Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 716719, 19 pages
doi:10.1155/2012/716719

Research Article

Symbolic Computation and the Extended Hyperbolic Function Method for Constructing Exact Traveling Solutions of Nonlinear PDEs

Huang Yong,[1] Shang Yadong,[2,3] and Yuan Wenjun[2,3]

[1] *School of Computer Science and Educational Software, Guangzhou University, Guangzhou 510006, China*
[2] *School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China*
[3] *Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, China*

Correspondence should be addressed to Shang Yadong, gzydshang@126.com
and Yuan Wenjun, wjyuan1957@126.com

Received 31 May 2012; Revised 6 August 2012; Accepted 13 August 2012

Academic Editor: Renat Zhdanov

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On the basis of the computer symbolic system Maple and the extended hyperbolic function method, we develop a more mathematically rigorous and systematic procedure for constructing exact solitary wave solutions and exact periodic traveling wave solutions in triangle form of various nonlinear partial differential equations that are with physical backgrounds. Compared with the existing methods, the proposed method gives new and more general solutions. More importantly, the method provides a straightforward and effective algorithm to obtain abundant explicit and exact particular solutions for large nonlinear mathematical physics equations. We apply the presented method to two variant Boussinesq equations and give a series of exact explicit traveling wave solutions that have some more general forms. So consequently, the efficiency and the generality of the proposed method are demonstrated.

1. Introduction

The nonlinear phenomena are very important in a variety of scientific fields, especially in fluid dynamics, solid-state physics, hydrodynamics, plasma physics, elastic dynamics, acoustics, chemical physics, and nonlinear optics. Nonlinear evolution partial differential equations are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid-state mechanics, atmospheric physics, chemical reaction-diffusion dynamics, ion acoustics, and nonlinear vibration. The investigation of exact traveling wave solutions to nonlinear evolution partial differential equations plays an important role in the study of nonlinear science. The exact solution, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. It can also provide much physical information and more insights into the physical aspects of the nonlinear physical problem. During the past decades, much effort has been spent on the subject of obtaining the exact analytical solutions to the nonlinear evolution PDEs. Many powerful methods have been proposed such as the inverse scattering transformation method [1], the Bäcklund and Darboux transformation method [2, 3], the Hirota bilinear method [4], the Lie group reduction method [5], the tanh method [6], the tanh-coth method [7], the sine-cosine method [8, 9], the homogeneous balance method [10–12], the Jacobi elliptic function method [13, 14], the extended tanh method [15, 16], the F-expansion method and its extension [17, 18], the Riccati method [19, 20], and extended improved tanh-function method [21, 22]. With the development of symbolic computation, the tanh method, the F-expansion method, the sine-Gordon equation expansion method, and all kinds of auxiliary equation methods attract more and more researchers. In this paper, we present an effective extension to the projective Riccati equation method [19, 20] and extended improved tanh-function method [21, 22] and develop an effective Maple software package “PDESolver” to uniformly construct a series of traveling wave solutions including solitary wave solutions, singular traveling, rational, triangular periodic solitons for general nonlinear evolution equations. Our method can be regarded as an extension of the works by Wazwaz [23, 24] and Soliman [21]. For illustration, we apply the presented method to two variant Boussinesq equations

$$\begin{aligned} &H_t + (Hu)_x + u_{xxx} = 0, \\ &u_t + H_x + uu_x = 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &H_t + u_x + (Hu)_x - \alpha u_{xxx} = 0, \\ &u_t + H_x + uu_x - 3\alpha u_{xxt} = 0, \end{aligned} \tag{1.2}$$

where $H(x,t)$, $u(x,t)$ are the unknown functions depending on the temporal variable t and the spatial variable x. These two equations were introduced as models for water waves and called variant Boussinesq equations I and II, respectively [25]. Their symmetries, conservation laws, and inverse scattering transformation, and soliton solutions have been investigated by many authors (see [8, 16] and references therein). Many exact solutions have been obtained by many researchers using the sine-cosine method [8], the homogeneous balance method [10], the extended tanh method [16], and the the F-expansion method [17], respectively. We will give a series of new traveling wave solutions for the two equations. Some entirely new exact solitary wave solutions and periodic wave solutions of the equations are obtained.

The paper is organized as follows: in Section 2, we briefly describe what is the extended hyperbolic function method and how to use it to derive the traveling solutions of nonlinear PDEs. In Section 3, we apply the extended hyperbolic function method to (1.1) and (1.2) and establish many rational form solitary wave, rational form triangular periodic wave solutions. In Section 4, we briefly make a summary to the results that we have obtained.

2. The Extended Hyperbolic Function Method

Now we would like to outline the main steps of our method for solving nonlinear PDEs.

Consider the coupled Riccati equations

$$f'(\xi) = -f(\xi)g(\xi),$$
$$g'(\xi) = \varepsilon - r\varepsilon f(\xi) - g^2(\xi), \tag{2.1}$$

where $\varepsilon = \pm 1$ or $0, r$ is a constant. When $\varepsilon \neq 0$, we can obtain the following first integral as given:

$$g^2(\xi) = \varepsilon - 2r\varepsilon f(\xi) + Cf^2(\xi). \tag{2.2}$$

Step 1. Consider a system of N nonlinear evolution equations ($N \geq 1$) with n independent variables $x = (x_1, x_2, \ldots, x_n)$ and m dependent variables $u = (u_1, u_2, \ldots, u_m)$, given by

$$P_i\left(u, \partial u, \partial^2 u, \ldots, \partial^r u\right) = 0, \quad i = 1, 2, \ldots, N, \tag{2.3}$$

where $P_i, i = 1, 2, \ldots, N$, are in general nonlinear functions of their arguments, $\partial^j u$ denotes the coordinates with components $\partial^j u / (\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_j}) = u_{i_1 i_2 \cdots i_j}$, $i_j = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, k$, corresponding to all jth-order partial derivatives of u with respect to x.

We seek the following formal traveling wave solutions which are of important physical significance:

$$u(x,t) = u(\xi), \qquad \xi = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n + \xi_0, \quad \left(\sum_{i=1}^{n} k_i^2 \neq 0\right), \tag{2.4}$$

where k_i are constants to be determined later and ξ_0 is an arbitrary constant.

Then the nonlinear partial differential equations (2.3) reduce to a nonlinear ordinary differential equations

$$Q_i\left(u, u', u'', \ldots u^k\right) = 0, \quad i = 1, 2, \ldots, N, \tag{2.5}$$

where $'$ denotes $d/d\xi$.

Step 2. To seek the exact solutions of nonlinear partial differential equations (2.3), we assume that the solution of the ODEs (2.5) is of the following form.

(a) When $\varepsilon = \pm 1$ in (2.1), (2.2),

$$u_l(x,t) = u_l(\xi) = \sum_{i=0}^{l_n} a_{l,i}(f(\xi))^i + \sum_{j=1}^{l_n} b_{l,j}(f(\xi))^{j-1} g(\xi), \quad l = 1, 2, \ldots, m, \tag{2.6}$$

where the coefficients $a_{l,i}$ $(i = 0, 1, 2 \ldots, l_n)$ and $b_{l,j}$ $(j = 1, 2, \ldots, l_n)$ are constants to be determined latter.

(b) When $\varepsilon = 0$ in (2.1),

$$u_l(x,t) = \sum_{i=0}^{l_n} a_{l,i}\left(g(\xi)\right)^i, \quad l = 1,2,\ldots,m, \tag{2.7}$$

where $g'(\xi) = -g^2(\xi)$ and the coefficients $a_{l,i}$ $(i = 0,1,2\ldots,l_n)$ are constants to be determined.

Step 3. Balancing the highest-order derivative term and the nonlinear terms in (2.5), we get balance powers l_n (usually positive integer). If some one of l_n is a fraction or a negative integer, say s_n is negative, we make the following transformation:

$$u_s(\xi) = v_s^{s_n}, \tag{2.8}$$

then return to determine balance power n again.

Step 4. (a) When $\varepsilon \neq 0$, substituting (2.6) along with the conditions (2.1) and (2.2) into (2.5).

(b) When $\varepsilon = 0$, substituting (2.7) along with the condition $g'(\xi) = -g^2(\xi)$ into (2.5).

Then eliminating any derivative of (f,g) and any power of g higher than one and setting the coefficients of powers f_i $(i = 0,1,\ldots;)$ and $f_j g$ $(j = 0,1,\ldots)$ in the case (a) (setting the coefficients of the different powers g in the case (b)) to zero yield a set of overdetermined nonlinear algebraic equations with respect to the unknown variables k_i, $i = 1,2,\ldots,n$, $a_{l,i}$ $(i = 0,1,2\ldots,l_n)$, $b_{l,j}$ $(j = 1,2,\ldots,l_n)$, r, C. With the aid of Maple, we apply the Wu-elimination method [26] to solve the above overdetermined system of nonlinear algebraic equations, that yields the values of k_i, $i = 1,2,\ldots,n$, $a_{l,i}$ $(i = 0,1,2\ldots,l_n)$, $b_{l,j}$ $(j = 1,2,\ldots,l_n)$, r, C.

Step 5. We know that the coupled Riccati equations (2.1) admit the following special solutions.

(a) When $\varepsilon = 1$,

$$f(\xi) = \frac{1}{a\cosh\xi + b\sinh\xi + r}, \qquad g(\xi) = \frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi + r}, \tag{2.9}$$

and then $g^2(\xi) = 1 - 2rf(\xi) + (b^2 - a^2 + r^2)f^2(\xi)$.

(b) When $\varepsilon = -1$,

$$f(\xi) = \frac{1}{a\cos\xi + b\sin\xi + r}, \qquad g(\xi) = \frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi + r}, \tag{2.10}$$

and then $g^2(\xi) = -1 + 2rf(\xi) + (b^2 + a^2 - r^2)f^2(\xi)$.

(c) When $\varepsilon = 0$,

$$f(\xi) = \frac{C}{\xi}, \qquad g(\xi) = \frac{1}{\xi}, \tag{2.11}$$

where C is constant.

Thus according to (2.4), (2.6), (2.9), (2.10) (or (2.4), (2.7), (2.11)) and the conclusions in Step 4, we can obtain the multiple more general exact explicit traveling wave solutions of nonlinear partial differential equations (2.3).

Remark 2.1. Many types of traveling wave solutions such as $\tanh\xi$, $\coth\xi$, $\operatorname{sech}\xi$, $\operatorname{cosech}\xi$, $\tan\xi$, $\cot\xi$, $\sec\xi$, $\operatorname{cosec}\xi$, and $1/\xi$ can be obtained by considering the different values of a, b in (2.9), (2.10). Our proposed method is a generalization of the tanh method [6], the extended tanh method, the tanh-coth method [7, 15], the extended tanh-function method [27], the extended improved tanh-function method [21], and the recent works of Wazwaz [23, 24].

The proposed method supplies a unified formulation to construct abundant traveling wave solutions to nonlinear evolution partial differential equations of special physical significance. Furthermore, the presented method is readily computerizable by using symbolic software Maple. Based on the extended hyperbolic function method and computer symbolic software, we develop a Maple software package "PDESolver." Compared with packages RATH, ERATH, AJFM, TRWS, and RAEEM, "PDESolver" is more effective. "PDESolver" can obtain more exact traveling wave solutions.

3. Exact Solutions of the Variant Boussinesq Equations

As an example of the use of Maple software package "PDESolver," we first consider the variant Boussinesq equations (1.1).

According to the above method, to seek traveling wave solutions of (1.1), we make the transformation

$$u(x,t) = u(\xi), \quad H(x,t) = H(\xi), \quad \xi = kx + \omega t + \xi_0, \tag{3.1}$$

and thus (1.1) becomes

$$\begin{aligned} &\omega H'(\xi) + k(Hu)'(\xi) + k^3 u'''(\xi) = 0, \\ &\omega u'(\xi) + kH'(\xi) + ku(\xi)u'(\xi) = 0. \end{aligned} \tag{3.2}$$

Firstly we assume that the solutions of (3.1) are of the form

$$\begin{aligned} H(x,t) = H(\xi) &= \sum_{i=0}^{m} a_{1,i}(f(\xi))^i + \sum_{j=1}^{m} b_{1,j}(f(\xi))^{j-1} g, \\ u(x,t) = u(\xi) &= \sum_{i=0}^{n} a_{1,i}(f(\xi))^i + \sum_{j=1}^{n} b_{1,j}(f(\xi))^{j-1} g, \end{aligned} \tag{3.3}$$

where $a_{1,i}$, $b_{1,j}$, $i = 0, 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$, and $a_{2,i}$, $b_{2,j}$, $i = 0, 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$, are constants to be determined latter, f and g satisfy (2.1), (2.2), and $\varepsilon = \pm 1$. We can get the balancing powers $m = 2$, $n = 1$. So we have

$$\begin{aligned} H(\xi) &= a_{1,0} + a_{1,1} f(\xi) + a_{1,2} f^2(\xi) + b_{1,1} g(\xi) + b_{1,2} f(\xi) g(\xi), \\ u(\xi) &= a_{2,0} + a_{2,1} f(\xi) + b_{2,1} g(\xi), \end{aligned} \tag{3.4}$$

where $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $b_{1,1}$, $b_{1,2}$, $a_{2,0}$, $a_{2,1}$, $b_{2,1}$ are constants to be determined later.

With the aid of Maple, substituting (3.4) along with (2.1) and (2.2) into (3.2), yields a set of algebraic equations for $f^i(\xi) g^j(\xi)$. Setting the coefficients of these terms $f^i(\xi) g^j(\xi)$ to zero yields a set of overdetermined nonlinear algebraic equations with respect to $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $b_{1,1}$, $b_{1,2}$, $a_{2,0}$, $a_{2,1}$, $b_{2,1}$, k, ω, and r:

$$\begin{gathered} -2k(b_{1,2} + a_{2,1} b_{2,1}) = 0, \\ -k\left(b_{2,1}^2 + 2a_{1,2} + a_{2,1}^2\right) = 0, \\ -3k\left(2k^2 a_{2,1} + b_{1,2} b_{2,1} + a_{1,2} a_{2,1}\right) = 0, \\ -3k\left(b_{1,2} a_{2,1} + a_{1,2} b_{2,1} + 2k^2 b_{2,1}\right) = 0, \\ k b_{2,1}^2 \varepsilon r - k a_{2,0} a_{2,1} - \omega a_{2,1} - k a_{1,1} = 0, \\ \varepsilon(-k a_{2,1} b_{2,1} + k a_{2,0} b_{2,1} r + k b_{1,1} r - k b_{1,2} + \omega b_{2,1} r) = 0, \\ 3k b_{1,2} \varepsilon r + 3k a_{2,1} b_{2,1} \varepsilon r - k b_{1,1} - k a_{2,0} b_{2,1} - \omega b_{2,1} = 0, \\ -k a_{1,1} a_{2,0} - k b_{1,2} \varepsilon b_{2,1} - k^3 a_{2,1} \varepsilon - k a_{1,0} a_{2,1} - \omega a_{1,1} + 2k b_{1,1} \varepsilon r b_{2,1} = 0, \\ -2k b_{1,1} b_{2,1} - 2k a_{1,1} a_{2,1} - 2k a_{1,2} a_{2,0} + 6k^3 a_{2,1} \varepsilon r - 2\omega a_{1,2} + 4k b_{1,2} \varepsilon r b_{2,1} = 0, \\ 5k b_{1,2} \varepsilon r a_{2,1} + 5k a_{1,2} b_{2,1} \varepsilon r - 2\omega b_{1,2} + 12k^3 b_{2,1} \varepsilon r - 2k b_{1,2} a_{2,0} - 2k b_{1,1} a_{2,1} - 2k a_{1,1} b_{2,1} = 0, \\ \varepsilon\left(k^3 b_{2,1} \varepsilon r - \omega b_{1,2} + k a_{1,0} b_{2,1} r + k b_{1,1} r a_{2,0} + \omega b_{1,1} r - k b_{1,2} a_{2,0} - k a_{1,1} b_{2,1} - k b_{1,1} a_{2,1}\right) = 0, \\ 3k a_{1,1} b_{2,1} \varepsilon r - k a_{1,0} b_{2,1} - 3k^3 b_{2,1} \varepsilon^2 r^2 - k b_{1,1} a_{2,0} - 2k b_{1,2} \varepsilon a_{2,1} - \omega b_{1,1} \\ + 3\omega b_{1,2} \varepsilon r - 4k^3 b_{2,1} \varepsilon + 3k b_{1,2} \varepsilon r a_{2,0} + 3k b_{1,1} \varepsilon r a_{2,1} - 2k a_{1,2} b_{2,1} \varepsilon = 0. \end{gathered} \tag{3.5}$$

To get a nontrivial solution of (1.1), we assume that $k \neq 0$ and $a_{1,1}^2 + a_{1,2}^2 + b_{1,1}^2 + b_{1,2}^2 + a_{2,1}^2 + b_{2,1}^2 \neq 0$. By making use of the Maple software package "PDESolver" which is based on the Wu-elimination method [26], we solve (3.5) and get the following nontrivial solutions with the aid of the computer program Maple 12.

Set $\xi = kx + \omega t + \xi_0$, and $k(\neq 0)$, ω, r are real constants.

When $\varepsilon = 1$.

Case 1. $r = r$, $a_{1,0} = 0$, $a_{1,1} = k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = k$, $b_{1,1} = 0$, $b_{1,2} = k^2$, $b_{2,1} = -k$. By applying these to (3.4), one gets a solution of (1.1):

$$H_1(x,t) = \frac{k^2 r}{a\cosh\xi + b\sinh\xi + r} - \frac{k^2}{(a\cosh\xi + b\sinh\xi + r)^2} + \frac{k^2(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi + r)^2},$$
$$u_1(x,t) = -\frac{\omega}{k} + \frac{k}{a\cosh\xi + b\sinh\xi + r} - \frac{k(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi + r)}. \tag{3.6}$$

Case 2. $r = r$, $a_{1,0} = 0$, $a_{1,1} = k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = k$, $b_{1,1} = 0$, $b_{1,2} = -k^2$, $b_{2,1} = k$. By applying these to (3.4), one gets a solution of (1.1):

$$H_2(x,t) = \frac{k^2 r}{a\cosh\xi + b\sinh\xi + r} - \frac{k^2}{(a\cosh\xi + b\sinh\xi + r)^2} - \frac{k^2(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi + r)^2},$$
$$u_2(x,t) = -\frac{\omega}{k} + \frac{k}{a\cosh\xi + b\sinh\xi + r} + \frac{k(a\sinh\xi + b\cosh\xi)}{a\cosh\xi + b\sinh\xi + r}. \tag{3.7}$$

Case 3. $r = r$, $a_{1,0} = 0$, $a_{1,1} = k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = -k$, $b_{1,1} = 0$, $b_{1,2} = k^2$, $b_{2,1} = k$. By applying these to (3.4), one gets a solution of (1.1):

$$H_3(x,t) = \frac{k^2 r}{a\cosh\xi + b\sinh\xi + r} - \frac{k^2}{(a\cosh\xi + b\sinh\xi + r)^2} + \frac{k^2(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi + r)^2},$$
$$u_3(x,t) = -\frac{\omega}{k} - \frac{k}{a\cosh\xi + b\sinh\xi + r} + \frac{k(a\sinh\xi + b\cosh\xi)}{a\cosh\xi + b\sinh\xi + r}. \tag{3.8}$$

Case 4. $r = r$, $a_{1,0} = 0$, $a_{1,1} = k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = -k$, $b_{1,1} = 0$, $b_{1,2} = -k^2$, $b_{2,1} = -k$. By applying these to (3.4), one gets a solution of (1.1):

$$H_4(x,t) = \frac{k^2 r}{a\cosh\xi + b\sinh\xi + r} - \frac{k^2}{(a\cosh\xi + b\sinh\xi + r)^2} - \frac{k^2(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi + r)^2},$$
$$u_4(x,t) = -\frac{\omega}{k} - \frac{k}{a\cosh\xi + b\sinh\xi + r} - \frac{k(a\sinh\xi + b\cosh\xi)}{a\cosh\xi + b\sinh\xi + r}. \tag{3.9}$$

Case 5. $r = (\omega + ka_{2,0})/k^2$, $a_{1,0} = (-k^4+\omega^2+2\omega k a_{2,0}+k^2 a_{2,0}^2)/k^2$, $a_{1,1} = 2\omega+2ka_{2,0}$, $a_{1,2} = -2k^2$, $a_{2,0} = a_{2,0}$, $a_{2,1} = -2k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_5(x,t) &= \frac{-k^4 + (\omega + ka_{2,0})^2}{k^2} + \frac{2\omega + 2ka_{2,0}}{(a\cosh\xi + b\sinh\xi + ((\omega + ka_{2,0})/k^2))} \\ &\quad - \frac{2k^2}{(a\cosh\xi + b\sinh\xi + ((\omega + ka_{2,0})/k^2))^2}, \\ u_5(x,t) &= a_{2,0} - \frac{2k}{a\cosh\xi + b\sinh\xi + ((\omega + ka_{2,0})/k^2)}, \end{aligned} \tag{3.10}$$

where $a_{2,0}$ is an arbitrary real constant.

Case 6. $r = -((\omega + ka_{2,0})/k^2)$, $a_{1,0} = (-k^4 + \omega^2 + 2\omega k a_{2,0} + k^2 a_{2,0}^2)/k^2$, $a_{1,1} = -2\omega - 2ka_{2,0}$, $a_{1,2} = -2k^2$, $a_{2,0} = a_{2,0}$, $a_{2,1} = 2k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_6(x,t) &= \frac{-k^4 + (\omega + ka_{2,0})^2}{k^2} - \frac{2(\omega + ka_{2,0})}{a\cosh\xi + b\sinh\xi - ((\omega + ka_{2,0})/k^2)} \\ &\quad - \frac{2k^2}{(a\cosh\xi + b\sinh\xi - ((\omega + ka_{2,0})/k^2))^2}, \\ u_6(x,t) &= a_{2,0} + \frac{2k}{a\cosh\xi + b\sinh\xi - ((\omega + ka_{2,0})/k^2)}. \end{aligned} \tag{3.11}$$

Case 7. $r = 0$, $a_{1,0} = 0$, $a_{1,1} = 0$, $a_{1,2} = -2k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = 0$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 2k$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_7(x,t) &= -\frac{2k^2}{(a\cosh\xi + b\sinh\xi)^2}, \\ u_7(x,t) &= -\frac{\omega}{k} + \frac{2k(a\sinh\xi + b\cosh\xi)}{a\cosh\xi + b\sinh\xi}. \end{aligned} \tag{3.12}$$

Case 8. $r = 0$, $a_{1,0} = 0$, $a_{1,1} = 0$, $a_{1,2} = -2k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = 0$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = -2k$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_8(x,t) &= -\frac{2k^2}{(a\cosh\xi + b\sinh\xi)^2}, \\ u_8(x,t) &= -\frac{\omega}{k} - \frac{2k(a\sinh\xi + b\cosh\xi)}{a\cosh\xi + b\sinh\xi}. \end{aligned} \tag{3.13}$$

While taking $a = 1, b = 0$ in solutions H_7, u_7 (3.12) and H_8, u_8 (3.13), we can obtain the exact soliton solutions obtained in [10] by homogeneous balance method. While taking $a = 0$,

$b = 1$ in solutions H_7, u_7 (3.12) and H_8, u_8 (3.13), we obtain singular traveling wave solutions of $\operatorname{csch}^2\xi$, $\coth\xi$ type corresponding to the soliton solutions of $\operatorname{sech}^2\xi$, $\tanh\xi$ type. Thus the results obtained in [10] are special cases of this paper. We would like to emphasize that the solutions H_5, u_5 and H_6, u_6 are solutions that have an entirely new form and proposed firstly in this paper. It should be pointed out that the solutions H_1, u_1(3.6) to H_4, u_4 (3.9) have a more general form than those that appeared in previous literatures. While setting $a = 1, b = 0$ (or $a = 0$, $b = 1$, resp.) in some of these solutions, we can obtain all solutions in hyperbolic function form presented in literatures [8, 16]. Since there are some parameters that can take different values, these solutions include abundant new information.

When $\varepsilon = -1$.

Case 9. $r = r$, $a_{1,0} = 0$, $a_{1,1} = -k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = k$, $b_{1,1} = 0$, $b_{1,2} = k^2$, $b_{2,1} = -k$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_9(x,t) &= -\frac{k^2 r}{a\cos\xi + b\sin\xi + r} - \frac{k^2}{(a\cos\xi + b\sin\xi + r)^2} + \frac{k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}, \\ u_9(x,t) &= -\frac{\omega}{k} + \frac{k}{a\cos\xi + b\sin\xi + r} - \frac{k(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}. \end{aligned} \tag{3.14}$$

Case 10. $r = r$, $a_{1,0} = 0$, $a_{1,1} = -k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = k$, $b_{1,1} = 0$, $b_{1,2} = -k^2$, $b_{2,1} = k$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_{10}(x,t) &= -\frac{k^2 r}{a\cos\xi + b\sin\xi + r} - \frac{k^2}{(a\cos\xi + b\sin\xi + r)^2} - \frac{k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}, \\ u_{10}(x,t) &= -\frac{\omega}{k} + \frac{k}{a\cos\xi + b\sin\xi + r} + \frac{k(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}. \end{aligned} \tag{3.15}$$

Case 11. $r = r$, $a_{1,0} = 0$, $a_{1,1} = -k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = -k$, $b_{1,1} = 0$, $b_{1,2} = k^2$, $b_{2,1} = k$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_{11}(x,t) &= -\frac{k^2 r}{a\cos\xi + b\sin\xi + r} - \frac{k^2}{(a\cos\xi + b\sin\xi + r)^2} + \frac{k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}, \\ u_{11}(x,t) &= -\frac{\omega}{k} - \frac{k}{a\cos\xi + b\sin\xi + r} + \frac{k(-a\sin\xi + b\cos\xi)}{a\cos\xi + b\sin\xi + r}. \end{aligned} \tag{3.16}$$

Case 12. $r = r$, $a_{1,0} = 0$, $a_{1,1} = -k^2 r$, $a_{1,2} = -k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = -k$, $b_{1,1} = 0$, $b_{1,2} = -k^2$, $b_{2,1} = -k$. By applying these to (3.4), one gets a solution of (1.1):

$$\begin{aligned} H_{12}(x,t) &= -\frac{k^2 r}{a\cos\xi + b\sin\xi + r} - \frac{k^2}{(a\cos\xi + b\sin\xi + r)^2} - \frac{k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}, \\ u_{12}(x,t) &= -\frac{\omega}{k} - \frac{k}{a\cos\xi + b\sin\xi + r} - \frac{k(-a\sin\xi + b\cos\xi)}{a\cos\xi + b\sin\xi + r}. \end{aligned} \tag{3.17}$$

Case 13. $r = (ka_{2,0}+\omega)/k^2$, $a_{1,0} = (k^4+k^2a_{2,0}^2+2ka_{2,0}\,\omega+\omega^2)/k^2$, $a_{1,1} = -2ka_{2,0}-2\omega$, $a_{1,2} = -2k^2$, $a_{2,0} = a_{2,0}$, $a_{2,1} = 2k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$. By applying these to (3.4), one gets a solution of (1.1):

$$H_{13}(x,t) = \frac{k^4 + (\omega + ka_{2,0})^2}{k^2} + \frac{-2ka_{2,0} - 2\omega}{a\cos\xi + b\sin\xi + ((ka_{2,0}+\omega)/k^2)} - \frac{2k^2}{\left(a\cos\xi + b\sin\xi + ((ka_{2,0}+\omega)/k^2)\right)^2},$$
$$u_{13}(x,t) = a_{2,0} + \frac{2k}{a\cos\xi + b\sin\xi + ((ka_{2,0}+\omega)/k^2)}, \tag{3.18}$$

where $a_{2,0}$ is an arbitrary real constant.

Case 14. $r = -((ka_{2,0}+\omega)/k^2)$, $a_{1,0} = (k^4 + k^2a_{2,0}^2 + 2ka_{2,0}\,\omega + \omega^2)/k^2$, $a_{1,1} = 2ka_{2,0} + 2\omega$, $a_{1,2} = -2k^2$, $a_{2,0} = a_{2,0}$, $a_{2,1} = -2k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$. By applying these to (3.4), one gets a solution of (1.1):

$$H_{14}(x,t) = \frac{k^4 + (\omega + ka_{2,0})^2}{k^2} + \frac{2ka_{2,0} + 2\omega}{a\cos\xi + b\sin\xi - ((ka_{2,0}+\omega)/k^2)} - \frac{2k^2}{\left(a\cos\xi + b\sin\xi - ((ka_{2,0}+\omega)/k^2)\right)^2},$$
$$u_{14}(x,t) = a_{2,0} - \frac{2k}{a\cos\xi + b\sin\xi - ((ka_{2,0}+\omega)/k^2)}, \tag{3.19}$$

where $a_{2,0}$ is an arbitrary real constant.

Case 15. $r = 0$, $a_{1,0} = 0$, $a_{1,1} = 0$, $a_{1,2} = -2k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = 0$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 2k$. By applying these to (3.4), one gets a solution of (1.1):

$$H_{15}(x,t) = -\frac{2k^2}{(a\cos\xi + b\sin\xi)^2},$$
$$u_{15}(x,t) = -\frac{\omega}{k} + \frac{2k(-a\sin\xi + b\cos\xi)}{a\cos\xi + b\sin\xi}. \tag{3.20}$$

Case 16. $r = 0$, $a_{1,0} = 0$, $a_{1,1} = 0$, $a_{1,2} = -2k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = 0$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = -2k$. By applying these to (3.4), one gets a solution of (1.1):

$$H_{16}(x,t) = -\frac{2k^2}{(a\cos\xi + b\sin\xi)^2},$$
$$u_{16}(x,t) = -\frac{\omega}{k} - \frac{2k(-a\sin\xi + b\cos\xi)}{a\cos\xi + b\sin\xi}. \tag{3.21}$$

When $\varepsilon = 0$.

Balancing the highest-order derivative term and the nonlinear terms in (2.5), we get balance powers $m = 2$, $n = 1$. By the method described in Section 2 Step 4(b), we get a set of overdetermined nonlinear algebraic equations with respect to $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $a_{2,0}$, $a_{2,1}$, k, ω, and r:

$$\begin{aligned} &-k\left(2a_{1,2} + a_{2,1}^2\right) = 0, \\ &-3ka_{2,1}\left(a_{1,2} + 2k^2\right) = 0, \\ &-\omega a_{1,1} - ka_{1,1}a_{2,0} - a_{2,1}ka_{1,0} = 0, \\ &-2\omega a_{1,2} - 2ka_{1,1}a_{2,1} - 2ka_{1,2}a_{2,0} = 0, \\ &-a_{2,1}\omega - ka_{1,1} - a_{2,1}ka_{2,0} = 0. \end{aligned} \tag{3.22}$$

Solving (3.22), we get the following.

Case 17. $a_{1,0} = 0$, $a_{1,1} = 0$, $a_{1,2} = -2k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = 2k$. By applying these to (2.7), one gets a solution of (1.1):

$$\begin{aligned} H_{17}(x,t) &= -\frac{2k^2}{(kx + \omega t + \xi_0)^2}, \\ u_{17}(x,t) &= -\frac{\omega}{k} + \frac{2k}{kx + \omega t + \xi_0}. \end{aligned} \tag{3.23}$$

Case 18. $a_{1,0} = 0$, $a_{1,1} = 0$, $a_{1,2} = -2k^2$, $a_{2,0} = -(\omega/k)$, $a_{2,1} = -2k$. By applying these to (2.7), one gets a solution of (1.1):

$$\begin{aligned} H_{18}(x,t) &= -\frac{2k^2}{(kx + \omega t + \xi_0)^2}, \\ u_{18}(x,t) &= -\frac{\omega}{k} + \frac{2k}{kx + \omega t + \xi_0}. \end{aligned} \tag{3.24}$$

The periodic wave solutions of triangle function types H_9, u_9 (3.14) to H_{16}, u_{16} (3.21) and the solitary wave solutions of rational function form H_{17}, u_{17} (3.23), H_{18}, u_{18} (3.24) have not been appeared in [10]. The periodic wave solutions H_{13}, u_{13} (3.18) and H_{14}, u_{14} (3.19) have the entirely new form and proposed firstly in this paper. The periodic wave solutions of triangle function types H_9, u_9 (3.14) to H_{12}, u_{12} (3.17) have more general form than those obtained in literatures [8, 16]. While taking $a = 1$, $b = 0$ (or $a = 0$, $b = 1$, resp.) in some of these solutions, we will obtain all the periodic wave solutions in triangle function form that have been obtained in literatures [8, 16, 17].

We next deal with the variant Boussinesq equation (1.2) in a similar way to (1.1).

Using transformation (3.1), we reduce (1.2) to a system of nonlinear ODEs:

$$\begin{aligned} &\omega H'(\xi) + ku' + (Hu)'(\xi) - \alpha k^3 u'''(\xi) = 0, \\ &\omega u'(\xi) + kH'(\xi) + ku(\xi)u'(\xi) - 3\alpha k^3 u'''(\xi) = 0. \end{aligned} \tag{3.25}$$

Firstly we assume that the solutions of (3.25) are of the form (3.3) with f and g satisfying (2.1), (2.2), and $\varepsilon = \pm 1$. We can get the balancing powers $m = 2$, $n = 2$. So we have

$$\begin{aligned} H(\xi) &= a_{1,0} + a_{1,1}f(\xi) + a_{1,2}f(\xi)^2 + b_{1,1}g(\xi) + b_{1,2}f(\xi)g(\xi), \\ u(\xi) &= a_{2,0} + a_{2,1}f(\xi) + a_{2,2}f(\xi)^2 + b_{2,1}g(\xi) + b_{2,2}f(\xi)g(\xi), \end{aligned} \tag{3.26}$$

where $\xi = kx + \omega t + \xi_0$, and $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $a_{2,0}$, $a_{2,1}$, $a_{2,2}$, $b_{1,1}$, $b_{1,2}$, $b_{2,1}$, $b_{2,2}$ are constants to be determined later.

Substituting (3.26) with (2.1) and (2.2) into (3.25) and using the Maple yield

$$\begin{gathered}
4kb_{2,2}(18\gamma\omega k - a_{2,2}) = 0, \\
2k\left(36\gamma\omega a_{2,2}k - b_{2,2}^2 - a_{2,2}^2\right) = 0, \\
4k\left(6\gamma b_{2,2}k^2 - a_{1,2}b_{2,2} - a_{2,2}b_{1,2}\right) = 0, \\
4k\left(6\gamma a_{2,2}k^2 - b_{2,2}b_{1,2} - a_{1,2}a_{2,2}\right) = 0, \\
-3k\left(30k\gamma\omega a_{2,2}r - 6k\gamma\omega a_{2,1} + b_{2,2}b_{2,1} - b_{2,2}^2 r + a_{2,1}a_{2,2}\right) = 0, \\
-k(-18k\gamma\omega b_{2,1} + 180k\gamma\omega b_{2,2}r + 3a_{2,2}b_{2,1} + 3a_{2,1}b_{2,2} - 7a_{2,2}b_{2,2}r) = 0, \\
-ka_{1,1} - kb_{2,1}b_{2,2} - ka_{2,0}a_{2,1} + 3\gamma\omega k^2 a_{2,1} - \omega a_{2,1} + kb_{2,1}^2 r = 0, \\
-2k^2\gamma a_{2,1} + 10k^2\gamma a_{2,2}r + a_{1,1}a_{2,2} - 2b_{1,2}rb_{2,2} + a_{1,2}a_{2,1} + b_{2,1}b_{1,2} + b_{2,2}b_{1,1} = 0, \\
-ka_{1,0}a_{2,1} + \gamma k^3 a_{2,1} - kb_{1,2}b_{2,1} - kb_{1,1}b_{2,2} - ka_{1,1}a_{2,0} - \omega a_{1,1} - ka_{2,1} + 2kb_{1,1}rb_{2,1} = 0, \\
-k\Big(60k^2\gamma b_{2,2}r - 6k^2\gamma b_{2,1} - 7b_{1,2}ra_{2,2} + 3a_{1,2}b_{2,1} + 3a_{2,1}b_{1,2} \\
-7a_{1,2}b_{2,2}r + 3a_{2,2}b_{1,1} + 3a_{1,1}b_{2,2}\Big) = 0, \\
-2ka_{1,2} - 2\omega a_{2,2} - 2ka_{2,0}a_{2,2} - 18\gamma\omega k^2 a_{2,1}r + 4kb_{2,1}b_{2,2}r - kb_{2,2}^2 \\
-kb_{2,1}^2 + 24\gamma\omega k^2 a_{2,2} - ka_{2,1}^2 = 0,
\end{gathered}$$

$$
\begin{aligned}
&-ka_{2,0}b_{2,2}+kb_{1,1}r-kb_{1,2}+3\gamma\omega k^2b_{2,2}+\omega b_{2,1}r-ka_{2,1}b_{2,1}\\
&\quad-\omega b_{2,2}-3\gamma\omega k^2b_{2,1}r+ka_{2,0}b_{2,1}r=0,\\
&-2kb_{1,2}+60\gamma\omega k^2b_{2,2}+5ka_{2,1}b_{2,2}r-2ka_{2,1}b_{2,1}-2ka_{2,0}b_{2,2}\\
&\quad-36\gamma\omega k^2b_{2,1}r+90\gamma\omega k^2b_{2,2}r^2-3ka_{2,2}b_{2,2}-2\omega b_{2,2}+5ka_{2,2}b_{2,1}r=0,\\
&-2ka_{1,2}a_{2,0}-2kb_{1,1}b_{2,1}-2kb_{1,2}b_{2,2}+4kb_{1,2}rb_{2,1}-2ka_{1,0}a_{2,2}\\
&\quad+8\gamma k^3a_{2,2}-2\omega a_{1,2}+4kb_{1,1}rb_{2,2}-6\gamma k^3a_{2,1}r-2ka_{1,1}a_{2,1}-2ka_{2,2}=0,\\
&-kb_{1,2}a_{2,0}+ka_{1,0}b_{2,1}r-ka_{1,1}b_{2,1}-kb_{1,1}a_{2,1}+kb_{2,1}r+\gamma k^3b_{2,2}\\
&\quad-\omega b_{1,2}+kb_{1,1}ra_{2,0}-kb_{2,2}-ka_{1,0}b_{2,2}+\omega b_{1,1}r-\gamma k^3b_{2,1}r=0,\\
&12\gamma\omega k^2b_{2,1}+3ka_{2,0}b_{2,2}r+3ka_{2,1}b_{2,1}r+9\gamma\omega k^2b_{2,1}r^2-2ka_{2,1}b_{2,2}\\
&\quad-45\gamma\omega k^2b_{2,2}r-\omega b_{2,1}-kb_{1,1}+3kb_{1,2}r-ka_{2,0}b_{2,1}-2ka_{2,2}b_{2,1}+3\omega b_{2,2}r=0,\\
&5kb_{1,2}ra_{2,1}-2ka_{1,0}b_{2,2}-2kb_{2,2}-3kb_{1,2}a_{2,2}-2ka_{1,1}b_{2,1}\\
&\quad-2\omega b_{1,2}-12\gamma k^3b_{2,1}r+5ka_{1,1}b_{2,2}r+5kb_{1,1}ra_{2,2}+30\gamma k^3b_{2,2}r^2\\
&\quad-2kb_{1,1}a_{2,1}+20\gamma k^3b_{2,2}+5ka_{1,2}b_{2,1}r-2kb_{1,2}a_{2,0}-3ka_{1,2}b_{2,2}=0,\\
&-ka_{1,0}b_{2,1}-kb_{1,1}a_{2,0}+3kb_{1,2}ra_{2,0}-2ka_{1,2}b_{2,1}-kb_{2,1}-2kb_{1,2}a_{2,1}\\
&\quad+3\omega b_{1,2}r-2kb_{1,1}a_{2,2}+4\gamma k^3b_{2,1}+3kb_{1,1}ra_{2,1}-\omega b_{1,1}+3ka_{1,1}b_{2,1}r\\
&\quad-15\gamma k^3b_{2,2}r+3ka_{1,0}b_{2,2}r-2ka_{1,1}b_{2,2}+3kb_{2,2}r+3\gamma k^3b_{2,1}r^2=0.
\end{aligned}
\tag{3.27}
$$

With the aid of the computer program Maple 12, make use of the Maple software package PDESolver developed by the authors, which is based on the Wu-elimination method [26]; solving (3.27), one gets the following nontrivial solutions.

Set $\xi = kx + \omega t + \xi_0$, and $k(\neq 0)$, ω, r are real constants.

For $\varepsilon = 1$, there are 7 solutions.

Case 1. $\varepsilon = 1$, $r = r$, $a_{1,0} = (-36\omega^2 + 18\alpha\omega^2k^2 + k^2)/36\omega^2$, $a_{1,1} = -3k^2\alpha r$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = (-k^2 + 18\alpha\omega^2k^2 - 6\omega^2)/6\omega k$, $a_{2,1} = -18k\alpha\omega r$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = 18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$
\begin{aligned}
H_1(x,t) &= \frac{-36\omega^2+18\alpha\omega^2k^2+k^2}{36\omega^2}-\frac{3k^2\alpha r}{a\cosh\xi+b\sinh\xi+r}+\frac{3\alpha k^2(1+a\sinh\xi+b\cosh\xi)}{(a\cosh\xi+b\sinh\xi+r)^2},\\
u_1(x,t) &= \frac{-k^2+18\alpha\omega^2k^2-6\omega^2}{6\omega k}-\frac{18k\alpha\omega r}{a\cosh\xi+b\sinh\xi+r}+\frac{18\alpha\omega k(1+a\sinh\xi+b\cosh\xi)}{(a\cosh\xi+b\sinh\xi+r)^2}.
\end{aligned}
\tag{3.28}
$$

Case 2. $\varepsilon = 1, r = r$, $a_{1,0} = (-36\omega^2 + 18\alpha\omega^2 k^2 + k^2)/36\omega^2$, $a_{1,1} = -3k^2\alpha r$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = (-k^2 + 18\gamma\omega^2 k^2 - 6\omega^2)/6\omega k$, $a_{2,1} = -18k\alpha\omega r$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = -3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = -18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$H_2(x,t) = \frac{-36\omega^2 + 18\alpha\omega^2 k^2 + k^2}{36\omega^2} - \frac{3k^2\alpha r}{a\cosh\xi + b\sinh\xi + r} - \frac{3\alpha k^2(a\sinh\xi + b\cosh\xi - 1)}{(a\cosh\xi + b\sinh\xi + r)^2},$$

$$u_2(x,t) = \frac{-k^2 + 18\alpha\omega^2 k^2 - 6\omega^2}{6\omega k} - \frac{18k\alpha\omega r}{a\cosh\xi + b\sinh\xi + r} - \frac{18\alpha\omega k(a\sinh\xi + b\cosh\xi - 1)}{(a\cosh\xi + b\sinh\xi + r)^2}. \tag{3.29}$$

Case 3. $\varepsilon = 1$, $r = 0$, $a_{1,0} = (-36\omega^2 + 18\alpha\omega^2 k^2 + k^2)/36\omega^2$, $a_{1,1} = 0$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = (-k^2 + 18\alpha\omega^2 k^2 - 6\omega^2)/6\omega k$, $a_{2,1} = 0$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = 18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$H_3(x,t) = \frac{-36\omega^2 + 18\alpha\omega^2 k^2 + k^2}{36\omega^2} + \frac{3\alpha k^2}{(a\cosh\xi + b\sinh\xi)^2} + \frac{3\alpha k^2(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi)^2},$$

$$u_3(x,t) = \frac{-k^2 + 18\alpha\omega^2 k^2 - 6\omega^2}{6\omega k} + \frac{18\alpha\omega k}{(a\cosh\xi + b\sinh\xi)^2} + \frac{18\alpha\omega k(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi)^2}. \tag{3.30}$$

Case 4. $\varepsilon = 1$, $r = 0$, $a_{1,0} = (-36\omega^2 + 18\alpha\omega^2 k^2 + k^2)/36\omega^2$, $a_{1,1} = 0$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = (-k^2 + 18\alpha\omega^2 k^2 - 6\omega^2)/6\omega k$, $a_{2,1} = 0$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = -3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = -18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$H_4(x,t) = \frac{-36\omega^2 + 18\alpha\omega^2 k^2 + k^2}{36\omega^2} + \frac{3\alpha k^2}{(a\cosh\xi + b\sinh\xi)^2} - \frac{3\alpha k^2(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi)^2},$$

$$u_4(x,t) = \frac{-k^2 + 18\alpha\omega^2 k^2 - 6\omega^2}{6\omega k} + \frac{18\alpha\omega k}{(a\cosh\xi + b\sinh\xi)^2} - \frac{18\alpha\omega k(a\sinh\xi + b\cosh\xi)}{(a\cosh\xi + b\sinh\xi)^2}. \tag{3.31}$$

Case 5. $\varepsilon = 1$, $r = 0$, $a_{1,0} = (-36\omega^2 + 72\alpha\omega^2 k^2 + k^2)/36\omega^2$, $a_{1,1} = 0$, $a_{1,2} = 6\alpha k^2$, $a_{2,0} = (-k^2 - 6\omega^2 + 72\alpha\omega^2 k^2)/6\omega k$, $a_{2,1} = 0$, $a_{2,2} = 36\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$, $b_{2,2} = 0$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$H_5(x,t) = \frac{-36\omega^2 + 72\alpha\omega^2 k^2 + k^2}{36\omega^2} + \frac{6\alpha k^2}{(a\cosh\xi + b\sinh\xi)^2},$$

$$u_5(x,t) = \frac{-k^2 - 6\omega^2 + 72\alpha\omega^2 k^2}{6\omega k} + \frac{36\alpha\omega k}{(a\cosh\xi + b\sinh\xi)^2}. \tag{3.32}$$

Case 6. $\varepsilon = 1$, $r = 1$, $a_{1,0} = (-36\omega^2 + 18\alpha\omega^2k^2 + k^2)/36\omega^2$, $a_{1,1} = -6\alpha k^2$, $a_{1,2} = 6\alpha k^2$, $a_{2,0} = (-k^2 + 18\alpha\omega^2k^2 - 6\omega^2)/6\omega k$, $a_{2,1} = -36\alpha\omega k$, $a_{2,2} = 36\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$, $b_{2,2} = 0$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$H_6(x,t) = \frac{-36\omega^2 + 18\alpha\omega^2k^2 + k^2}{36\omega^2} - \frac{6\alpha k^2}{(a\cosh\xi + b\sinh\xi + 1)} + \frac{6\alpha k^2}{(a\cosh\xi + b\sinh\xi + 1)^2},$$
$$u_6(x,t) = \frac{-k^2 + 18\alpha\omega^2k^2 - 6\omega^2}{6\omega k} - \frac{36\alpha\omega k}{(a\cosh\xi + b\sinh\xi + 1)} + \frac{36\alpha\omega k}{(a\cosh\xi + b\sinh\xi + 1)^2}. \tag{3.33}$$

Case 7. $\varepsilon = 1$, $r = -1$, $a_{1,0} = (-36\omega^2 + 18\alpha\omega^2k^2 + k^2)/36\omega^2$, $a_{1,1} = 6\alpha k^2$, $a_{1,2} = 6\alpha k^2$, $a_{2,0} = (-k^2 + 18\alpha\omega^2k^2 - 6\omega^2)/6\omega k$, $a_{2,1} = 36\alpha\omega k$, $a_{2,2} = 36\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$, $b_{2,2} = 0$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$H_7(x,t) = \frac{-36\omega^2 + 18\alpha\omega^2k^2 + k^2}{36\omega^2} + \frac{6\alpha k^2}{(a\cosh\xi + b\sinh\xi - 1)} + \frac{6\alpha k^2}{(a\cosh\xi + b\sinh\xi - 1)^2},$$
$$u_7(x,t) = \frac{-k^2 + 18\alpha\omega^2k^2 - 6\omega^2}{6\omega k} + \frac{36\alpha\omega k}{(a\cosh\xi + b\sinh\xi - 1)} + \frac{36\alpha\omega k}{(a\cosh\xi + b\sinh\xi - 1)^2}. \tag{3.34}$$

While taking $a = 1, b = 0$ in solutions H_5, u_5 (3.32), we can get the soliton solutions in $\operatorname{sech}^2\xi, \operatorname{sech}^2\xi$ form obtained in [10] by homogeneous balance method. While taking $a = 1, b = 0$ in solutions H_5, u_5 (3.32), we can get the singular traveling soliton-like solutions in $\operatorname{csch}^2\xi, \operatorname{csch}^2\xi$ form. It should be emphasized that the solutions H_1, u_1 (3.28) and H_2, u_2 (3.29) are entirely new solutions that have not been proposed in previous literatures. We also should point out that H_3, u_3 (3.30), H_4, u_4 (3.31), H_7, u_7 (3.33), H_8, u_8 (3.34) include all soliton solutions in hyperbolic function form obtained in [8, 16]. Because there are some parameters that can take different values, these solutions possess more new information.

For $\varepsilon = -1$.

Case 8. $\varepsilon = -1$, $r = r$, $a_{1,0} = -((36\omega^2 + 18\alpha\omega^2k^2 - k^2)/36\omega^2)$, $a_{1,1} = 3k^2\alpha r$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = -(k^2 + 18\alpha\omega^2k^2 + 6\omega^2)/6\omega k$, $a_{2,1} = 18k\alpha\omega r$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = 18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$\begin{aligned} H_8(x,t) = &-\frac{36\omega^2 + 18\alpha\omega^2k^2 - k^2}{36\omega^2} + \frac{3k^2\alpha r}{(a\cos\xi + b\sin\xi + r)} \\ &+ \frac{3\alpha k^2}{(a\cos\xi + b\sin\xi + r)^2} + \frac{3\alpha k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}, \\ u_8(x,t) = &-\frac{k^2 + 18\alpha\omega^2k^2 + 6\omega^2}{6\omega k} + \frac{18k\alpha\omega r}{(a\cos\xi + b\sin\xi + r)} \\ &+ \frac{18\alpha\omega k}{(a\cos\xi + b\sin\xi + r)^2} + \frac{18\alpha\omega k(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}. \end{aligned} \tag{3.35}$$

Case 9. $\varepsilon = -1$, $r = r$, $a_{1,0} = -((36\omega^2 + 18\alpha\omega^2k^2 - k^2)/36\omega^2)$, $a_{1,1} = 3k^2\alpha r$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = -((k^2 + 18\alpha\omega^2k^2 + 6\omega^2)/6\omega k)$, $a_{2,1} = 18k\alpha\omega r$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = -3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = -18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.1):

$$\begin{aligned} H_9(x,t) &= -\frac{36\omega^2 + 18\alpha\omega^2k^2 - k^2}{36\omega^2} + \frac{3k^2\alpha r}{(a\cos\xi + b\sin\xi + r)} \\ &\quad + \frac{3\alpha k^2}{(a\cos\xi + b\sin\xi + r)^2} - \frac{3\alpha k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}, \\ u_9(x,t) &= -\frac{k^2 + 18\alpha\omega^2k^2 + 6\omega^2}{6\omega k} + \frac{18k\alpha\omega r}{(a\cos\xi + b\sin\xi + r)} \\ &\quad + \frac{18\alpha\omega k}{(a\cos\xi + b\sin\xi + r)^2} - \frac{18\alpha\omega k(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi + r)^2}. \end{aligned} \tag{3.36}$$

Case 10. $\varepsilon = -1$, $r = 0$, $a_{1,0} = -((36\omega^2 + 18\alpha\omega^2k^2 - k^2)/36\omega^2)$, $a_{1,1} = 0$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = -((k^2+18\alpha\omega^2k^2+6\omega^2)/6\omega k)$, $a_{2,1} = 0$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = 18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$\begin{aligned} H_{10}(x,t) &= -\frac{36\omega^2 + 18\alpha\omega^2k^2 - k^2}{36\omega^2} + \frac{3\alpha k^2}{(a\cos\xi + b\sin\xi)^2} + \frac{3\alpha k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi)^2}, \\ u_{10}(x,t) &= -\frac{k^2 + 18\alpha\omega^2k^2 + 6\omega^2}{6\omega k} + \frac{18\alpha\omega k}{(a\cos\xi + b\sin\xi)^2} + \frac{18\alpha\omega k(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi)^2}. \end{aligned} \tag{3.37}$$

Case 11. $\varepsilon = -1$, $r = 0$, $a_{1,0} = -((36\omega^2+18\alpha\omega^2k^2-k^2)/36\omega^2)$, $a_{1,1} = 0$, $a_{1,2} = 3\alpha k^2$, $a_{2,0} = -((k^2 + 18\alpha\omega^2k^2 + 6\omega^2)/6\omega k)$, $a_{2,1} = 0$, $a_{2,2} = 18\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = -3\alpha k^2$, $b_{2,1} = 0$, $b_{2,2} = -18\alpha\omega k$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$\begin{aligned} H_{11}(x,t) &= -\frac{36\omega^2 + 18\alpha\omega^2k^2 - k^2}{36\omega^2} + \frac{3\alpha k^2}{(a\cos\xi + b\sin\xi)^2} - \frac{3\alpha k^2(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi)^2}, \\ u_{11}(x,t) &= -\frac{k^2 + 18\alpha\omega^2k^2 + 6\omega^2}{6\omega k} + \frac{18\alpha\omega k}{(a\cos\xi + b\sin\xi)^2} - \frac{18\alpha\omega k(-a\sin\xi + b\cos\xi)}{(a\cos\xi + b\sin\xi)^2}. \end{aligned} \tag{3.38}$$

Case 12. $\varepsilon = -1$, $r = 0$, $a_{1,0} = -((36\omega^2 + 72\alpha\omega^2k^2 - k^2)/36\omega^2)$, $a_{1,1} = 0$, $a_{1,2} = 6\alpha k^2$, $a_{2,0} = -((k^2 + 72\alpha\omega^2k^2 + 6\omega^2)/6\omega k)$, $a_{2,1} = 0$, $a_{2,2} = 36\alpha\omega k$, $b_{1,1} = 0$, $b_{1,2} = 0$, $b_{2,1} = 0$, $b_{2,2} = 0$. By applying these to (3.26), one gets a solution of the PDEs (1.2):

$$\begin{aligned} H_{12}(x,t) &= -\frac{36\omega^2 + 72\alpha\omega^2k^2 - k^2}{36\omega^2} + \frac{6\alpha k^2}{(a\cos\xi + b\sin\xi)^2}, \\ u_{12}(x,t) &= -\frac{k^2 + 72\alpha\omega^2k^2 + 6\omega^2}{6\omega k} + \frac{36\alpha\omega k}{(a\cos\xi + b\sin\xi)^2}. \end{aligned} \tag{3.39}$$

The periodic wave solutions of triangle function H_8, u_8 (3.35) and H_9, u_9 (3.36) are entirely new periodic wave solutions that have not been obtained in other literatures. The periodic wave solutions of triangle function H_{10}, u_{10} (3.37) to H_{12}, u_{12} have more general form than those obtained in literatures [8, 16]. All of the above periodic wave solutions have not been obtained in literature [10].

For $R = 0$, we can deduce that the solution of (3.25) is of the form

$$\begin{aligned} H(\xi) &= a_{1,0} + a_{1,1}g(\xi) + a_{1,2}g(\xi)^2, \\ u(\xi) &= a_{2,0} + a_{2,1}g(\xi) + a_{2,2}g(\xi)^2, \end{aligned} \tag{3.40}$$

where $\xi = kx + \omega t + \xi_0$ and $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $a_{2,0}$, $a_{2,1}$, $a_{2,2}$ are constants to be determined later.

Substituting (3.40) along with the condition $g'(\xi) = -g^2(\xi)$ into (3.25) yields a set of nonlinear algebraic equations:

$$\begin{gathered} 2ka_{2,2}(36\alpha\omega k - a_{2,2}) = 0, \\ 4ka_{2,2}\left(6\alpha k^2 - a_{1,2}\right) = 0, \\ 3ka_{2,1}(6\alpha\omega k - a_{2,2}) = 0, \\ -\omega a_{2,1} - ka_{2,0}a_{2,1} - ka_{1,1} = 0, \\ 3k\left(2\alpha a_{2,1}k^2 - a_{1,2}a_{2,1} - a_{2,2}a_{1,1}\right) = 0, \\ -\omega a_{1,1} - ka_{2,1} - ka_{1,1}a_{2,0} - ka_{1,0}a_{2,1} = 0, \\ -2\omega a_{2,2} - ka_{2,1}^2 - 2ka_{2,0}a_{2,2} - 2ka_{1,2} = 0, \\ -2\omega a_{1,2} - 2ka_{2,2} - 2ka_{1,1}a_{2,1} - 2ka_{1,2}a_{2,0} - 2ka_{1,0}a_{2,2} = 0. \end{gathered} \tag{3.41}$$

With the aid of the computer program Maple 12, make use of the Maple software package PDESolver by the authors, which is based on the Wu-elimination method [26]; solving (3.41), one gets the following nontrivial solutions.

Case 13. $\varepsilon = 0$, $r = r$, $a_{1,0} = (-36\omega^2 + k^2)/36\omega^2$, $a_{1,1} = 0$, $a_{1,2} = 6\alpha k^2$, $a_{2,0} = -((k^2 + 6\omega^2)/6\omega k)$, $a_{2,1} = 0$, $a_{2,2} = 36\alpha\omega k$. By applying these to (3.40), one gets a solution of the PDEs (1.2):

$$\begin{aligned} H_{13}(x,t) &= \frac{-36\omega^2 + k^2}{36\omega^2} + \frac{6\alpha k^2}{(kx + \omega t + \xi_0)^2}, \\ u_{13}(x,t) &= -\frac{k^2 + 6\omega^2}{6\omega k} + \frac{36\alpha\omega k}{(kx + \omega t + \xi_0)^2}. \end{aligned} \tag{3.42}$$

The solitary wave solutions H_{13}, u_{13} are in rational function form and have not been obtained in literatures [8, 10, 16].

4. Summary and Conclusions

In summary, we propose an effective and a readily computerizable method to uniformly construct a series of traveling wave solutions for nonlinear evolution partial differential equations in mathematical physic, and develop a symbolic computational software package "PDESolver." As an application of the proposed method, we deal with two types of variant Boussinesq equations. Some more general forms of exact explicit traveling wave solutions for two variant Boussinesq equations are obtained. We not only obtain all known exact solitary wave solutions, periodic wave solutions, and singular traveling wave solutions but also find abundant new exact solitary wave solutions, singular traveling wave solutions, and periodic traveling wave solutions of triangle function. Since there are some parameters that can take different values, these solutions admit abundant new information. Because these two variant Boussinesq equations are integrable, all solitary wave solutions are stable. All known results in [8, 10, 16, 17] are improved and completed. Although the procedure has many advantages than other methods such as the tanh method [6], the extended tanh method, the tanh-coth method [7, 15], the extended tanh-function method [27], the extended improved tanh-function method [21], and the recent method proposed by Wazwaz [23, 24], there exist some limitations in our procedure. We cannot obtain nontraveling wave solution by using the method presented in this paper. Comparing with the Jacobi elliptic function method, the F-expansion method, and the improved F-expansion method, our method failed to obtain elliptic function doubly periodic wave solutions.

Acknowledgments

This work is supported by the NSF of China (10771041, 40890153, 11271090), Doctorial Point Fund of National Education Ministry of China (200810780002) and NSF of Guangdong (S2012010010121). It is also supported by the Visiting Scholar Program of Chern Institute of Mathematics at Nankai University when the authors worked as visiting scholars. The authors would like to express their hearty thanks to Chern Institute of Mathematics that provided very comfortable research environments to them. The authors would like to thank Professor Wang Mingliang for his helpful suggestions. The authors wish to thank the referee for his or her very helpful comments and useful suggestions.

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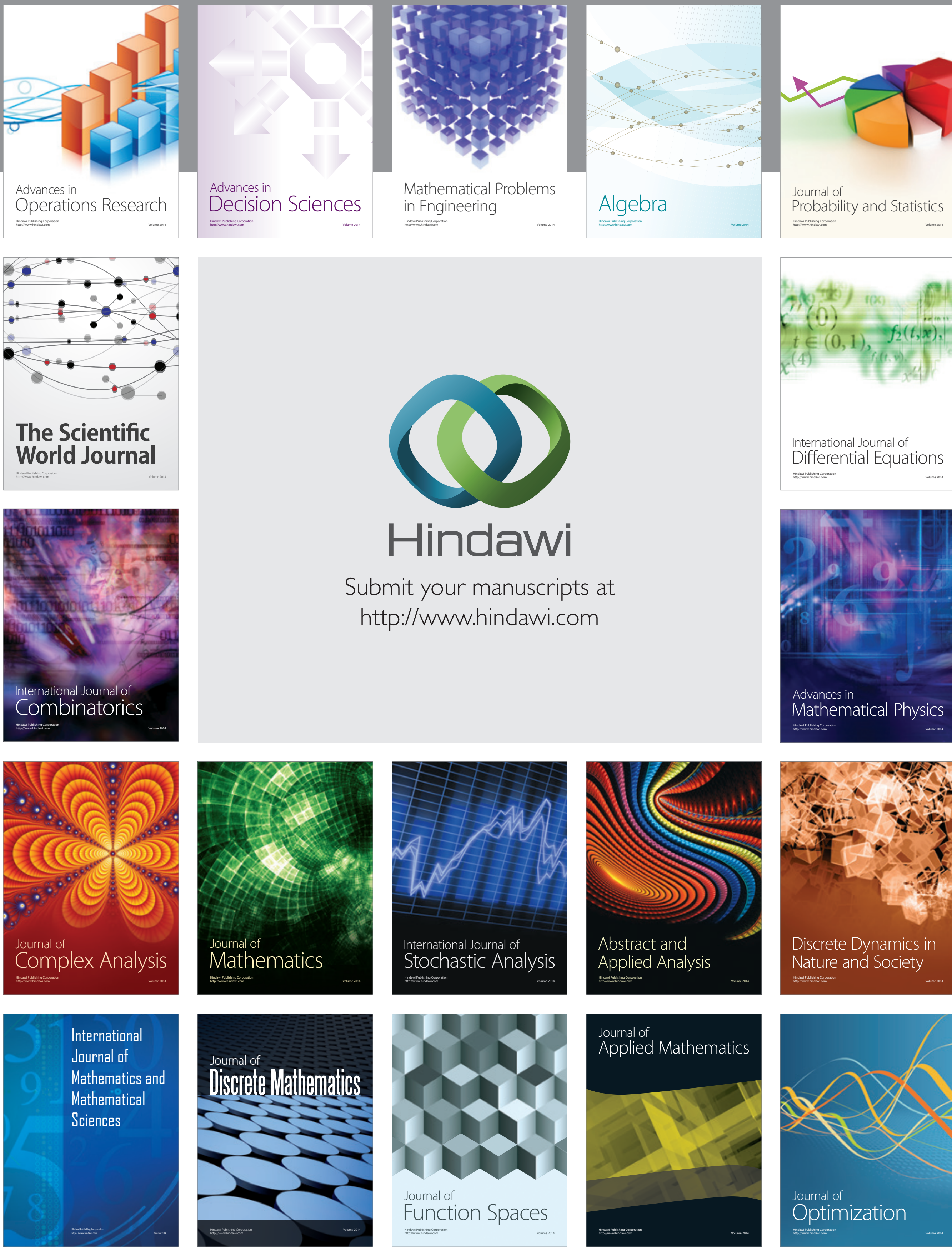
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