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Research Article

A New Iterative Scheme for Generalized Mixed Equilibrium, Variational Inequality Problems, and a Zero Point of Maximal Monotone Operators

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The purpose of this paper is to introduce a new iterative scheme for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of solutions of variational inequality problems, the zero point of maximal monotone operators, and the set of two countable families of quasi-ϕ-nonexpansive mappings in Banach spaces. Moreover, the strong convergence theorems of this method are established under the suitable conditions of the parameter imposed on the algorithm. Finally, we apply our results to finding a zero point of inverse-strongly monotone operators and complementarity problems. Our results presented in this paper improve and extend the recently results by many others.

1. Introduction

Equilibrium problem theory is the most important area of mathematical sciences and widely popular among mathematicians and researchers in other fields due to its applications in a wide class of problems which arise in economics, finance, optimization, network and transportation, image reconstruction, ecology, and many others. It has been improved and extended in many directions. Furthermore, equilibrium problems are related to the problem of finding fixed point of nonexpansive mappings. In this way, they have been extensively studied by many authors; see [1–9]. They introduced new iterative schemes for finding a common element of the set of the solutions of equilibrium problems and the set of fixed points. In this paper, we are interested a new hybrid iterative method for finding a common elements of the set of solutions of generalized mixed equilibrium problems, the set of solutions of variational inequality problems, the zero point of maximal monotone operators, and the set of two countable families of quasi-ϕ-nonexpansive mappings in the framework of Banach spaces.

Let E be a Banach space with norm $\|\cdot\|$ and C a nonempty closed convex subset of E and let E^* denote the dual of E.

A mapping $S : C \to C$ is said to be

(1) *nonexpansive* [1] if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$,

(2) *relatively nonexpansive* [10–12] if $\widehat{F(S)} = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$, where the functional ϕ defined by (2.6). The asymptotic behavior of a relatively nonexpansive mapping was studied in [13, 14],

(3) *ϕ-nonexpansive*, if $\phi(Sx, Sy) \leq \phi(x, y)$ for $x, y \in C$,

(4) *quasi- ϕ-nonexpansive* if $F(S) \neq \emptyset$ and $\phi(p, Sx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(S)$.

In the sequel, we denote $F(T)$ as the set of fixed points of S. If C is a bounded closed convex set and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty (see [15]).

A point p in C is said to be an asymptotic fixed point of S [16] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widehat{F(S)}$.

Let B be an operator from C into E^*, and B is said to be *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Bx - By\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C. \tag{1.1}$$

If an operator B is an α-inverse-strongly monotone, then we can said that B is *Lipschitz continuous*; that is, $\|Bx - By\| \leq (1/\alpha)\ \|x - y\|$ for all $x, y \in C$.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ a real-valued function, and $B : C \to E^*$ be a nonlinear mapping. The *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$f(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

We denote Ω as the set of solutions to (1.2) that is,

$$\Omega = \{x \in C : f(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in C\}. \tag{1.3}$$

If $B \equiv 0$, the problem (1.2) reduced into the *mixed equilibrium problem for f*, denoted by $\mathrm{MEP}(f, \varphi)$, is to find $x \in C$ such that

$$f(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $f \equiv 0$, the problem (1.2) reduced into the *mixed variational inequality* of Browder type, denoted by $\mathrm{VI}(C, B, \varphi)$, is to find $x \in C$ such that

$$\langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.5}$$

If $B \equiv 0$ and $\varphi \equiv 0$ the problem (1.2), reduced into the *equilibrium problem for* F, denoted by $\mathrm{EP}(f)$, is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

In addition, fixed point problem, optimization problem, and many problems can be written in the form of $\mathrm{EP}(f)$. There are the development of researches in this area as seen in many papers which appeared in the literature on the existence of the solutions of $\mathrm{EP}(f)$; see, for example [17–21] and reference therein. Furthermore, there are many solution methods proposed continuously to solve the $\mathrm{EP}(f)$ as shown in [2, 3, 18, 20, 22–26] and many others.

Next, we let B be a *monotone* operator of C into E^*. The so-called *variational inequality problem* is to find a point $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0 \quad \forall \mathrm{y} \in C. \tag{1.7}$$

The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, B)$.

As we know that the classical variational inequality was first introduced and studied by Stampacchia [27] in 1964. Its solution can be computed by using iterative projection method. There are many results with corresponding to variational inequality; for example, Yao et al. [28] proposed the strong convergence theorem for a system of nonlinear variational inequalities in Banach spaces, and then, they studied the two-step projection methods, and they established the convergence theorem for a system of variational inequality problems in the framework of Banach spaces. Moreover, the important generalized variational inequalities called variational inclusion also have been extensively studied and extended in many different directions. Yao et al. [29] considered the algorithm and proved the strong convergence of common solutions for variational inclusions, mixed equilibrium problems, and fixed point problems.

The one classical way to approximate a fixed point of a nonlinear self mapping T on C was firstly introduced by Halpern [30], and then, Aoyama et al. [31] extended the mapping in the Halpern-type iterative sequence to be a countable family of nonexpansive mappings. They introduced the following iterative sequence: let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n, \tag{1.8}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. They proved that $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Recently, Nakajo et al. [32] introduced the more general condition so-called the NST*-condition, and $\{T_n\}$ is said to satisfy the NST*-condition if for every bounded sequence $\{z_n\}$ in C,

$$\lim_{n \to \infty} \|z_n - T_n z_n\| = \lim_{n \to \infty} \|z_n - z_{n+1}\| = 0 \quad \text{implies} \quad \omega_w(z_n) \subset F. \tag{1.9}$$

They also prove strong convergence theorems by the hybrid method for families of mappings in a uniformly convex Banach space E whose norm is Gâteaux differentiable.

In Hilbert space H, Iiduka et al. [33] introduced an iterative scheme and proved that the sequence $\{x_n\}$ generated by the following algorithm: $x_1 = x \in C$, and

$$x_{n+1} = P_C(x_n - \lambda_n B x_n), \tag{1.10}$$

where P_C is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive real numbers, converges weakly to some element of $\mathrm{VI}(C, B)$.

Later, Iiduka and Takahashi [34] are interested in the similar problem in the framework of Banach spaces, they introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator $B: \ x_1 = x \in C$, and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n B x_n), \tag{1.11}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^*, and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.11) converges weakly to some element of $\mathrm{VI}(C, B)$.

In 1974, Rockafellar interested in the following problem of finding:

$$v \in E \quad \text{such that} \quad 0 \in A(v), \tag{1.12}$$

where A is an operator from E into E^*. Such $v \in E$ is called a *zero point* of A. He introduced a well-known method, *proximal point algorithm*, for solving (1.12) in a Hilbert space H as shown in the following: $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \ldots, \tag{1.13}$$

where $\{r_n\} \subset (0, \infty)$, A is a maximal monotone and $J_{r_n} = (I + r_n A)^{-1}$. He proved that the sequence $\{x_n\}$ converges weakly to an element of $A^{-1}(0)$.

In 2004, Kamimura et al. [35] considered the algorithm (1.14) in a uniformly smooth and uniformly convex Banach space E; namely,

$$x_{n+1} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J(J_{r_n} x_n)), \quad n = 1, 2, 3, \ldots. \tag{1.14}$$

They proved that the algorithm $\{x_n\}$ generated by (1.14) converges weakly to some element of $A^{-1}0$.

In 2008, Li and Song [36] established a strong convergence theorem in a Banach space. They introduced the following algorithm: $x_1 = x \in E$ and

$$\begin{aligned} y_n &= J^{-1}\left(\beta_n J(x_n) + (1 - \beta_n) J(J_{r_n} x_n)\right), \\ x_{n+1} &= J^{-1}\left(\alpha_n J x + (1 - \alpha_n) J(y_n)\right). \end{aligned} \tag{1.15}$$

Under the suitable conditions of the coefficient sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$, they proved that the sequence $\{x_n\}$ generated by the above scheme converges strongly to $\Pi_C x$, where Π_C is the generalized projection from E onto C.

In 2010, Petrot et al. [37] introduced a hybrid projection iterative scheme for approximating a common element of the set of solutions of a generalized mixed equilibrium problem and the set of fixed points of two quasi-ϕ-nonexpansive mappings in a real uniformly convex and uniformly smooth Banach space by the following manner:

$$\begin{aligned}
x_0 &= x \in C, \\
y_n &= J^{-1}(\delta_n J x_n + (1-\delta_n) J z_n), \\
z_n &= J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\
u_n &= K_{r_n} y_n, \\
C_n &= \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\}, \\
Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x.
\end{aligned} \tag{1.16}$$

They proved that $\{x_n\}$ converges strongly to $p \in F(T) \cap F(S) \cap \Omega$, where $p \in \Pi_{F(T)\cap F(S)\cap\Omega} x$.

Recently, Klin-eam et al. [38], obtained the strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of two relatively nonexpansive mappings in a Banach space by using a new hybrid method. Saewan and Kumam [39] introduced a new hybrid projection method for finding a common solution of the set of common fixed points of two countable families of relatively quasi-nonexpansive mappings, the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of the generalized mixed equilibrium problem, and zeros of a maximal monotone operator in a real uniformly smooth and 2-uniformly convex Banach space. Wattanawitoon and Kumam [40] proved the strong convergence theorem by using modified hybrid projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of solution of variational inequality operators of an inverse strongly monotone, the zero point of a maximal monotone operator, and the set of fixed point of two relatively quasi-nonexpansive mappings in Banach space.

Motivated and inspired by the ongoing research and the above-mentioned results, we are also interested in generalized mixed equilibrium problem, variational inequality problems, and the zero point of maximal monotone operators. In this paper, we extend the fixed point problems of two relatively quasi-nonexpansive mappings in [40] to the countable families of two quasi-ϕ-nonexpansive mappings and improve the iterative scheme to be more general as shown in the following: $x_1 = x \in C$,

$$\begin{aligned}
w_n &= \Pi_C J^{-1}(Jx_n - \lambda_n B x_n), \\
z_n &= J^{-1}(\delta_n J(x_n) + (1-\delta_n) J(J_{r_n} w_n)), \\
y_n &= J^{-1}(\alpha_n J x_n + \beta_n J T_n x_n + \gamma_n J S_n z_n), \\
u_n &\in C \text{ such that } F(u_n, y) + \langle \Upsilon u_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) \\
&\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall \quad y \in C,
\end{aligned}$$

$$C_{n+1} = \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\},$$
$$x_{n+1} = \Pi_{C_{n+1}} x. \tag{1.17}$$

By the new iterative scheme, we will prove the strong convergence theorems of the sequence $\{x_n\}$ which could be converged to the point $\Pi_{(\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \cap \Omega \cap VI(C,B) \cap A^{-1}(0)} x$. Furthermore, we propose the new better appropriate conditions of the coefficient sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{r_n\}$. Finally, we will apply our result to find a zero point of inverse-strongly monotone operators and complementarity problem in the last section. The results presented in this paper extend and improve the corresponding ones announced by Kamimura et al. [35], Petrot et al. [37], Wattanawitoon and Kumam [40], and some authors in the literature.

2. Preliminaries

In this section, we propose the following preliminaries and lemmas which will be used in our proof.

Throughout this paper, we let E be a Banach space with norm $\|\cdot\|$, and C a nonempty closed convex subset of E, and let E^* denote the dual of E. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that the sequence $\{x_n\}$ converges strongly to x.

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if for any $x, y \in U$,

$$x \ne y \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$

$$\|x - y\| \ge \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| < 1 - \delta. \tag{2.2}$$

We know that a uniformly convex Banach space is reflexive and strictly convex; see [41, 42] for more details.

The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x - y\| \ge \varepsilon\right\}. \tag{2.3}$$

Furthermore, it is said to be *smooth*, provided that

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$.

Let p be a fixed real number with $p \geq 2$. Observe that every p-uniformly convex is uniformly convex. One should note that no a Banach space is p-uniformly convex for $1 < p < 2$. It is well known that a Hilbert space is 2-uniformly convex and uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1} \right\}, \tag{2.5}$$

for all $x \in E$.

In particular, $J = J_2$ is called *the normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We know the following (see [43]):

(1) if E is smooth, then J is single-valued,

(2) if E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$,

(3) if E is reflexive, then J is surjective,

(4) if E is uniformly convex, then it is reflexive,

(5) if E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [44] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Let E be a smooth, strictly convex and reflexive Banach space, and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \text{for } x, y \in E. \tag{2.6}$$

Remark 2.1. We know the following: for each $x, y, z \in E$,

(i) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

(ii) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,

(iii) $\phi(x, y) = \|x - y\|^2$ in a real Hilbert space.

The *generalized projection*, introduced by Alber [45], $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(x, y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x), \tag{2.7}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J.

If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$, if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From Remark 2.1 (i), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [43, 46] for more details.

Lemma 2.2 (see [47, 48]). *If E be a 2-uniformly convex Banach space, then for all $x, y \in E$, one has*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \tag{2.8}$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $1/c$ in the Lemma is called the 2-uniformly convex constant of E; see [41].

Lemma 2.3 (see [47, 49]). *If E is a p-uniformly convex Banach space and p a given real number with $p \geq 2$, then for all $x, y \in E, J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x - y, Jx - Jy\rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p, \tag{2.9}$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Lemma 2.4 (Xu [48]). *Let E be a uniformly convex Banach space, then for each $r > 0$, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda y)\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|), \tag{2.10}$$

for all $x, y \in \{z \in E : \|z\| \leq r\}$ and $\lambda \in [0, 1]$.

Lemma 2.5 (Kamimura and Takahashi [50]). *Let E be a uniformly convex and smooth real Banach space and $\{x_n\}, \{y_n\}$ two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.6 (Alber [45]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \tag{2.11}$$

Lemma 2.7 (Alber [45]). *Let E be a reflexive, strictly convex and smooth Banach space and C a nonempty closed convex subset of E and let $x \in E$. Then,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.12}$$

Let E be a strictly convex, smooth, and reflexive Banach space and J the duality mapping from E into E^*. Then, J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. Define a function $V : E \times E^* \to \mathbb{R}$ as follows (see [51]):

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{2.13}$$

for all $x \in E$ and $x^* \in E^*$. Then, it is obvious that $V(x, x^*) = \phi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \phi(x, y)$.

Lemma 2.8 (Kohsaka and Takahashi [51, Lemma 3.2]). *Let E be a strictly convex, smooth, and reflexive Banach space and V as in* (2.13). *Then,*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^*\right\rangle \leq V\left(x, x^* + y^*\right), \tag{2.14}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ and $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous, satisfying the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F\left(tz + (1-t)x, y\right) \leq F\left(x, y\right), \tag{2.15}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.9 (Blum and Oettli [17]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E and F a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F\left(z, y\right) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in C. \tag{2.16}$$

Lemma 2.10 (Takahashi and Zembayashi [52]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E and F a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{z \in C : F\left(z, y\right) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\right\}, \tag{2.17}$$

for all $x \in E$. Then, the following hold:

(1) *T_r is single-valued,*

(2) *T_r is a firmly nonexpansive-type mapping, that is, for all $x, y \in E$,*

$$\left\langle T_r x - T_r y, JT_r x - JT_r y\right\rangle \leq \left\langle T_r x - T_r y, Jx - Jy\right\rangle, \tag{2.18}$$

(3) *$F(T_r) = EP(F)$,*

(4) *$EP(F)$ is closed and convex.*

Lemma 2.11 (Takahashi and Zembayashi [52]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and F a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.19}$$

Lemma 2.12 (Zhang [53]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Let $B : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ convex and lower semi-continuous, and F a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$F(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.20}$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \left\{u \in C : F(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \forall y \in C\right\}, \tag{2.21}$$

for all $x \in E$. Then, the following hold:

(i) *K_r is single-valued,*

(ii) *K_r is firmly nonexpansive, that is, for all $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$,*

(iii) *$F(K_r) = \Omega$,*

(iv) *Ω is closed and convex,*

(v) *$\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$ for all $p \in F(K_r)$, $z \in E$.*

It follows from Lemma 2.10 that the mapping $K_r : C \to C$ defined by (2.21) is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a closed convex subset of E. Because $\phi(x, y)$ is strictly convex and coercive in the first variable, we know that the minimization problem $\inf_{y\in C}\phi(x, y)$ has a unique solution. The operator $\Pi_C x := \arg\min_{y\in C}\phi(x, y)$ is said to be the generalized projection of x on C.

Let A be a set-valued mapping from E to E^* with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{x \in E : A(x) \neq \emptyset\}$, and range $R(A) = \{x^* \in E^* : x^* \in A(x), x \in D(A)\}$. We denote a set-valued operator A from E to E^* by $A \subset E \times E^*$. A is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be *maximal monotone* if it graph is not properly contained in the graph of any other monotone operator. We know that if A is maximal monotone, then the solution set $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is closed and convex.

Let E be a reflexive, strictly convex and smooth Banach space, it is known that A is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.

Define the *resolvent* of A by $J_r x = x_r$. In other words, $J_r = (J + rA)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(A)$. Also, $A^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of A by $A_r = (J - JJ_r)/r$. We know that $A_r x \in A(J_r x)$ for all $r > 0$ and $x \in E$.

Lemma 2.13 (Kohsaka and Takahashi [51, Lemma 3.1]). *Let E be a smooth, strictly convex, and reflexive Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$, $r > 0$, and $J_r = (J + rT)^{-1}J$. Then,*

$$\phi(x, J_r y) + \phi(J_r y, y) \leq \phi(x, y) \tag{2.22}$$

for all $x \in A^{-1}0$ and $y \in E$.

Let B be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0,1]$ into E^*, defined by $F(t) = B(tx + (1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the *normal cone* for C at a point $v \in C$; that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}. \tag{2.23}$$

Theorem 2.14. *(Rockafellar [54]). Let C be a nonempty, closed convex subset of a Banach space E and B a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Bv + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2.24}$$

Then T, is maximal monotone and $T^{-1}0 = \mathrm{VI}(C, B)$.

Lemma 2.15 (Tan and Xu [55]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0. \tag{2.25}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. The Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of mixed equilibrium problems, the set of solutions of the variational inequality problem, the zero point of a maximal monotone operator, and the set of two families of quasi-ϕ-nonexpansive mappings in a Banach space by using the shrinking hybrid projection method.

Theorem 3.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function, and let $A : E \to E^*$ be a maximal monotone operator satisfying $D(A) \subset E$. Let $J_r = (J + rA)^{-1}J$ for $r > 0$, let B be an α-inverse-strongly monotone* *operator of E into E^*, and let $\Upsilon : E \to E^*$ be a continuous and monotone mapping. Let $\{T_n\}$ and $\{S_n\}$ be two families of quasi-ϕ-nonexpansive mappings of E into itself satisfies the NST *-condition, with $\Theta := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \cap \Omega \cap VI(C,B) \cap A^{-1}(0) \neq \emptyset$ and $\|By\| \leq \|By - Bu\|$ for all $y \in C$ and $u \in \Theta$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and*

$$\begin{aligned}
w_n &= \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n),\\
z_n &= J^{-1}(\delta_n J(x_n) + (1-\delta_n) J(J_{r_n} w_n)),\\
y_n &= J^{-1}(\alpha_n Jx_n + \beta_n JT_n x_n + \gamma_n JS_n z_n),\\
u_n &\in C \text{ such that } F(u_n, y) + \langle \Upsilon u_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)\\
&\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,\\
C_{n+1} &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},\\
x_{n+1} &= \Pi_{C_{n+1}} x,
\end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$. If the coefficient sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$, $\liminf_{n\to\infty}\gamma_n(1-\delta_n) > 0$, $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $1/c$ is the 2-uniformly convexity constant of E. Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Theta x$.

Proof. We first show that $\{x_n\}$ is bounded. Let $p \in \Theta := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \cap \Omega \cap \mathrm{VI}(C,B) \cap A^{-1}(0)$, and let

$$\begin{aligned}
H(u_n, y) &= F(u_n, y) + \langle \Upsilon u_n, y - u_n\rangle + \varphi(y) - \varphi(u_n), \quad y \in C,\\
K_{r_n} &= \left\{u \in C : H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C\right\}.
\end{aligned} \tag{3.2}$$

Put $v_n = J^{-1}(Jx_n - \lambda_n Bx_n)$ and $u_n = K_{r_n} y_n$.

With its relatively nonexpansiveness of J_{r_n} and by Lemma 2.8, the convexity of the function V in the second variable, we have

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, \Pi_C v_n)\\
&\leq \phi(p, v_n) = \phi\left(p, J^{-1}(Jx_n - \lambda_n Bx_n)\right)\\
&\leq V(p, Jx_n - \lambda_n Bx_n + \lambda_n Bx_n) - 2\left\langle J^{-1}(Jx_n - \lambda_n Bx_n) - p, \lambda_n Bx_n\right\rangle\\
&= V(p, Jx_n) - 2\lambda_n\langle v_n - p, Bx_n\rangle\\
&= \phi(p, x_n) - 2\lambda_n\langle x_n - p, Bx_n\rangle + 2\langle v_n - x_n, -\lambda_n Bx_n\rangle.
\end{aligned} \tag{3.3}$$

Since $p \in \mathrm{VI}(C, B)$ and B is α-inverse-strongly monotone, we consider

$$\begin{aligned}-2\lambda_n\langle x_n - p, Bx_n\rangle &= -2\lambda_n\langle x_n - p, Bx_n - Bp\rangle - 2\lambda_n\langle x_n - p, Bp\rangle \\ &\leq -2\alpha\lambda_n\|Bx_n - Bp\|^2.\end{aligned} \tag{3.4}$$

Therefore, by Lemma 2.2, we obtain

$$\begin{aligned}2\langle v_n - x_n, -\lambda_n Bx_n\rangle &= 2\left\langle J^{-1}(Jx_n - \lambda_n Bx_n) - J^{-1}(Jx_n), -\lambda_n Bx_n\right\rangle \\ &\leq 2\left\|J^{-1}(Jx_n - \lambda_n Bx_n) - J^{-1}(Jx_n)\right\|\|\lambda_n Bx_n\| \\ &\leq \frac{4}{c^2}\|Jx_n - \lambda_n Bx_n - Jx_n\|\|\lambda_n Bx_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|Bx_n\|^2 \\ &\leq \frac{4}{c^2}\lambda_n^2\|Bx_n - Bp\|^2.\end{aligned} \tag{3.5}$$

We can rewrite (3.3), which yield that

$$\begin{aligned}\phi(p, w_n) &\leq \phi(p, x_n) - 2\alpha\lambda_n\|Bx_n - Bp\|^2 + \frac{4}{c^2}\lambda_n^2\|Bx_n - Bp\|^2 \\ &\leq \phi(p, x_n) + 2\lambda_n\left(\frac{2}{c^2}\lambda_n - \alpha\right)\|Bx_n - Bp\|^2 \\ &\leq \phi(p, x_n).\end{aligned} \tag{3.6}$$

Apply the Lemma 2.8, Lemma 2.13 and (3.6), we consider

$$\begin{aligned}\phi(p, z_n) &= \phi\Big(p, J^{-1}(\delta_n J(x_n) + (1 - \delta_n)J(J_{r_n}w_n))\Big) \\ &= V(p, \delta_n J(x_n) + (1 - \delta_n)J(J_{r_n}w_n)) \\ &\leq \delta_n V(p, J(x_n)) + (1 - \delta_n)V(p, J(J_{r_n}w_n)) \\ &= \delta_n\phi(p, x_n) + (1 - \delta_n)\phi(p, J_{r_n}w_n) \\ &\leq \delta_n\phi(p, x_n) + (1 - \delta_n)(\phi(p, w_n) - \phi(J_{r_n}w_n, w_n)) \\ &\leq \delta_n\phi(p, x_n) + (1 - \delta_n)\phi(p, w_n) \\ &\leq \delta_n\phi(p, x_n) + (1 - \delta_n)\phi(p, x_n) \\ &= \phi(p, x_n),\end{aligned} \tag{3.7}$$

hence, we obtain

$$\begin{aligned}
\phi(p, y_n) &= \phi\Big(p, J^{-1}(\alpha_n J x_n + \beta_n J T_n x_n + \gamma_n J S_n z_n)\Big) \\
&= \|p\|^2 - 2\alpha_n\langle p, J x_n\rangle - 2\beta_n\langle p, J T_n x_n\rangle - 2\gamma_n\langle p, J S_n z_n\rangle \\
&\quad + \|\alpha_n J x_n + \beta_n J T_n x_n + \gamma_n J S_n z_n\|^2 \\
&\le \|p\|^2 - 2\alpha_n\langle p, J x_n\rangle - 2\beta_n\langle p, J T_n x_n\rangle - 2\gamma_n\langle p, J S_n z_n\rangle \\
&\quad + \alpha_n\|J x_n\|^2 + \beta_n\|J T_n x_n\|^2 + \gamma_n\|J S_n z_n\|^2 \\
&= \alpha_n\phi(p, x_n) + \beta_n\phi(p, T_n x_n) + \gamma_n\phi(p, S_n z_n) \\
&\le \alpha_n\phi(p, x_n) + \beta_n\phi(p, x_n) + \gamma_n\phi(p, z_n) \\
&\le \alpha_n\phi(p, x_n) + \beta_n\phi(p, x_n) + \gamma_n\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned} \tag{3.8}$$

By (3.1), again,

$$\phi(p, u_n) = \phi(p, K_{r_n} y_n) \le \phi(p, y_n) \le \phi(p, x_n). \tag{3.9}$$

This shows that $p \in C_{n+1}$. Consequently, $\Theta \subset C_n$, for all $n \in \mathbb{N}$.

Next, we show that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. Since $x_n = \Pi_{C_n} x$, it follows from Lemma 2.7 that

$$\phi(x_n, x) \le \phi(p, x) - \phi(p, x_n) \le \phi(p, x), \tag{3.10}$$

for each $p \in \Theta \subset C_n$. Then, $\phi(x_n, x)$ is bounded. It implies that $\{x_n\}$ is bounded and $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, and $\{J_{r_n} w_n\}$ are also bounded.

From $x_n = \Pi_{C_n} x$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x) \le \phi(x_{n+1}, x), \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Therefore, $\{\phi(x_n, x)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x)\}$ exists, and from Lemma 2.7, we have

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x) \le \phi(x_{n+1}, x) - \phi(\Pi_{C_n} x, x) = \phi(x_{n+1}, x) - \phi(x_n, x), \tag{3.12}$$

for all $n \in \mathbb{N}$. Thus, we have

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{3.13}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x \in C_{n+1}$, it follows from the definition of C_{n+1} that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \longrightarrow 0. \tag{3.14}$$

By Lemma 2.5, (3.13), and (3.14), we note that

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.15}$$

Since J is uniformly norm-to-norm continuous on the bounded set, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.16}$$

Since $x_m = \Pi_{C_m} \subset C_n$ for any positive integer $m \geq n$, it follows from Lemma 2.7 that

$$\begin{aligned} \phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_n) &\leq \phi(x_m, x) - \phi(\Pi_{C_n} x_n, x) \\ &= \phi(x_m, x) - \phi(x_n, x). \end{aligned} \tag{3.17}$$

Taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$ as $n \to \infty$. It follows from Lemma 2.5, that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we can assume that $x_n \to u \in C$ as $n \to \infty$.

Next, we show that $u \in (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n))$.

Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Let $r = \sup_{n\in\mathbb{N}} \{\|x_n\|, \|T_n x_n\|, \|S_n z_n\|\}$. From Lemma 2.4, we have

$$\begin{aligned} \phi(p, u_n) &\leq \phi(p, y_n) \\ &= \phi\Big(p, J^{-1}(\alpha_n Jx_n + \beta_n JT_n x_n + \gamma_n JS_n z_n)\Big) \\ &= \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2\beta_n \langle p, JT_n x_n \rangle - 2\gamma_n \langle p, JS_n z_n \rangle \\ &\quad + \|\alpha_n Jx_n + \beta_n JT_n x_n + \gamma_n JS_n z_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2\beta_n \langle p, JT_n x_n \rangle - 2\gamma_n \langle p, JS_n z_n \rangle \\ &\quad + \alpha_n \|Jx_n\|^2 + \beta_n \|JT_n x_n\|^2 + \gamma_n \|JS_n z_n\|^2 - \alpha_n \beta_n g(\|JT_n x_n - Jx_n\|) \\ &= \alpha_n \phi(p, x_n) + \beta_n \phi(p, T_n x_n) + \gamma_n \phi(p, S_n z_n) - \alpha_n \beta_n g(\|JT_n x_n - Jx_n\|) \\ &\leq \alpha_n \phi(p, x_n) + \beta_n \phi(p, x_n) + \gamma_n \phi(p, z_n) - \alpha_n \beta_n g(\|JT_n x_n - Jx_n\|) \\ &\leq \alpha_n \phi(p, x_n) + \beta_n \phi(p, x_n) + \gamma_n \phi(p, x_n) - \alpha_n \beta_n g(\|JT_n x_n - Jx_n\|) \\ &= \phi(p, x_n) - \alpha_n \beta_n g(\|JT_n x_n - Jx_n\|). \end{aligned} \tag{3.18}$$

This implies that

$$\alpha_n \beta_n g(\|JT_n x_n - Jx_n\|) \leq \phi(p, x_n) - \phi(p, u_n). \tag{3.19}$$

On the other hand, we have

$$\begin{aligned}\phi(p,x_n)-\phi(p,u_n)&=\|x_n\|^2-\|u_n\|^2-2\langle p,Jx_n-Ju_n\rangle\\&=\|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|p\|\|Jx_n-Ju_n\|.\end{aligned}\tag{3.20}$$

Noticing (3.15) and (3.16), we obtain

$$\phi(p,x_n)-\phi(p,u_n)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\tag{3.21}$$

Since $\liminf_{n\to\infty}\alpha_n\beta_n>0$ and (3.21), it follows from (3.19) that

$$g(\|JT_nx_n-Jx_n\|)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\tag{3.22}$$

It follows from the property of g that

$$\|JT_nx_n-Jx_n\|\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\tag{3.23}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\to\infty}\|T_nx_n-x_n\|=0.\tag{3.24}$$

Similarly, using the condition $\limsup_{n\to\infty}\alpha_n\gamma_n>0$, one can obtain

$$\lim_{n\to\infty}\|S_nz_n-x_n\|=0.\tag{3.25}$$

By (3.6), (3.8), and (3.18), we have

$$\begin{aligned}\phi(p,u_n)&\le\alpha_n\phi(p,x_n)+\beta_n\phi(p,x_n)+\gamma_n\phi(p,z_n)-\alpha_n\beta_ng(\|JT_nx_n-Jx_n\|)\\&\le\alpha_n\phi(p,x_n)+\beta_n\phi(p,x_n)\\&\quad+\gamma_n\left[\delta_n\phi(p,x_n)+(1-\delta_n)\phi(p,w_n)\right]-\alpha_n\beta_ng(\|JT_nx_n-Jx_n\|)\\&\le\alpha_n\phi(p,x_n)+\beta_n\phi(p,x_n)+\gamma_n\delta_n\phi(p,x_n)\\&\quad+\gamma_n(1-\delta_n)\phi(p,w_n)-\alpha_n\beta_ng(\|JT_nx_n-Jx_n\|)\\&\le\alpha_n\phi(p,x_n)+\beta_n\phi(p,x_n)\\&\quad+\gamma_n\delta_n\phi(p,x_n)+\gamma_n(1-\delta_n)\left[\phi(p,x_n)+2\lambda_n\left(\frac{2}{c^2}\lambda_n-\alpha\right)\left\|Bx_n-Bp\right\|^2\right]\\&\quad-\alpha_n\beta_ng(\|JT_nx_n-Jx_n\|)\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\phi(p,x_n)+\beta_n\phi(p,x_n)+\gamma_n\delta_n\phi(p,x_n)+\gamma_n(1-\delta_n)\phi(p,x_n)\\
&\quad+\gamma_n(1-\delta_n)2\lambda_n\left(\frac{2}{c^2}\lambda_n-\alpha\right)\|Bx_n-Bp\|^2-\alpha_n\beta_n g(\|JT_nx_n-Jx_n\|)\\
&\leq \phi(p,x_n)+\gamma_n(1-\delta_n)2\lambda_n\left(\frac{2}{c^2}\lambda_n-\alpha\right)\|Bx_n-Bp\|^2.
\end{aligned}\tag{3.26}$$

This implies that

$$2\lambda_n\left(\alpha-\frac{2}{c^2}\lambda_n\right)\|Bx_n-Bp\|^2\leq\frac{1}{\gamma_n(1-\delta_n)}\left[\phi(p,x_n)-\phi(p,u_n)\right].\tag{3.27}$$

By assumption, $\liminf_{n\to\infty}\gamma_n(1-\delta_n)>0$ and (3.21), we get that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0.\tag{3.28}$$

From Lemma 2.7, Lemma 2.8, and (3.5), we have

$$\begin{aligned}
\phi(x_n,w_n)&=\phi(x_n,\Pi_C v_n)\\
&\leq\phi(x_n,v_n)\\
&=\phi\left(x_n,J^{-1}(Jx_n-\lambda_nBx_n)\right)\\
&=V(x_n,Jx_n-\lambda_nBx_n)\\
&\leq V(x_n,(Jx_n-\lambda_nBx_n)+\lambda_nBx_n)-2\left\langle J^{-1}(Jx_n-\lambda_nBx_n)-x_n,\lambda_nBx_n\right\rangle\\
&=V(x_n,Jx_n)-2\lambda_n\langle v_n-x_n,Bx_n\rangle\\
&=\phi(x_n,x_n)+2\langle v_n-x_n,\lambda_nBx_n\rangle\\
&\leq\frac{4}{c^2}\lambda_n^2\|Bx_n-Bp\|^2.
\end{aligned}\tag{3.29}$$

By Lemma 2.8 and Lemma 2.13, we have

$$\begin{aligned}
\phi(x_n,z_n)&=\phi\left(x_n,J^{-1}(\delta_nJx_n+(1-\delta_n)J(J_{r_n}w_n))\right)\\
&=V(x_n,\delta_nJx_n+(1-\delta_n)J(J_{r_n}w_n))\\
&\leq\delta_nV(x_n,Jx_n)+(1-\delta_n)V(x_n,J(J_{r_n}w_n))\\
&=\delta_n\phi(x_n,x_n)+(1-\delta_n)\phi(x_n,J_{r_n}w_n)\\
&=\delta_n\phi(x_n,x_n)+(1-\delta_n)(\phi(x_n,w_n)-\phi(J_{r_n}w_n,w_n))\\
&=(1-\delta_n)\phi(x_n,w_n)\\
&\leq(1-\delta_n)\frac{4}{c^2}\lambda_n^2\|Bx_n-Bp\|^2.
\end{aligned}\tag{3.30}$$

From Lemma 2.5 and (3.28), we obtain

$$\lim_{n\to\infty}\|x_n - w_n\| = \lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.31}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we note that

$$\lim_{n\to\infty}\|Jx_n - Jw_n\| = \lim_{n\to\infty}\|Jx_n - Jz_n\| = 0. \tag{3.32}$$

Since $x_n \to u$ as $n \to \infty$, $z_n \to u$ as $n \to \infty$. Combining (3.15), (3.25), and (3.28), we also obtain

$$\|S_n z_n - z_n\| \le \|S_n z_n - x_n\| + \|x_n - z_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.33}$$

By (3.15) and (3.31), we obtain that

$$\|z_{n+1} - z_n\| \le \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.34}$$

By (3.15), (3.24), (3.33), and (3.34), and $\{T_n\}$, $\{S_n\}$ satisfies the NST*-condition and $x_n \to p$, then we have $p \in (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n))$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u \in C$. It follows from (3.31) that we have $w_{n_i} \rightharpoonup u$ as $i \to \infty$. Next, we show that $u \in A^{-1}0$.

By (3.6), (3.8), and (3.9), we obtain

$$\begin{aligned}
\phi(p, u_n) &\le \phi(p, y_n) \\
&\le \alpha_n \phi(p, x_n) + \beta_n \phi(p, x_n) + \gamma_n \phi(p, z_n) \\
&\le \alpha_n \phi(p, x_n) + \beta_n \phi(p, x_n) + \gamma_n \left[\delta_n \phi(p, x_n) + (1-\delta_n)(\phi(p, w_n) - \phi(J_{r_n} w_n, w_n))\right] \\
&\le \alpha_n \phi(p, x_n) + \beta_n \phi(p, x_n) + \gamma_n \left[\delta_n \phi(p, x_n) + (1-\delta_n)(\phi(p, x_n) - \phi(J_{r_n} w_n, w_n))\right] \\
&\le \phi(p, x_n) - \gamma_n (1-\delta_n)\phi(J_{r_n} w_n, w_n).
\end{aligned} \tag{3.35}$$

This implies that

$$\gamma_n (1-\delta_n)\phi(J_{r_n} w_n, w_n) \le \phi(p, x_n) - \phi(p, u_n). \tag{3.36}$$

By (3.21), we have

$$\lim_{n\to\infty}\|J_{r_n} w_n - w_n\| = 0. \tag{3.37}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we note that

$$\lim_{n\to\infty}\|JJ_{r_n} w_n - Jw_n\| = 0. \tag{3.38}$$

Indeed, since $\liminf_{n\to\infty} r_n > 0$, it follows from (3.38) that

$$\lim_{n\to\infty}\|A_{r_n}w_n\| = \lim_{n\to\infty}\frac{1}{r_n}\|Jw_n - J(J_{r_n}w_n)\| = 0. \tag{3.39}$$

If $(w, w^*) \in A$, then it holds from the monotonicity of A that

$$\left\langle w - w_{n_i}, w^* - A_{r_{n_i}}w_{n_i}\right\rangle \geq 0, \tag{3.40}$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle w - u, w^*\rangle \geq 0$. Then, the maximality of A implies $u \in A^{-1}0$.

Next, we show that $u \in \mathrm{VI}(C, B)$. Let $K \subset E \times E^*$ be an operator as follows:

$$Kv = \begin{cases} Bv + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{3.41}$$

By Theorem 2.14, K is maximal monotone and $K^{-1}0 = \mathrm{VI}(C, B)$.

Let $(v, w) \in G(K)$. Since $w \in Kv = Bv + N_C(v)$, we get $w - Bv \in N_C(v)$. From $w_n \in C$, we have

$$\langle v - w_n, w - Kv\rangle \geq 0. \tag{3.42}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n)$, then by Lemma 2.6, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Bx_n)\rangle \geq 0, \tag{3.43}$$

thus

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Bx_n\right\rangle \leq 0. \tag{3.44}$$

It follows from (3.42) and (3.44) that

$$\begin{aligned}
\langle v - w_n, w\rangle &\geq \langle v - w_n, Bv\rangle \\
&\geq \langle v - w_n, Bv\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Bx_n\right\rangle \\
&= \langle v - w_n, Bv - Bx_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\right\rangle \\
&= \langle v - w_n, Bv - Bw_n\rangle + \langle v - w_n, Bw_n - Bx_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\right\rangle \\
&\geq -\|v - w_n\|\frac{\|w_n - x_n\|}{\alpha} - \|v - w_n\|\frac{\|Jx_n - Jw_n\|}{a} \\
&\geq -M\left(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{a}\right),
\end{aligned} \tag{3.45}$$

where $M = \sup_{n\geq 1}\{\|v - w_n\|\}$. From (3.31) and (3.32), we obtain $\langle v - u, w\rangle \geq 0$. By the maximality of K, we have $u \in K^{-1}0$ and hence $u \in \mathrm{VI}(C, B)$.

Next, we show that $u \in \Omega$. From $u_n = K_{r_n} y_n$ and Lemma 2.12, we obtain

$$\begin{aligned}\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n)\\ &\leq \phi(u, y_n) - \phi(u, K_{r_n} y_n)\\ &\leq \phi(u, x_n) - \phi(u, K_{r_n} y_n)\\ &\leq \phi(u, x_n) - \phi(u, u_n).\end{aligned} \tag{3.46}$$

On the other hand, we have

$$\begin{aligned}\phi(u, x_n) - \phi(u, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle\\ &= \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|.\end{aligned} \tag{3.47}$$

Noticing (3.15) and (3.16), we obtain

$$\phi(u, x_n) - \phi(u, u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.48}$$

It follows that

$$\phi(u_n, y_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.49}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.50}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we get

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{3.51}$$

From the assumption $\liminf_{n\to\infty} r_n > a$, we get

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.52}$$

Noticing that $u_n = K_{r_n} y_n$, we have

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C. \tag{3.53}$$

Hence,

$$H(u_{n_i}, y) + \frac{1}{r_{n_i}} \langle y - u_{n_i}, Ju_{n_i} - Jy_{n_i} \rangle \geq 0, \quad \forall y \in C. \tag{3.54}$$

From the (A2), we note that

$$\|y - u_{n_i}\| \frac{\|Ju_{n_i} - Jy_{n_i}\|}{r_{n_i}} \geq \frac{1}{r_{n_i}} \langle y - u_{n_i}, Ju_{n_i} - Jy_{n_i} \rangle \geq -H(u_{n_i}, y) \geq H(y, u_{n_i}), \quad \forall y \in C. \tag{3.55}$$

Taking the limit as $n \to \infty$ in the above inequality, and from (A4) and $u_{n_i} \rightharpoonup u$, we have $H(y,u) \leq 0$, for all $y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)u$. Noticing that $y, u \in C$, we obtain $y_t \in C$, which yields that $H(y_t, u) \leq 0$. It follows from (A1) that

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1-t)H(y_t, u) \leq tH(y_t, y). \tag{3.56}$$

That is, $H(y_t, y) \geq 0$.

Let $t \downarrow 0$, from (A3), we obtain $H(u, y) \geq 0$, for all $y \in C$. This implies that $u \in \Omega$. Hence, $u \in \Theta := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \cap \Omega \cap VI(C, B) \cap A^{-1}(0)$.

Finally, we show that $u = \Pi_{\Theta} x$. Indeed, from $x_n = \Pi_{C_n} x$ and Lemma 2.6, we have

$$\langle Jx - Jx_n, x_n - z \rangle \geq 0, \quad \forall z \in C_n. \tag{3.57}$$

Since $\Theta \subset C_n$, we also have

$$\langle Jx - Jx_n, x_n - p \rangle \geq 0, \quad \forall p \in \Theta. \tag{3.58}$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx - Ju, u - p \rangle \geq 0, \quad \forall p \in \Theta. \tag{3.59}$$

By again Lemma 2.6, we can conclude that $u = \Pi_{\Theta} x_0$. This completes the proof. □

Corollary 3.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function, and let $A : E \to E^*$ be a maximal monotone operator satisfying $D(A) \subset E$. Let $J_r = (J + rA)^{-1} J$ for $r > 0$, let B be an α-inverse-strongly monotone operator of E into E^*, and let $\Upsilon : C \to E^*$ be a continuous and monotone mapping. Let T and S be two quasi-ϕ-nonexpansive mappings of E into itself with*

$F := F(T) \cap F(S) \cap \Omega \cap \mathrm{VI}(C,B) \cap A^{-1}(0) \neq \emptyset$ and $\|By\| \leq \|By - Bu\|$ for all $y \in C$ and $u \in \Theta$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and

$$
\begin{aligned}
w_n &= \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n),\\
z_n &= J^{-1}(\delta_n J(x_n) + (1-\delta_n)J(J_{r_n} w_n)),\\
y_n &= J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSz_n),\\
u_n &\in C \text{ such that } F(u_n, y) + \langle Yu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)\\
&\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,\\
C_{n+1} &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},\\
x_{n+1} &= \Pi_{C_{n+1}} x,
\end{aligned}
\tag{3.60}
$$

for all $n \in \mathbb{N}$. If the coefficient sequence $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ and $\{\delta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$, $\liminf_{n\to\infty}\gamma_n(1-\delta_n) > 0$, $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < (c^2\alpha)/2$, $1/c$ is the 2-uniformly convexity constant of E. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Theta x$.

Remark 3.3. Theorem 3.1 and Corollary 3.2 extended and improved the results of [40] by extending the mapping from two-relatively quasi-nonexpansive mappings to two countable families of quasi-ϕ-nonexpansive mappings and improving the iterative scheme to be more general, and finally, we proposed the better new conditions for the coefficient sequences which was imposed in our algorithm.

4. Applications

4.1. A Zero Point of Inverse-Strongly Monotone Operators

Next, we consider the problem of finding a zero point of an inverse-strongly monotone operator of E into E^*. Assume that B satisfies the following conditions:

(C1) B is α-inverse-strongly monotone,

(C2) $B^{-1}0 = \{u \in E : Bu = 0\} \neq \emptyset$.

Hence, we also have the following result.

Corollary 4.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function, and let $A : E \to E^*$ be a maximal monotone operator satisfying $D(A) \subset E$. Let $J_r = (J + rA)^{-1}J$ for $r > 0$, let B be an α-inverse-strongly monotone operator of E into E^*, and let $Y : C \to E^*$ be a continuous and monotone mapping. Let T and S be two quasi-ϕ-nonexpansive mappings of E into itself with*

$$\Theta := F(T) \cap F(S) \cap \Omega \cap B^{-1}(0) \cap A^{-1}(0) \neq \emptyset. \tag{4.1}$$

Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and

$$\begin{aligned}
&w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n),\\
&z_n = J^{-1}(\delta_n J(x_n) + (1-\delta_n)J(J_{r_n}w_n)),\\
&y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSz_n),\\
&u_n \in C \text{ such that } F(u_n, y) + \langle \Upsilon u_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)\\
&\qquad\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,\\
&C_{n+1} = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\},\\
&x_{n+1} = \Pi_{C_{n+1}} x,
\end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. If the coefficient sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$, $\liminf_{n\to\infty}\gamma_n(1-\delta_n) > 0$, $\liminf_{n\to\infty} r_n > 0$, and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $1/c$ is the 2-uniformly convexity constant of E, then the sequence $\{x_n\}$ converges strongly to $\Pi_\Theta x$.

Proof. Setting $\widetilde{B}x \equiv 0$, for all $x \in C$, then $D(\widetilde{B}) = E$ and hence $C = E$ in Corollary 3.2, we also get $\Pi_E = I$. We also have $\mathrm{VI}(B, C) = \mathrm{VI}(B, E) = \{x \in E : Bx = 0\} \neq \emptyset$, and then, the condition $\|By\| \leq \|By - Bu\|$ holds for all $y \in E$ and $u \in B^{-1}0$. So, we obtain the result. □

4.2. Complementarity Problems

Let K be a nonempty, closed convex cone in E. We define the *polar* K^* of K as follows:

$$K^* = \{y^* \in E^* : \langle x, y^*\rangle \geq 0, \forall x \in K\}. \tag{4.3}$$

If $A : K \to E^*$ is an operator, then an element $u \in K$ is called a solution of the *complementarity problem* ([43]) if

$$Au \in K^*, \qquad \langle u, Au\rangle = 0. \tag{4.4}$$

The set of solutions of the complementarity problem is denoted by $C(A, K)$.

Corollary 4.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, and let K be a nonempty closed convex subset of E. Let F be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\varphi : K \to \mathbb{R}$ be a proper lower semicontinuous and convex function, and let $A : E \to E^*$ be a maximal monotone operator satisfying $D(A) \subset E$. Let $J_r = (J + rA)^{-1}J$ for $r > 0$, let B be an α-inverse-strongly monotone operator of E into E^*, and let $\Upsilon : K \to E^*$ be a continuous and monotone mapping. Let T and S be two quasi-ϕ-nonexpansive mappings of E into itself with*

$$\Theta := F(T) \cap F(S) \cap \Omega \cap C(B, K) \cap A^{-1}(0) \neq \emptyset, \tag{4.5}$$

and $\|By\| \leq \|By - Bu\|$ *for all* $y \in K$ *and* $u \in \Theta$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in E$, *and*

$$\begin{aligned}
w_n &= \Pi_K J^{-1}(Jx_n - \lambda_n Bx_n),\\
z_n &= J^{-1}(\delta_n J(x_n) + (1-\delta_n)J(J_{r_n}w_n)),\\
y_n &= J^{-1}\left(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSz_n\right),\\
u_n &\in K \text{ such that } F(u_n, y) + \langle Yu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n)\\
&\qquad + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in K,\\
K_{n+1} &= \{z \in K : \phi(z, u_n) \leq \phi(z, x_n)\}\\
x_{n+1} &= \Pi_{K_{n+1}} x,
\end{aligned} \tag{4.6}$$

for all $n \in \mathbb{N}$, *where* Π_K *is the generalized projection from E onto* K, J *is the duality mapping on E. If the coefficient sequence* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, *and* $\{\delta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$, $\liminf_{n\to\infty}\gamma_n(1-\delta_n) > 0$, $\liminf_{n\to\infty} r_n > 0$, *and* $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, $1/c$ *is the 2-uniformly convexity constant of E, then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_\Theta x$.

Proof. As in the proof of Takahashi in [43, Lemma 7.11], we have $\mathrm{VI}(B,K) = C(B,K)$. So, we obtain the above result. □

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