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Research Article

Wave Equations in Bianchi Space-Times

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We investigate the wave equation in Bianchi type III space-time. We construct a Lagrangian of the model, calculate and classify the Noether symmetry generators, and construct corresponding conserved forms. A reduction of the underlying equations is performed to obtain invariant solutions.

1. Introduction

The study of partial differential equations (PDEs) in terms of Lie point symmetries is well known and well established [1–5], where these symmetries can be used to obtain, inter alia, exact analytic solutions of the PDEs. In addition, Noether symmetries are also widely investigated and are associated with PDEs that possess a Lagrangian. Noether [6] discovered the interesting link between symmetries and conservation laws showing that for every infinitesimal transformation admitted by the action integral of a system there exists a conservation law. Investigations have been devoted to understand Noether symmetries of Lagrangians that arise from certain pseudo-Riemannian metrics of interest [7, 8]. Recently, a study was aimed at understanding the effect of gravity on the solutions of the wave equation by solving the wave equation in various space-time geometries [9].

In [10], the Bianchi universes were investigated using Noether symmetries. The authors of [11] studied the Noether symmetries of Bianchi type I and III space-times in scalar coupled theories. Therein, they obtained the exact solutions for potential functions, scalar field, and the scale factors, see also [12].

We pursue an investigation of the symmetries of the wave equation in Bianchi III space-time. We construct solutions of these equations and find conservation laws associated with Noether symmetries. The plan of the paper is as follows.

In Section 2, we discuss the procedure to obtain an expression representing Noether symmetries and conservation laws. In Section 3, we derive and classify strict Noether symmetries of the Bianchi III space-time. Also in Section 3, we briefly describe the relation of Noether symmetries to conservation laws. We then illustrate the reduction of the wave equation and obtain invariant solutions.

2. Definitions and Notation

We briefly outline the notation and pertinent results used in this work. In this regard, the reader is referred to [13].

The convention that repeated indices imply summation is used. Let $x = (x^1, x^2, \ldots, x^n) \in R^n$ be independent variable with x^i, and let $u = (u^1, u^2, \ldots, u^m) \in R^m$ be the dependent variable with coordinates u^α. Furthermore, let $\pi : R^{n+m} \longrightarrow R^n$ be the projection map $\pi(x, u) = x$. Also, suppose that $s : \chi \subset R^n \longrightarrow \mathcal{U} \subset R^{n+m}$ is a smooth map such that $\pi \circ s = 1_\chi$, where 1_χ is the identity map on χ. The r-jet bundle $\mathcal{J}^r(\mathcal{U})$ is given by the equivalence classes of sections of $\mathcal{U}$. The coordinates on $\mathcal{J}^r(\mathcal{U})$ are denoted by $(x^i, u^\alpha, \ldots, u^\alpha_{i_1\cdots i_r})$, where $1 \le i_1 \le \cdots \le i_r \le n$ and $u^\alpha_{i_1\cdots i_r}$ corresponds to the partial derivatives of u^α with respect to $x^{i_1}, \ldots, x^{i_r}$. The partial derivatives of u with respect to x are connected by the operator of total differentiation

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots, \quad i = 1, \ldots, n, \tag{2.1}$$

as

$$u^\alpha_i = D_i(u^\alpha), \qquad u^\alpha_{ij} = D_j D_i(u^\alpha), \ldots \tag{2.2}$$

The collection of all first-order derivatives u^α_i will be denoted by $u_{(1)}$. Similarly, the collections of all higher order derivatives will be denoted by $u_{(2)}, u_{(3)}, \ldots$.

The r-jet bundle on U will be written as $\mathcal{J}^r(\mathcal{U}) = \{(x, u, u_{(1)}, \ldots, u_{(r)})/(x, u) \in \mathcal{U}\}$. We now review the space of differential forms on $\mathcal{J}^r(\mathcal{U})$. To this end, let $\Omega^r_k(\mathcal{U})$ be the vector space of differential k-forms on $\mathcal{J}^r(\mathcal{U})$ with differential d. A smooth differential k-form on $\mathcal{J}^r(\mathcal{U})$ is given by

$$\omega = f_{i1,i2,\ldots,ik}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \tag{2.3}$$

where each component $f_{i1,i2,\ldots,ik} \in \Omega^r_0(\mathcal{U})$. Note that for differential functions $f \in \Omega^r_0(\mathcal{U})$,

$$Df = D_j f dx^j, \tag{2.4}$$

where D is the total differential or the total exterior derivative. Moreover, the total exterior derivative of ω is

$$D\omega = Df_{i1,i2,\ldots,ik}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \tag{2.5}$$

and by invoking (2.4) one has

$$D\omega = Df_{i1,i2,\ldots,ik}\, dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}. \tag{2.6}$$

The total differential D has properties analogous to the algebraic properties of the usual exterior derivative d:

$$D(\omega \wedge \upsilon) = D\omega \wedge \upsilon + (-1)^k \omega \wedge D\upsilon, \tag{2.7}$$

for ω a k-form and υ an l-form and $D(D\omega) = 0$. Also, it is known that if $D(D\omega) = 0$, then ω is a locally exact k-form, that is, $\omega = D\upsilon$ for some $(k-1)$-form υ, [14].

2.1. Action of Symmetries

Consider an rth-order system of partial differential equations of n independent variables and m dependent variables:

$$E^\beta\left(x, u, u_{(1)}, \ldots, u_{(r)}\right) = 0, \quad \beta = 1, \ldots, \tilde{m}. \tag{2.8}$$

Definition 2.1. A conserved form of (2.8) is a differential $(n-1)$-form,

$$\omega = T^i\left(x, u, u_{(1)}, \ldots, u_{(r-1)}\right)\left(\frac{\partial}{\partial x^i} \lrcorner \left(dx^1 \wedge \cdots \wedge dx^n\right)\right), \tag{2.9}$$

defined on $\mathcal{J}^{r-1}(\mathcal{U})$ if

$$D\omega = 0 \tag{2.10}$$

is satisfied on the surface given by (2.8).

Remark 2.2. When Definition 2.1 is satisfied, (2.10) is called a conservation law for (2.8).

It is clear that (2.10) evaluated on the surface (2.8) implies that

$$D_i T^i = 0 \tag{2.11}$$

on the surface given by (2.8), which is also referred to as a conservation law of (2.8). The tuple $T = (T^1, \ldots, T^n)$, $T^j \in \Omega_0^{r-1}(\mathcal{U})$, $j = 1, \ldots, n$, is called a conserved vector of (2.8).

We now review some definitions and results relating to Euler-Lagrange, Lie, and Noether operators ([15, 16] and references therein).

Let $\mathcal{A} = \bigcup_{r=0}^{p} \Omega_0^r(\mathcal{U})$ for some $p < \infty$. Then $\mathcal{A}$ is the universal space of differential functions of finite orders.

Consider a symmetry operator given by the infinite formal sum:

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1\cdots i_s} \frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \tag{2.12}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$, and the additional coefficients are determined uniquely by the prolongation formulae

$$\zeta^\alpha_{i_1\cdots i_s} = D_{i_1}\cdots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s > 1. \tag{2.13}$$

In (2.13), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.14}$$

In particular, a symmetry operator of the form $\widetilde{X} = Q^\alpha \partial/\partial u^\alpha + \cdots$, where $Q^\alpha \in \mathcal{A}$, is called a *canonical* or *evolutionary* representation of X, and Q^α is called its *characteristic*.

An operator X is said to be a Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$, if there exists a vector $B^i = (B^1, \ldots, B^n), B^i \in \mathcal{A}$, such that

$$X(L) + LD_i\left(\xi^i\right) = D_i\left(B^i\right). \tag{2.15}$$

If $B^i = 0, (i = 1, \ldots, n)$, then X is referred to as a strict Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$. This case is also obtained by setting the Lie derivative on the n-form $Ldx^1 \wedge \cdots \wedge dx^n$ in the direction of X to zero, that is,

$$\mathcal{L}_X L dx^1 \wedge \cdots \wedge dx^n = X\left(L dx^1 \wedge \cdots \wedge dx^n\right) = 0, \tag{2.16}$$

where $\mathcal{L}$ is the Lie derivative operator.

In view of the above discussions and definitions, the Noether theorem [6] is formulated as follows.

Noether's Theorem

For any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $T^i = (T^1, \ldots, T^n),\ T^i \in \mathcal{A}$, defined by

$$T^i = B^i - N^i(L), \quad i = 1, \ldots, n, \tag{2.17}$$

which is a conserved current of the Euler-Lagrange equations $\delta L/\delta u^\alpha = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1}\cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \ldots, m, \tag{2.18}$$

and the Noether operator associated with the operator X is given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\cdots D_{i_s}(W^\alpha)\frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}}, \quad i = 1,\ldots,n, \tag{2.19}$$

in which the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (2.18) by replacing u^α by the corresponding derivatives, for example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}, \quad i = 1,\ldots,n,\ \alpha = 1,\ldots,m. \tag{2.20}$$

3. Bianchi III Space-Time

Consider the Bianchi III metric:

$$ds^2 = -\beta^2 dt^2 + t^{2L}\left(dx^2 + e^{-2ax/N}dy^2\right) + t^{2L/m}dz^2. \tag{3.1}$$

The wave equation in (3.1) takes the form [17]

$$\begin{aligned} &-\frac{1}{\beta}(2L+1)t^{2L}e^{-(a/N)x}u_t - \frac{1}{\beta}t^{2L+1}e^{-(a/N)x}u_{tt} \\ &\quad -\frac{a\beta}{N}te^{-(a/N)x}u_x + \beta te^{-(a/N)x}u_{xx} \\ &\quad + \beta te^{(a/N)x}u_{yy} + \beta t^{2L+1-(2L/m)}e^{-(a/N)x}u_{zz} = 0. \end{aligned} \tag{3.2}$$

In [17], some aspects of the wave equation on the Bianchi metric were studied. The *multiplier method* [1] was adopted to determine some of the conserved densities. This lengthy procedure ultimately leads to the construction of only three symmetries and its associated conserved vectors.

In this paper, we investigate the wave equation on the Bianchi III metric using Noether's theorem and the method of differential forms. We obtain a wide range of results and also perform symmetry reductions of the wave equation for some cases to obtain invariant solutions. For the purposes of Sections 3.1 and 3.2, we denote the Lagrangian by **L**.

3.1. The Strict Noether Symmetries of (3.2)

We classify the cases that yield strict Noether symmetries (gauge is zero) of (3.2), via the Lagrangian

$$\mathbf{L} = -\frac{\beta}{2}te^{-(a/N)x}{u_x}^2 - \frac{\beta}{2}te^{(a/N)x}{u_y}^2 - \frac{\beta}{2}t^{2L+1-(2L/m)}e^{-(a/N)x}{u_z}^2 + \frac{1}{2\beta}t^{2L+1}e^{-(a/N)x}{u_t}^2. \tag{3.3}$$

Many of the calculations have been left out as they are tedious—the details are available to the reader in a number of texts that have been cited here.

The principle Noether algebra is

$$\begin{aligned} X_1 &= \partial_y, \\ X_2 &= \partial_z, \\ X_3 &= \partial_u, \\ X_4 &= \frac{N}{a}\partial_x + y\partial_y, \\ X_5 &= \frac{-2ay}{N}\partial_x + \frac{1}{N^2}\left(-a^2y^2 + N^2e^{2ax/N}\right)\partial_y. \end{aligned} \tag{3.4}$$

Furthermore, specific cases of L and m give rise to the symmetries X_1 to X_5 from above, and some additional symmetries.

Case 1 ($L = 1$, $m = 1/3$). The additional symmetries are,

$$\begin{aligned} X_6 &= -2z\partial_z + t\partial_t, \\ X_7 &= -\left(\frac{4z^2t^4 + \beta^2}{4t^4}\right)\partial_z + zt\partial_t. \end{aligned} \tag{3.5}$$

Case 2 ($L = 1$, $m = -1$). The additional symmetries are,

$$X_8 = -z\partial_z - \frac{t}{2}\partial_t + u\partial_u. \tag{3.6}$$

Case 3 ($L = 1$, $m = 1$). The additional symmetries are,

$$X_9 = -t\partial_t + u\partial_u. \tag{3.7}$$

Table 1 contains the conserved forms corresponding to each Noether symmetry X_i ($i = 1, \ldots, 9$), that is, it lists the three form ω. The four form is $D\omega$, and $D\omega$ vanishes on the solutions of (3.2). Thus

$$\omega = -T^t dx \wedge dy \wedge dz + T^z dx \wedge dy \wedge dt - T^y dx \wedge dz \wedge dt + T^x dy \wedge dz \wedge dt, \tag{3.8}$$

so that

$$D_t T^t + D_x T^x + D_y T^y + D_z T^z = 0 \tag{3.9}$$

in (3.2).

3.2. Symmetry Reduction and Invariant Solutions

We briefly show how the order of the (1+3) wave equation (3.2) can be reduced. Ultimately the equation with four independent variables is reduced to an ordinary differential equation.

3.2.1. Reduction—Using the Principle Noether Algebra

We begin reducing (3.2) using X_2 followed by X_4. The characteristic equations are

$$\frac{a\,dx}{N} = \frac{dt}{0} = \frac{dy}{y} = \frac{du}{0}. \tag{3.10}$$

Integrating yields $s = ye^{-(a/N)x}$ and (3.2) is reduced to

$$-\frac{1}{\beta}t^{2L}u_{tt} - \frac{1}{\beta}(2L+1)t^{2L}u_t + \beta t\left(1 + \frac{a^2}{N^2}s^2\right)u_{ss} + 2s\frac{a^2}{N^2}\beta t u_s = 0, \tag{3.11}$$

with $u = u(s,t)$.

A Lagrangian of (3.11) is

$$\mathbf{L} = -\frac{1}{\beta}t^{2L+1}\frac{u_t^2}{2} + \beta t\left(1 + \frac{a^2}{N^2}s^2\right)\frac{u_s^2}{2}. \tag{3.12}$$

It turns out that we if we let $L = 1$, $N = 1$ in (3.11), we can obtain its Noether symmetries, namely,

$$X_1^* = t\partial_t - u\partial_u, \qquad X_2^* = \partial_u. \tag{3.13}$$

We reduce (3.11) with X_1^*, and the characteristic equations are

$$\frac{dt}{t} = \frac{ds}{0} = \frac{du}{-u}. \tag{3.14}$$

Integrating yields $\Upsilon = tu$ and (3.11) is reduced to the ordinary differential equation:

$$\frac{1}{\beta}\Upsilon + \beta\left(1 + s^2a^2\right)\Upsilon'' + 2a^2\beta s\Upsilon' = 0, \tag{3.15}$$

Table 1: Conserved Form ω.

X_1	$T^t = \frac{e^{-ax/N}t^{1+2L}(-u_y u_t + uu_{ty})}{2\beta}, \qquad T^x = \frac{1}{2}e^{-ax/N}t\beta(u_y u_x - uu_{xy}),$ $T^y = \frac{e^{-ax/N}t^{-2L/m}(e^{2ax/N}Nt^{1+(2L/m)}\beta^2 {u_y}^2 + u(Nt^{1+2L}\beta^2 u_{zz} - t^{2L/m}(at\beta^2 u_x + N(-t\beta^2 u_{xx} + t^{2L}((1+2L)u_t + tu_{tt})))))}{2N\beta},$ $T^z = \frac{1}{2}e^{-ax/N}t^{1+(2L(-1+m)/m)}\beta(u_z u_y - uu_{yz})$
X_2	$T^t = \frac{e^{-ax/N}t^{1+2L}(-u_z u_t + uu_{tz})}{2\beta}, \qquad T^x = \frac{1}{2}e^{-ax/N}t\beta(u_z u_x - uu_{xz}),$ $T^y = \frac{1}{2}e^{ax/N}t\beta(u_z u_y - uu_{yz}),$ $T^z = \frac{e^{-ax/N}t^{-2L/m}(Nt^{1+2L}\beta^2 {u_z}^2 + t^{2L/m}u(e^{2ax/N}Nt\beta^2 u_{yy} - at\beta^2 u_x + N(t\beta^2 u_{xx} - t^{2L}((1+2L)u_t + tu_{tt}))))}{2N\beta}$
X_3	$T^t = -\frac{e^{-ax/N}t^{1+2L}u_t}{\beta}, \qquad T^x = e^{-ax/N}t\beta u_x,$ $T^y = e^{ax/N}t\beta u_y, \qquad T^z = e^{-ax/N}t^{1+2L-(2L/m)}\beta u_z$
X_4	$T^t = -\frac{e^{-ax/N}t^{1+2L}(ayu_y u_t + Nu_x u_t - u(ayu_{ty} + Nu_{tx}))}{2a\beta},$ $T^x = \frac{1}{2a\beta}\left(e^{-ax/N}t^{-2L/m}\left(t^{1+(2L/m)}\beta^2 u_x\left(ayu_y + Nu_x\right)\right.\right.$ $\left.\left. + u\left(Nt^{1+2L}\beta^2 u_{zz} + t^{2L/m}\left(e^{2ax/N}Nt\beta^2 u_{yy} - at\beta^2 u_x - aty\beta^2 u_{xy} - Nt^{2L}u_t - 2LNt^{2L}u_t - Nt^{1+2L}u_{tt}\right)\right)\right)\right),$ $T^y = \frac{1}{2aN\beta}e^{-ax/N}t^{-2L/m}\left(e^{2ax/N}Nt^{1+(2L/m)}\beta^2 u_y\left(ayu_y + Nu_x\right)\right.$ $\left. + u\left(aNt^{1+2L}y\beta^2 u_{zz} - t^{2L/m}\left(ae^{2ax/N}Nt\beta^2 u_y + a^2ty\beta^2 u_x + N\left(e^{2ax/N}Nt\beta^2 u_{xy} + ay\left(-t\beta^2 u_{xx} + t^{2L}((1+2L)u_t + tu_{tt})\right)\right)\right)\right)\right),$ $T^z = \frac{e^{-ax/N}t^{1+(2L(-1+m)/m)}\beta(u_z(ayu_y + Nu_x) - u(ayu_{yz} + Nu_{xz}))}{2a}$
X_5	$T^t = \frac{1}{2N^2\beta}\left(e^{-ax/N}t^{1+2L}\left(-\left(e^{2ax/N}N^2 - a^2y^2\right)u_y u_t + 2aNyu_x u_t + u\left(\left(e^{2ax/N}N^2 - a^2y^2\right)u_{ty} - 2aNyu_{tx}\right)\right)\right),$ $T^x = \frac{1}{2N^2\beta}e^{-ax/N}t^{-2L/m}\left(t^{1+(2L/m)}\beta^2 u_x\left(\left(e^{2ax/N}N^2 - a^2y^2\right)u_y - 2aNyu_x\right)\right.$ $+ u\left(-2aNt^{1+2L}y\beta^2 u_{zz} + t^{2L/m}\left(-2ae^{2ax/N}Nt\beta^2 u_y - 2ae^{2ax/N}Nty\beta^2 u_{yy} + 2a^2ty\beta^2 u_x - e^{2ax/N}N^2t\beta^2 u_{xy} + a^2ty^2\beta^2 u_{xy}\right.\right.$ $\left.\left.\left. + 2aNt^{2L}yu_t + 4aLNt^{2L}yu_t + 2aNt^{1+2L}yu_{tt}\right)\right)\right),$

Table 1: Continued.

	$T^y = \frac{1}{2N^3\beta}e^{-ax/N}t^{-2L/m}\big(e^{2ax/N}Nt^{1+(2L/m)}\beta^2u_y\big((e^{2ax/N}N^2-a^2y^2)u_y-2aNyu_x\big)$ $+u\big(Nt^{1+2L}(e^{2ax/N}N^2-a^2y^2)\beta^2u_{zz}+t^{2L/m}\big(2a^2e^{2ax/N}Nty\beta^2u_y+at(e^{2ax/N}N^2+a^2y^2)\beta^2u_x$ $+N\big(2ae^{2ax/N}Nty\beta^2u_{xy}+(e^{2ax/N}N^2-a^2y^2)(t\beta^2u_{xx}-t^{2L}((1+2L)u_t+tu_{tt}))\big)\big)\big)\big)$, $T^z = \frac{e^{-ax/N}t^{1+(2L(-1+m)/m)}\beta(u_z((e^{2ax/N}N^2-a^2y^2)u_y-2aNyu_x)-u((e^{2ax/N}N^2-a^2y^2)u_{yz}-2aNyu_{xz}))}{2N^2}$
X_6	$T^t = \frac{1}{2Nt^2\beta}\big(e^{-ax/N}\big(Nt^5u_t(2zu_z-tu_t)+u\big(N\beta^2u_{zz}+t^4\big(e^{2ax/N}N\beta^2u_{yy}-a\beta^2u_x+N\beta^2u_{xx}-2Ntu_t-2Ntzu_{tz}\big)\big)\big)\big)$, $T^x = \frac{1}{2}e^{-ax/N}t\beta(-2zu_zu_x+tu_xu_t+u(2zu_{xz}-tu_{tx}))$, $T^y = \frac{1}{2}e^{ax/N}t\beta(-2zu_zu_y+tu_yu_t+u(2zu_{yz}-tu_{ty}))$, $T^z = \frac{1}{2Nt^3\beta}\big(e^{-ax/N}\big(N\beta^2u_z(-2zu_z+tu_t)+u\big(2N\beta^2u_z-2e^{2ax/N}Nt^4z\beta^2u_{yy}+2at^4z\beta^2u_x-2Nt^4z\beta^2u_{xx}+6Nt^5zu_t-Nt\beta^2u_{tz}+2Nt^6zu_{tt}\big)\big)\big)$
X_7	$T^t = \frac{1}{8Nt^2\beta}\big(e^{-ax/N}\big(Ntu_t\big((4t^4z^2+\beta^2)u_z-4t^5zu_t\big)+u\big(4N\beta^2u_z+4Nz\beta^2u_{zz}+4e^{2ax/N}Nt^4z\beta^2u_{yy}-4at^4z\beta^2u_x$ $+4Nt^4z\beta^2u_{xx}-8Nt^5zu_t-4Nt^5z^2u_{tz}-Nt\beta^2u_{tz}\big)\big)\big)$, $T^x = \frac{e^{-ax/N}\beta(-(4t^4z^2+\beta^2)u_zu_x+4t^5zu_xu_t+((4t^4z^2+\beta^2)u_{xz}-4t^5zu_{tx}))}{8t^3}$, $T^y = \frac{e^{ax/N}\beta(-(4t^4z^2+\beta^2)u_zu_y+4t^5zu_yu_t+u((4t^4z^2+\beta^2)u_{yz}-4t^5zu_{ty}))}{8t^3}$, $T^z = \frac{1}{8Nt^7\beta}e^{-ax/N}\big(-N\beta^2u_z\big((4t^4z^2+\beta^2)u_z-4t^5zu_t\big)+t^4u\big(8Nz\beta^2u_z-e^{2ax/N}N\beta^2(4t^4z^2+\beta^2)u_{yy}+4at^4z^2\beta^2u_x+a\beta^4u_x-4Nt^4z^2\beta^2u_{xx}-N\beta^4u_{xx}$ $+12Nt^5z^2u_t-Nt\beta^2u_t-4Ntz\beta^2u_{tz}+4Nt^6z^2u_{tt}+Nt^2\beta^2u_{tt})\big)$
X_8	$T^t = \frac{1}{4N\beta}\big(e^{-ax/N}t^2\big(-Ntu_t(2zu_z+tu_t)+u\big(Nt^4\beta^2u_{zz}+e^{2ax/N}N\beta^2u_{yy}-a\beta^2u_x+N\beta^2u_{xx}-2Ntu_t+2Ntzu_{tz}\big)\big)\big)$, $T^x = \frac{1}{4}e^{-ax/N}t\beta(2zu_zu_x+tu_xu_t-u(2zu_{xz}+tu_{tx}))$, $T^y = \frac{1}{4}e^{ax/N}t\beta(2zu_zu_y+tu_yu_t-u(2zu_{yz}+tu_{ty}))$, $T^z = \frac{1}{4N\beta}\big(e^{-ax/N}t\big(Nt^4\beta^2u_z(2zu_z+tu_t)-u\big(2Nt^4\beta^2u_z-2e^{2ax/N}nz\beta^2u_{yy}+2az\beta^2u_x-2Nz\beta^2u_{xx}+6Ntzu_t+Nt^5\beta^2u_{tz}+2Nt^2zu_{tt}\big)\big)\big)$
X_9	$T^t = \frac{1}{2N\beta}\big(e^{-ax/N}t^2\big(-Nt^2{u_t}^2+u\big(N\beta^2u_{zz}+e^{2ax/N}N\beta^2u_{yy}-a\beta^2u_x+N\beta^2u_{xx}-2Ntu_t\big)\big)\big)$, $T^x = \frac{1}{2}e^{-ax/N}t^2\beta(u_xu_t-uu_{tx})$, $T^y = \frac{1}{2}e^{ax/N}t^2\beta(u_yu_t-uu_{ty})$, $T^z = \frac{1}{2}e^{-ax/N}t^2\beta(u_zu_t-uu_{tz})$

with $Y = Y(s)$, and which has a solution in terms of special functions, that is,

$$Y(s) = C_1 \text{ Legendre}\, P\left[\frac{-a\beta + \sqrt{-4 + a^2\beta^2}}{2a\beta}, ias\right] + C_2 \text{ Legendre}\, Q\left[\frac{-a\beta + \sqrt{-4 + a^2\beta^2}}{2a\beta}, ias\right], \tag{3.16}$$

where C_1, C_2 are arbitrary constants, Legendre $P[n, x]$ refers to the Legendre polynomial $P_n(x)$, and $Q[n, z]$ refers to the Legendre function of the second kind $Q_n(z)$.

3.2.2. Reduction—Case 1, $L = 1$, $m = 1/3$.

We reduce (3.2) using X_1 followed by X_6 from Case 1. The characteristic equations are

$$\frac{dz}{-2z} = \frac{dt}{t} = \frac{dx}{0} = \frac{du}{0}. \tag{3.17}$$

Integrating yields $r = t^2 z$ and (3.2) is reduced to

$$-\frac{8}{\beta} r e^{-\frac{ax}{N}} u_r - \frac{a\beta}{N} e^{-\frac{ax}{N}} u_x + \beta e^{-\frac{ax}{N}} u_{xx} + e^{-\frac{ax}{N}} \left(\frac{\beta^2 - 4r^2}{\beta}\right) u_{rr} = 0, \tag{3.18}$$

with $u = u(r, x)$.

A Lagrangian of (3.18) is

$$\mathbf{L} = \beta e^{-ax/N} \frac{u_x^2}{2} + e^{-ax/N} \left(\frac{\beta^2 - 4r^2}{\beta}\right) \frac{u_r^2}{2}. \tag{3.19}$$

Hence, we obtain the Noether symmetries of (3.18), namely,

$$X_3^* = \partial_x + \frac{a}{2N} u \partial_u, \qquad X_4^* = \partial_u. \tag{3.20}$$

We reduce (3.18) with X_3^*, and the characteristic equations are

$$\frac{dx}{1} = \frac{dr}{0} = \frac{2N du}{au}. \tag{3.21}$$

Integrating yields $Z = e^{-ax/2N} u$ and (3.18) is reduced to the ordinary differential equation:

$$-\frac{8}{\beta} r Z' + \left(\frac{\beta^2 - 4r^2}{\beta}\right) Z'' - \frac{a^2\beta}{4N^2} Z = 0, \tag{3.22}$$

with $Z = Z(r)$, and which has a solution in terms of special functions, that is,

$$\begin{aligned} Z(r) = C_1 \text{ Legendre}\, P\left[\frac{-2N + \sqrt{4N^2 - a^2\beta^2}}{4N}, \frac{2r}{\beta}\right] \\ + C_2 \text{ Legendre}\, Q\left[\frac{-2N + \sqrt{4N^2 - a^2\beta^2}}{4N}, \frac{2r}{\beta}\right], \end{aligned} \tag{3.23}$$

where, as before, C_1, C_2 are arbitrary constants, Legendre $P[n, x]$ refers to the Legendre polynomial $P_n(x)$, and $Q[n, z]$ refers to the Legendre function of the second kind $Q_n(z)$.

4. Conclusion

We classified the Noether symmetry generators, determined some conserved forms, and reduced some cases of the underlying equations associated with the wave equation on the Bianchi III manifold. The first reduction done above involved the principle Noether algebra, whilst the second dealt with a particular case. To obtain other reductions, one needs to conclude a three-dimensional subalgebra of symmetries to reduce to an ordinary differential equation whose solution would be an invariant solution invariant under the subalgebra. Alternatively, a lower dimensional subalgebra can be used to reduce to a partial differential equation which may be tackled using other methods. The final solution in this case will be invariant only under the lower dimensional algebra. In general, the procedure performed above is the most convenient.

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