable results.

In the present paper, we aim to discuss the asymptotic stability of CNNs and DCNNs. Our method is based on the contraction mapping theory, which is different from the usual method of Lyapunov theory. Some new and easily checked algebraic criteria are presented ensuring the existence and uniqueness of solution and the asymptotic stability of trivial equilibrium at the same time. These sufficient conditions do not require even the differentiability of delays, let alone the monotone decreasing behavior of delays.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space and $\|\cdot\|$ represent the Euclidean norm. $\mathcal{N} \triangleq \{1, 2, \ldots, n\}$. $\mathrm{R}_+ = [0, \infty)$. $C(X, Y)$ corresponds to the space of continuous mappings from the topological space X to the topological space Y.

In this paper, we consider the cellular neural network described by

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right), \quad t \geq 0, \tag{2.1}$$

$$x_i(0) = x_{0i}, \tag{2.2}$$

and the following cellular neural network with time-varying delays as

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(t - \tau_j(t)\right)\right), \quad t \geq 0 \tag{2.3}$$

$$x_i(s) = \varphi_i(s), \quad -\tau \leq s \leq 0, \tag{2.4}$$

where $i \in \mathcal{N}$ and n is the number of neurons in the neural network. $x_i(t)$ corresponds to the state of the ith neuron at time t. $\mathbf{x}_0 = (x_{01}, \ldots, x_{0n})^{\mathrm{T}} \in \mathrm{R}^n$. $f_j(x_j(t))$ denotes the activation function of the jth neuron at time t and $g_j(x_j(t - \tau_j(t)))$ is the activation function of the jth neuron at time $t - \tau_j(t)$. The constant b_{ij} represents the connection weight of the jth neuron on the ith neuron at time t. The constant $a_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state when disconnected from the network and external inputs. The constant c_{ij} represents the connection strength of the jth neuron on the ith neuron at time $t - \tau_j(t)$, where $\tau_j(t)$ corresponds to the transmission delay along the axon of the jth neuron and satisfies $0 \le \tau_j(t) \le \tau$ (τ is a constant). $f_j(\cdot)$, $g_j(\cdot) \in C(\mathrm{R}, \mathrm{R})$. $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s))^{\mathrm{T}} \in \mathrm{R}^n$ and $\varphi_i(s) \in C([-\tau, 0], \mathrm{R})$. Denote $|\varphi| = \sup_{s\in[-\tau,0]} \|\varphi(s)\|$.

Throughout this paper, we always assume that $f_j(0) = g_j(0) = 0$ for $j \in \mathcal{N}$ and therefore (2.1) and (2.3) admit a trivial equilibrium $\mathbf{x} = 0$.

Denote by $\mathbf{x}(t; 0, \mathbf{x}_0) = (x_1(t; 0, x_{01}), \ldots, x_n(t; 0, x_{0n}))^{\mathrm{T}} \in \mathrm{R}^n$ the solution of (2.1) with the initial condition (2.2) and denote by $\mathbf{x}(t; s, \varphi) = (x_1(t; s, \varphi_1), \ldots, x_n(t; s, \varphi_n))^{\mathrm{T}} \in \mathrm{R}^n$ the solution of (2.3) with the initial condition (2.4).

Definition 2.1 (see [32]). The trivial equilibrium $\mathbf{x} = 0$ of (2.1) is said to be stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial condition $\mathbf{x}_0$ satisfying $\|\mathbf{x}_0\| < \delta$,

$$\|\mathbf{x}(t; 0, \mathbf{x}_0)\| < \varepsilon, \quad t \ge 0. \tag{2.5}$$

Definition 2.2 (see [32]). The trivial equilibrium $\mathbf{x} = 0$ of (2.1) is said to be asymptotically stable if it is stable and for any $\mathbf{x}_0 \in \mathrm{R}^n$,

$$\lim_{t\to\infty} \|\mathbf{x}(t; 0, \mathbf{x}_0)\| = 0. \tag{2.6}$$

Definition 2.3 (see [32]). The trivial equilibrium $\mathbf{x} = 0$ of (2.3) is said to be stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial condition $\varphi(s) \in C([-\tau, 0], \mathrm{R}^n)$ satisfying $|\varphi| < \delta$,

$$\|\mathbf{x}(t; s, \varphi)\| < \varepsilon, \quad t \ge 0. \tag{2.7}$$

Definition 2.4 (see [32]). The trivial equilibrium $\mathbf{x} = 0$ of (2.3) is said to be asymptotically stable if it is stable and for any initial condition $\varphi(s) \in C([-\tau, 0], \mathrm{R}^n)$,

$$\lim_{t\to\infty} \|\mathbf{x}(t; s, \varphi)\| = 0. \tag{2.8}$$

The consideration of this paper is based on the following fixed point theorem.

Lemma 2.5 (see [33]). *Let Υ be a contraction operator on a complete metric space Θ, then there exists a unique point $\zeta \in \Theta$ for which $\Upsilon(\zeta) = \zeta$.*

3. Asymptotic Stability of Cellular Neural Networks

In this section, we will simultaneously consider the existence and uniqueness of solution to (2.1)-(2.2) and the asymptotic stability of trivial equilibrium $\mathbf{x} = 0$ of (2.1) by means of the contraction mapping principle. Before proceeding, we firstly introduce the following assumption:

(A1) There exist nonnegative constants l_j such that for $\eta, \upsilon \in \mathrm{R}$,

$$\left|f_j(\eta) - f_j(\upsilon)\right| \le l_j\left|\eta - \upsilon\right|, \quad j \in \mathcal{N}. \tag{3.1}$$

Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$, where $\mathcal{S}_i$ $(i \in \mathcal{N})$ is the space consisting of continuous functions $\phi_i(t) : \mathrm{R}_+ \to \mathrm{R}$ such that $\phi_i(0) = x_{0i}$ and $\phi_i(t) \to 0$ as $t \to \infty$, here x_{0i} is the same as defined in Section 2. Also $\mathcal{S}$ is a complete metric space when it is equipped with a metric defined by

$$d\left(\overline{\mathbf{q}}(t), \overline{\mathbf{h}}(t)\right) = \sup_{t\ge 0}\sum_{i=1}^{n}\left|q_i(t) - h_i(t)\right|, \tag{3.2}$$

where $\overline{\mathbf{q}}(t) = (q_1(t), \dots, q_n(t)) \in \mathcal{S}$ and $\overline{\mathbf{h}}(t) = (h_1(t), \dots, h_n(t)) \in \mathcal{S}$.

Theorem 3.1. *Assume the condition (A1) holds. If the following inequalities hold*

$$\sum_{i=1}^{n}\left\{\frac{1}{a_i}\max_j\left\{\left|b_{ij}l_j\right|\right\}\right\} < 1, \qquad \max_{i\in\mathcal{N}}\{\lambda_i\} < \frac{1}{\sqrt{n}}, \tag{3.3}$$

where $\lambda_i = (1/a_i)\sum_{j=1}^{n}|b_{ij}l_j|$, *then the trivial equilibrium* $\mathbf{x} = 0$ *of* (2.1) *is asymptotically stable.*

Proof. Multiplying both sides of (2.1) with $\mathrm{e}^{a_i t}$ gives

$$\mathrm{d}\mathrm{e}^{a_i t}x_i(t) = \mathrm{e}^{a_i t}\mathrm{d}x_i(t) + a_i x_i(t)\mathrm{e}^{a_i t}\mathrm{d}t = \mathrm{e}^{a_i t}\sum_{j=1}^{n}b_{ij}f_j\left(x_j(t)\right)\mathrm{d}t, \quad t \ge 0,\ i \in \mathcal{N}, \tag{3.4}$$

which yields after integrating from 0 to t as

$$x_i(t) = x_{0i}\,\mathrm{e}^{-a_i t} + \mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n}b_{ij}f_j\left(x_j(s)\right)\mathrm{d}s, \quad t \ge 0,\ i \in \mathcal{N}. \tag{3.5}$$

Now, for any $\overline{\mathbf{y}}(t) = (y_1(t), \dots, y_n(t)) \in \mathcal{S}$, we define the following operator Φ acting on $\mathcal{S}$ as

$$\Phi(\overline{\mathbf{y}})(t) = \left(\Phi\left(y_1\right)(t), \dots, \Phi\left(y_n\right)(t)\right), \quad t \ge 0, \tag{3.6}$$

where

$$\Phi\left(y_i\right)(t) = x_{0i}\,\mathrm{e}^{-a_i t} + \mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n}b_{ij}f_j\left(y_j(s)\right)\mathrm{d}s, \quad i \in \mathcal{N}. \tag{3.7}$$

The following proof is based on the contraction mapping principle, which can be divided into two steps as follows.

Step 1. We need to prove $\Phi(\mathcal{S}) \subset \mathcal{S}$. Recalling the construction of $\mathcal{S}$, we know that it is necessary to show the continuity of Φ on $[0,\infty)$ and $\Phi(y_i)(t)|_{t=0} = x_{0i}$ as well as $\lim_{t\to\infty}\Phi(y_i)(t) = 0$ for $i \in \mathcal{N}$.

From (3.7), it is easy to see $\Phi(y_i)(t)|_{t=0} = x_{0i}$. Moreover, for a fixed time $t_1 \geq 0$, we have

$$\begin{aligned}\Phi(y_i)(t_1+r) - \Phi(y_i)(t_1) = x_{0i}\ \mathrm{e}^{-a_i(t_1+r)}& \\ - x_{0i}\ \mathrm{e}^{-a_it_1} + \mathrm{e}^{-a_i(t_1+r)}\int_0^{t_1+r} &\mathrm{e}^{a_is}\sum_{j=1}^n b_{ij}f_j(y_j(s))\mathrm{d}s \\ - \mathrm{e}^{-a_it_1}\int_0^{t_1}\mathrm{e}^{a_is}&\sum_{j=1}^n b_{ij}f_j(y_j(s))\mathrm{d}s.\end{aligned} \tag{3.8}$$

It is not difficult to see that $\Phi(y_i)(t_1+r) - \Phi(y_i)(t_1) \to 0$ as $r \to 0$ which implies Φ is continuous on $[0,\infty)$.

Next we shall prove $\lim_{t\to\infty}\Phi(y_i)(t) = 0$ for $y_i(t) \in \mathcal{S}_i$. Since $y_j(t) \in \mathcal{S}_j$, we get $\lim_{t\to\infty}y_j(t) = 0$. Then for any $\varepsilon > 0$, there exists a $T_j > 0$ such that $s \geq T_j$ implies $|y_j(s)| < \varepsilon$. Choose $T^* = \max_{j\in\mathcal{N}}\{T_j\}$. It is then derived form (A1) that

$$\begin{aligned}&\mathrm{e}^{-a_it}\int_0^t \mathrm{e}^{a_is}\sum_{j=1}^n b_{ij}f_j(y_j(s))\mathrm{d}s \\ &\quad\leq \mathrm{e}^{-a_it}\int_0^t \mathrm{e}^{a_is}\sum_{j=1}^n\{|b_{ij}l_j||y_j(s)|\}\mathrm{d}s \\ &\quad= \mathrm{e}^{-a_it}\int_0^{T^*} \mathrm{e}^{a_is}\sum_{j=1}^n\{|b_{ij}l_j||y_j(s)|\}\mathrm{d}s + \mathrm{e}^{-a_it}\int_{T^*}^t \mathrm{e}^{a_is}\sum_{j=1}^n\{|b_{ij}l_j||y_j(s)|\}\mathrm{d}s \\ &\quad\leq \mathrm{e}^{-a_it}\sum_{j=1}^n\left\{|b_{ij}l_j|\sup_{s\in[0,T^*]}|y_j(s)|\right\}\left\{\int_0^{T^*}\mathrm{e}^{a_is}\mathrm{d}s\right\} + \varepsilon\mathrm{e}^{-a_it}\sum_{j=1}^n\{|b_{ij}l_j|\}\int_{T^*}^t \mathrm{e}^{a_is}\mathrm{d}s \\ &\quad\leq \mathrm{e}^{-a_it}\sum_{j=1}^n\left\{|b_{ij}l_j|\sup_{s\in[0,T^*]}|y_j(s)|\right\}\left\{\int_0^{T^*}\mathrm{e}^{a_is}\mathrm{d}s\right\} + \frac{\varepsilon}{a_i}\sum_{j=1}^n\{|b_{ij}l_j|\}.\end{aligned} \tag{3.9}$$

As $a_i > 0$, we obtain $\mathrm{e}^{-a_it}\int_0^t \mathrm{e}^{a_is}\sum_{j=1}^n b_{ij}f_j(y_j(s))\mathrm{d}s \to 0$ as $t \to \infty$. So $\lim_{t\to\infty}\Phi(y_i)(t) = 0$ for $i \in \mathcal{N}$. We therefore conclude that $\Phi(\mathcal{S}) \subset \mathcal{S}$.

Step 2. We need to prove Φ is contractive. For any $\overline{\mathbf{y}} = (y_1(t), \ldots, y_n(t)) \in \mathcal{S}$ and $\overline{\mathbf{z}} = (z_1(t), \ldots, z_n(t)) \in \mathcal{S}$, we compute

$$\begin{aligned}
&\sup_{t\in[0,T]} \sum_{i=1}^{n} \left|\Phi(y_i)(t) - \Phi(z_i)(t)\right| \\
&\quad\le \sup_{t\in[0,T]} \sum_{i=1}^{n} \left\{ e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \left|b_{ij}\right| \left|f_j(y_j(s)) - f_j(z_j(s))\right| \right\} ds \right\} \\
&\quad\le \sup_{t\in[0,T]} \sum_{i=1}^{n} \left\{ e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \left|b_{ij} l_j\right| \left|y_j(s) - z_j(s)\right| \right\} ds \right\} \\
&\quad\le \sup_{t\in[0,T]} \sum_{i=1}^{n} \left\{ \max_{j\in\mathcal{N}} \left\{ \left|b_{ij} l_j\right| \right\} \sup_{s\in[0,T]} \left\{ \sum_{j=1}^{n} \left|y_j(s) - z_j(s)\right| \right\} e^{-a_i t} \int_0^t e^{a_i s} ds \right\} \\
&\quad\le \sum_{i=1}^{n} \left\{ \frac{1}{a_i} \max_{j\in\mathcal{N}} \left\{ \left|b_{ij} l_j\right| \right\} \right\} \sup_{s\in[0,T]} \left\{ \sum_{j=1}^{n} \left|y_j(s) - z_j(s)\right| \right\}.
\end{aligned} \tag{3.10}$$

As $\sum_{i=1}^{n}\{(1/a_i) \max_{j\in\mathcal{N}}\{|b_{ij}l_j|\}\} < 1$, Φ is a contraction mapping.

Therefore, by the contraction mapping principle, we see there must exist a unique fixed point $\overline{\mathbf{u}}(\cdot)$ of Φ in $\mathcal{S}$ which means $\overline{\mathbf{u}}^{\mathrm{T}}(\cdot)$ is the solution of (2.1)-(2.2) and $\|\overline{\mathbf{u}}^{\mathrm{T}}(\cdot)\| \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we still need to prove that the trivial equilibrium $\mathbf{x} = 0$ of (2.1) is stable. For any $\varepsilon > 0$, from the conditions of Theorem 3.1, we can find δ satisfying $0 < \delta < \varepsilon$ such that $\delta + \max_{i\in\mathcal{N}}\{\lambda_i\}\varepsilon \le \varepsilon/\sqrt{n}$.

Let $\|\mathbf{x}_0\| < \delta$. According to what have been discussed above, we know that there must exist a unique solution $\mathbf{x}(t; 0, \mathbf{x}_0) = (x_1(t; 0, x_{01}), \ldots, x_n(t; 0, x_{0n}))^{\mathrm{T}} \in \mathrm{R}^n$ to (2.1)-(2.2), and

$$x_i(t) = \Phi(x_i)(t) = J_1(t) + J_2(t), \quad t \ge 0, \tag{3.11}$$

where $J_1(t) = x_{0i}\, e^{-a_i t}$, $J_2(t) = e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) ds$.

Suppose there exists $t^* > 0$ such that $\|\mathbf{x}(t^*; 0, \mathbf{x}_0)\| = \varepsilon$ and $\|\mathbf{x}(t; 0, \mathbf{x}_0)\| < \varepsilon$ as $0 \le t < t^*$. It follows from (3.11) that $|x_i(t^*)| \le |J_1(t^*)| + |J_2(t^*)|$.

As $|J_1(t^*)| = |x_{0i} e^{-a_i t^*}| \le \delta$ and $|J_2(t^*)| \le e^{-a_i t^*} \int_0^{t^*} e^{a_i s} \sum_{j=1}^{n} |b_{ij} l_j x_j(s)| ds < (\varepsilon/a_i) \sum_{j=1}^{n} |b_{ij} l_j|$, we obtain $|x_i(t^*)| < \delta + \lambda_i \varepsilon$. Hence

$$\|\mathbf{x}(t^*; 0, \mathbf{x}_0)\|^2 = \sum_{i=1}^{n} \left\{ |x_i(t^*)|^2 \right\} < \sum_{i=1}^{n} \left\{ |\delta + \lambda_i \varepsilon|^2 \right\} \le n \left| \delta + \max_{i\in\mathcal{N}}\{\lambda_i\}\varepsilon \right|^2 \le \varepsilon^2. \tag{3.12}$$

This contradicts to the assumption of $\|\mathbf{x}(t^*; 0, \mathbf{x}_0)\| = \varepsilon$. Therefore, $\|\mathbf{x}(t; 0, \mathbf{x}_0)\| < \varepsilon$ holds for all $t \ge 0$. This completes the proof. $\square$

4. Asymptotic Stability of Delayed Cellular Neural Networks

In this section, we will simultaneously consider the existence and uniqueness of solution to (2.3)-(2.4) and the asymptotic stability of trivial equilibrium $\mathbf{x} = 0$ of (2.3) by means of the contraction mapping principle. Before proceeding, we give the assumption as follows.

(A2) There exist nonnegative constants k_j such that for $\eta, \upsilon \in \mathrm{R}$,

$$\left|g_j(\eta) - g_j(\upsilon)\right| \le k_j\left|\eta - \upsilon\right|, \quad j \in \mathcal{N}. \tag{4.1}$$

Let $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, where $\mathcal{H}_i$ $(i \in \mathcal{N})$ is the space consisting of continuous functions $\phi_i(t) : [-\tau, \infty) \to \mathrm{R}$ such that $\phi_i(s) = \varphi_i(s)$ on $s \in [-\tau, 0]$ and $\phi_i(t) \to 0$ as $t \to \infty$, here $\varphi_i(s)$ is the same as defined in Section 2. Also $\mathcal{H}$ is a complete metric space when it is equipped with a metric defined by

$$d\left(\overline{\mathbf{q}}(t), \overline{\mathbf{h}}(t)\right) = \sup_{t \ge -\tau} \sum_{i=1}^{n} \left|q_i(t) - h_i(t)\right|, \tag{4.2}$$

where $\overline{\mathbf{q}}(t) = (q_1(t), \dots, q_n(t)) \in \mathcal{H}$ and $\overline{\mathbf{h}}(t) = (h_1(t), \dots, h_n(t)) \in \mathcal{H}$.

Theorem 4.1. *Assume the conditions (A1)-(A2) hold. If the following inequalities hold*

$$\sum_{i=1}^{n} \left\{ \frac{1}{a_i} \left(\max_{j \in \mathcal{N}} \left|b_{ij} l_j\right| + \max_{j \in \mathcal{N}} \left|c_{ij} k_j\right| \right) \right\} < 1, \qquad \max_{i \in \mathcal{N}} \left\{\lambda_i^*\right\} < \frac{1}{\sqrt{n}} \tag{4.3}$$

where $\lambda_i^* = (1/a_i) \sum_{j=1}^{n} |b_{ij} l_j| + (1/a_i) \sum_{j=1}^{n} |c_{ij} k_j|$, *then the trivial equilibrium* $\mathbf{x} = 0$ *of* (2.3) *is asymptotically stable.*

Proof. Multiplying both sides of (2.3) with $\mathrm{e}^{a_i t}$ gives

$$\begin{aligned} \mathrm{d}\mathrm{e}^{a_i t} x_i(t) &= \mathrm{e}^{a_i t} \mathrm{d}x_i(t) + a_i x_i(t) \mathrm{e}^{a_i t} \mathrm{d}t \\ &= \mathrm{e}^{a_i t} \left\{ \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t - \tau_j(t))) \right\} \mathrm{d}t, \end{aligned} \tag{4.4}$$

which yields after integrating from 0 to t as

$$x_i(t) = \varphi_i(0) \mathrm{e}^{-a_i t} + \mathrm{e}^{-a_i t} \int_0^t \mathrm{e}^{a_i s} \left\{ \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s - \tau_j(s))) \right\} \mathrm{d}s. \tag{4.5}$$

Now for any $\overline{\mathbf{y}}(t) = (y_1(t), \dots, y_n(t)) \in \mathcal{H}$, we define the following operator π acting on $\mathcal{H}$

$$\pi(\overline{\mathbf{y}})(t) = \left(\pi(y_1)(t), \dots, \pi(y_n)(t)\right), \tag{4.6}$$

where

$$\pi(y_i)(t) = \varphi_i(0)\mathrm{e}^{-a_i t} + \mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\left\{\sum_{j=1}^{n} b_{ij} f_j(y_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(y_j(s-\tau_j(s)))\right\}\mathrm{d}s, \quad t \geq 0, \tag{4.7}$$

and $\pi(y_i)(s) = \varphi_i(s)$ on $s \in [-\tau, 0]$ for $i \in \mathcal{N}$.

Similar to the proof of Theorem 3.1, we shall apply the contraction mapping principle to prove Theorem 4.1. The subsequent proof can be divided into two steps.

Step 1. We need prove $\pi(\mathcal{H}) \subset \mathcal{H}$. To prove $\pi(\mathcal{H}) \subset \mathcal{H}$, it is necessary to show the continuity of π on $[-\tau, \infty)$ and $\lim_{t\to\infty}\pi(y_i)(t) = 0$ for $y_i(t) \in \mathcal{H}_i$ and $i \in \mathcal{N}$. In light of (4.7), we have, for a fixed time $t_1 \geq 0$,

$$\pi(y_i)(t_1 + r) - \pi(y_i)(t_1) = I_1 + I_2 + I_{3,} \tag{4.8}$$

where

$$\begin{aligned}
I_1 &= \varphi_i(0)\ \mathrm{e}^{-a_i(t_1+r)} - \varphi_i(0)\ \mathrm{e}^{-a_i t_1}, \\
I_2 &= \mathrm{e}^{-a_i(t_1+r)}\int_0^{t_1+r} \mathrm{e}^{a_i s}\sum_{j=1}^{n} b_{ij} f_j(y_j(s))\mathrm{d}s - \mathrm{e}^{-a_i t_1}\int_0^{t_1} \mathrm{e}^{a_i s}\sum_{j=1}^{n} b_{ij} f_j(y_j(s))\mathrm{d}s, \\
I_3 &= \mathrm{e}^{-a_i(t_1+r)}\int_0^{t_1+r} \mathrm{e}^{a_i s}\sum_{j=1}^{n} c_{ij} g_j(y_j(s-\tau_j(s)))\mathrm{d}s - \mathrm{e}^{-a_i t_1}\int_0^{t_1} \mathrm{e}^{a_i s}\sum_{j=1}^{n} c_{ij} g_j(y_j(s-\tau_j(s)))\mathrm{d}s.
\end{aligned} \tag{4.9}$$

It is easy to see that $\lim_{r\to 0}\{\pi(y_i)(t_1 + r) - \pi(y_i)(t_1)\} = 0$. Thus, π is continuous on $[0, \infty)$. Noting $\varphi_i(s) \in C([-\tau, 0], \mathrm{R})$ and $\pi(y_i)(0) = \varphi_i(0)$, we obtain π is indeed continuous on $[-\tau, \infty)$.

Next, we will prove $\lim_{t\to\infty}\pi(y_i)(t) = 0$ for $y_i(t) \in \mathcal{H}_i$. As we did in Section 3, we know $\lim_{t\to\infty}\mathrm{e}^{-a_i t} = 0$ and $\mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n} b_{ij} f_j(y_j(s))\mathrm{d}s \to 0$ as $t \to \infty$. In what follows, we will show $\mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s} \sum_{j=1}^{n} c_{ij} g_j(y_j(s-\tau_j(s)))\mathrm{d}s \to 0$ as $t \to \infty$. In fact, since $y_j(t) \in \mathcal{H}_j$, we have $\lim_{t\to\infty} y_j(t) = 0$. Then for any $\varepsilon > 0$, there exists a $T_j{}' > 0$ such that $s \geq T_j{}' - \tau$ implies $|y_j(s)| < \varepsilon$. Select $\widehat{T} = \max_{j\in\mathcal{N}}\{T_j{}'\}$. It is then derived from (A2) that

$$\begin{aligned}
&\mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n} c_{ij} g_j(y_j(s-\tau_j(s)))\mathrm{d}s \\
&\quad\leq \mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n}\{|c_{ij}k_j||y_j(s-\tau_j(s))|\}\mathrm{d}s \\
&\quad= \mathrm{e}^{-a_i t}\int_0^{\widehat{T}} \mathrm{e}^{a_i s}\sum_{j=1}^{n}\{|c_{ij}k_j||y_j(s-\tau_j(s))|\}\mathrm{d}s \\
&\qquad+ \mathrm{e}^{-a_i t}\int_{\widehat{T}}^t \mathrm{e}^{a_i s}\sum_{j=1}^{n}\{|c_{ij}k_j||y_j(s-\tau_j(s))|\}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathrm{e}^{-a_i t}\sum_{j=1}^{n}\left\{|c_{ij}k_j|\sup_{s\in[-\tau,\widehat{T}]}|y_j(s)|\right\}\left\{\int_0^{\widehat{T}}\mathrm{e}^{a_i s}\mathrm{d}s\right\}+\varepsilon\mathrm{e}^{-a_i t}\sum_{j=1}^{n}\{|c_{ij}k_j|\}\int_{\widehat{T}}^{t}\mathrm{e}^{a_i s}\mathrm{d}s\\
&\leq \mathrm{e}^{-a_i t}\sum_{j=1}^{n}\left\{|c_{ij}k_j|\sup_{s\in[-\tau,\widehat{T}]}|y_j(s)|\right\}\left\{\int_0^{\widehat{T}}\mathrm{e}^{a_i s}\mathrm{d}s\right\}+\frac{\varepsilon}{a_i}\sum_{j=1}^{n}\{|c_{ij}k_j|\}.
\end{aligned}
\tag{4.10}
$$

As $\lim_{t\to\infty}\mathrm{e}^{-a_i t}=0$, we obtain $\mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n}c_{ij}g_j(y_j(s-\tau_j(s)))\mathrm{d}s \to 0$ as $t\to\infty$, which leads to $\lim_{t\to\infty}\pi(y_i)(t)=0$ for $y_i(t)\in\mathscr{H}_i$ and $i\in\mathcal{N}$. We therefore conclude $\pi(\mathscr{H})\subset\mathscr{H}$.

Step 2. We need to prove π is contractive. For any $\overline{\mathbf{y}}=(y_1(t),\dots,y_n(t))\in\mathscr{H}$ and $\overline{\mathbf{z}}=(z_1(t),\dots,z_n(t))\in\mathscr{H}$, we estimate

$$
\begin{aligned}
&\sum_{i=1}^{n}|\pi(y_i)(t)-\pi(z_i)(t)|\\
&\quad\leq\sum_{i=1}^{n}\left\{\mathrm{e}^{-a_i t}\int_0^t\mathrm{e}^{a_i s}\sum_{j=1}^{n}[|b_{ij}||f_j(y_j(s))-f_j(z_j(s))|]\mathrm{d}s\right\}\\
&\quad\quad+\sum_{i=1}^{n}\left\{\mathrm{e}^{-a_i t}\int_0^t\mathrm{e}^{a_i s}\sum_{j=1}^{n}[|c_{ij}||g_j(y_j(s-\tau_j(s)))-g_j(z_j(s-\tau_j(s)))|]\mathrm{d}s\right\}\\
&\quad\leq\sum_{i=1}^{n}\left\{\mathrm{e}^{-a_i t}\int_0^t\mathrm{e}^{a_i s}\sum_{j=1}^{n}[|b_{ij}l_j||y_j(s)-z_j(s)|]\mathrm{d}s\right\}\\
&\quad\quad+\sum_{i=1}^{n}\left\{\mathrm{e}^{-a_i t}\int_0^t\mathrm{e}^{a_i s}\sum_{j=1}^{n}[|c_{ij}k_j||y_j(s-\tau_j(s))-z_j(s-\tau_j(s))|]\mathrm{d}s\right\}\\
&\quad\leq\sum_{i=1}^{n}\left\{\max_{j\in\mathcal{N}}|b_{ij}l_j|\sup_{s\in[0,t]}\left\{\sum_{j=1}^{n}|y_j(s)-z_j(s)|\right\}e^{-a_i t}\int_0^t e^{a_i s}\mathrm{d}s\right\}\\
&\quad\quad+\sum_{i=1}^{n}\left\{\max_{j\in\mathcal{N}}|c_{ij}k_j|\sup_{s\in[-\tau,t]}\left\{\sum_{j=1}^{n}|y_j(s)-z_j(s)|\right\}\mathrm{e}^{-a_i t}\int_0^t\mathrm{e}^{a_i s}\mathrm{d}s\right\}\\
&\quad\leq\sum_{i=1}^{n}\left\{\frac{1}{a_i}\max_{j\in\mathcal{N}}|b_{ij}l_j|\right\}\sup_{s\in[0,t]}\left\{\sum_{j=1}^{n}|y_j(s)-z_j(s)|\right\}\\
&\quad\quad+\sum_{i=1}^{n}\left\{\frac{1}{a_i}\max_{j\in\mathcal{N}}|c_{ij}k_j|\right\}\sup_{s\in[-\tau,t]}\left\{\sum_{j=1}^{n}|y_j(s)-z_j(s)|\right\}.
\end{aligned}
\tag{4.11}
$$

Hence,

$$\sup_{t\in[-\tau,T]}\sum_{i=1}^{n}\left|\pi(y_i)(t)-\pi(z_i)(t)\right| \le \left\{\sum_{i=1}^{n}\left\{\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|\right\}+\sum_{i=1}^{n}\left\{\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|\right\}\right\}\sup_{s\in[-\tau,T]}\left\{\sum_{j=1}^{n}\left|y_j(s)-z_j(s)\right|\right\}. \tag{4.12}$$

As $\sum_{i=1}^{n}\{(1/a_i)(\max_{j\in\mathcal{N}}|b_{ij}l_j|+\max_{j\in\mathcal{N}}|c_{ij}k_j|)\}<1$, π is a contraction mapping and hence there exists a unique fixed point $\overline{\mathbf{u}}(\cdot)$ of π in $\mathscr{H}$ which means $\overline{\mathbf{u}}^{\mathrm{T}}(\cdot)$ is the solution of (2.3)-(2.4) and $\|\overline{\mathbf{u}}^{\mathrm{T}}(\cdot)\|\to 0$ as $t\to\infty$.

To obtain the asymptotic stability, we still need to prove that the trivial equilibrium of (2.3) is stable. For any $\varepsilon>0$, from the conditions of Theorem 4.1, we can find δ satisfying $0<\delta<\varepsilon$ such that $\delta+\max_{i\in\mathcal{N}}\{\lambda_i^*\}\varepsilon\le\varepsilon/\sqrt{n}$.

Let $|\varphi|<\delta$. According to what have been discussed above, we know that there exists a unique solution $\mathbf{x}(t;s,\varphi)=(x_1(t;s,\varphi_1),\ldots,x_n(t;s,\varphi_n))^{\mathrm{T}}\in\mathrm{R}^n$ to (2.3)-(2.4), and

$$x_i(t)=\pi(x_i)(t)=J_1(t)+J_2(t)+J_3(t),\quad t\ge 0, \tag{4.13}$$

where

$$J_1(t)=x_{0i}\,\mathrm{e}^{-a_i t},\qquad J_2(t)=\mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n}b_{ij}f_j(x_j(s))\mathrm{d}s,$$
$$J_3=\mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\sum_{j=1}^{n}c_{ij}g_j(x_j(s-\tau_j(s)))\mathrm{d}s. \tag{4.14}$$

Suppose there exists $t^*>0$ such that $\|\mathbf{x}(t^*;s,\varphi)\|=\varepsilon$ and $\|\mathbf{x}(t;s,\varphi)\|<\varepsilon$ as $0\le t<t^*$. It follows from (4.13) that $|x_i(t^*)|\le|J_1(t^*)|+|J_2(t^*)|+|J_3(t^*)|$.

As $|J_1(t^*)|=|x_{0i}\mathrm{e}^{-a_i t^*}|\le\delta$, $|J_2(t^*)|<(\varepsilon/a_i)\sum_{j=1}^{n}|b_{ij}l_j|$ and

$$|J_3(t^*)|\le \mathrm{e}^{-a_i t^*}\int_0^{t^*}\mathrm{e}^{a_i s}\sum_{j=1}^{n}\left|c_{ij}k_jx_j(s-\tau_j(s))\right|\mathrm{d}s<\frac{\varepsilon}{a_i}\sum_{j=1}^{n}\left|c_{ij}k_j\right|, \tag{4.15}$$

we obtain $|x_i(t^*)|<\delta+\lambda_i{}^*\varepsilon$. Hence

$$\left\|\mathbf{x}(t^*;s,\varphi)\right\|^2=\sum_{i=1}^{n}\left\{|x_i(t^*)|^2\right\}<\sum_{i=1}^{n}\left\{\left|\delta+\lambda_i^*\varepsilon\right|^2\right\}\le n\left|\delta+\max_{i\in\mathcal{N}}\{\lambda_i^*\}\varepsilon\right|^2\le\varepsilon^2. \tag{4.16}$$

This contradicts to the assumption of $\|\mathbf{x}(t^*;s,\varphi)\|=\varepsilon$. Therefore, $\|\mathbf{x}(t;s,\varphi)\|<\varepsilon$ holds for all $t\ge 0$. This completes the proof. □

Remark 4.2. In Theorems 3.1 and 4.1, we use the contraction mapping principle to study the existence and uniqueness of solution and the asymptotic stability of trivial equilibrium at the same time, while Lyapunov method fails to do this.

Remark 4.3. The provided sufficient conditions in Theorem 4.1 do not require even the differentiability of delays, let alone the monotone decreasing behavior of delays which is necessary in some relevant works.

5. Example

Consider the following two-dimensional cellular neural network with time-varying delays

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{2} c_{ij} g_j\left(x_j\left(t - \tau_j(t)\right)\right), \tag{5.1}$$

with the initial conditions $x_1(s) = \cos(s)$, $x_2(s) = \sin(s)$ on $-\tau \leq s \leq 0$, where $a_1 = a_2 = 3$, $b_{11} = 0$, $b_{12} = 1/7$, $b_{21} = -1/7$, $b_{22} = -1/7$, $c_{11} = 3/7$, $c_{12} = 2/7$, $c_{21} = 0$, $c_{22} = 1/7$, $f_j(s) = g_j(s) = (|s+1| - |s-1|)/2$, $\tau_j(t)$ is bounded by τ.

It is easily to know that $l_j = k_j = 1$ for $j = 1, 2$. Compute

$$\sum_{i=1}^{2}\left\{\frac{1}{a_i}\left(\max_{j=1,2}\left|b_{ij}l_j\right| + \max_{j=1,2}\left|c_{ij}k_j\right|\right)\right\} < 1, \qquad \max_{i\in\mathcal{N}}\left\{\frac{1}{a_i}\sum_{j=1}^{n}\left|b_{ij}l_j\right| + \frac{1}{a_i}\sum_{j=1}^{n}\left|c_{ij}k_j\right|\right\} < \frac{1}{\sqrt{2}}. \tag{5.2}$$

From Theorem 4.1, we conclude that the trivial equilibrium $\mathbf{x} = 0$ of this two-dimensional cellular neural network is asymptotically stable.

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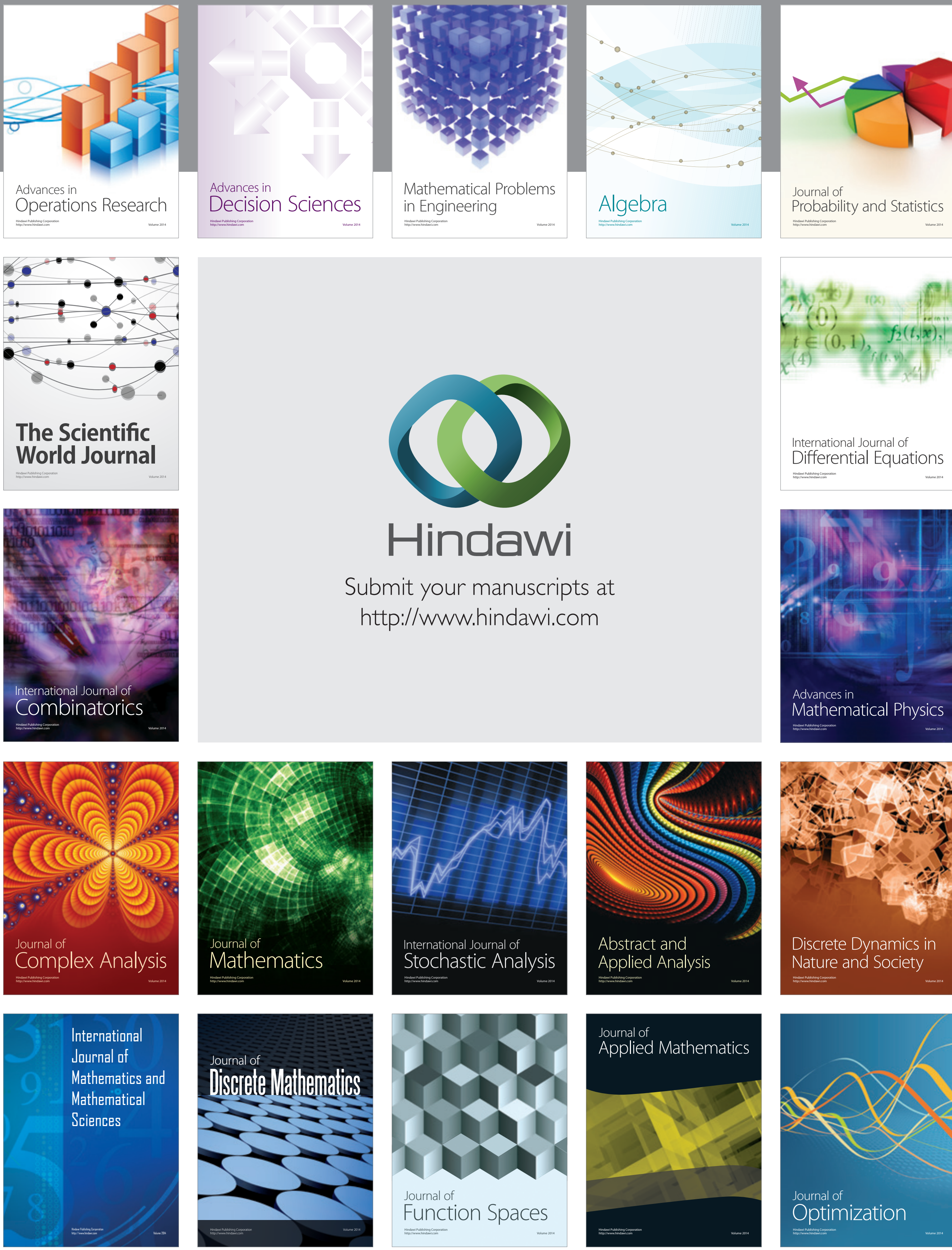
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