

*Research Article*

# **A Hybrid Gradient-Projection Algorithm for Averaged Mappings in Hilbert Spaces**

**Ming Tian and Min-Min Li**

*College of Science, Civil Aviation University of China, Tianjin 300300, China*

Correspondence should be addressed to Ming Tian, tianming1963@126.com

Received 18 April 2012; Accepted 21 June 2012

Academic Editor: Hong-Kun Xu

Copyright © 2012 M. Tian and M.-M. Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

It is well known that the gradient-projection algorithm (GPA) is very useful in solving constrained convex minimization problems. In this paper, we combine a general iterative method with the gradient-projection algorithm to propose a hybrid gradient-projection algorithm and prove that the sequence generated by the hybrid gradient-projection algorithm converges in norm to a minimizer of constrained convex minimization problems which solves a variational inequality.

## **1. Introduction**

Let  $H$  be a real Hilbert space and  $C$  a nonempty closed and convex subset of  $H$ . Consider the following constrained convex minimization problem:

$$\text{minimize}_{x \in C} f(x), \quad (1.1)$$

where  $f : C \rightarrow \mathbb{R}$  is a real-valued convex and continuously Fréchet differentiable function. The gradient  $\nabla f$  satisfies the following Lipschitz condition:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \forall x, y \in C, \quad (1.2)$$

where  $L > 0$ . Assume that the minimization problem (1.1) is consistent, and let  $S$  denote its solution set.

It is well known that the gradient-projection algorithm is very useful in dealing with constrained convex minimization problems and has extensively been studied ([1–5] and the

references therein). It has recently been applied to solve split feasibility problems [6–10]. Levitin and Polyak [1] consider the following gradient-projection algorithm:

$$x_{n+1} := \text{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \quad n \geq 0. \quad (1.3)$$

Let  $\{\lambda_n\}_{n=0}^{\infty}$  satisfy

$$0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < \frac{2}{L}. \quad (1.4)$$

It is proved that the sequence  $\{x_n\}$  generated by (1.3) converges weakly to a minimizer of (1.1).

Xu proved that under certain appropriate conditions on  $\{\alpha_n\}$  and  $\{\lambda_n\}$  the sequence  $\{x_n\}$  defined by the following relaxed gradient-projection algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \text{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \quad n \geq 0, \quad (1.5)$$

converges weakly to a minimizer of (1.1) [11].

Since the Lipschitz continuity of the gradient of  $f$  implies that it is indeed inverse strongly monotone (ism) [12, 13], its complement can be an averaged mapping. Recall that a mapping  $T$  is nonexpansive if and only if it is Lipschitz with Lipschitz constant not more than one, that a mapping is an averaged mapping if and only if it can be expressed as a proper convex combination of the identity mapping and a nonexpansive mapping, and that a mapping  $T$  is said to be  $\nu$ -inverse strongly monotone if and only if  $\langle x - y, Tx - Ty \rangle \geq \nu \|Tx - Ty\|^2$  for all  $x, y \in H$ , where the number  $\nu > 0$ . Recall also that the composite of finitely many averaged mappings is averaged. That is, if each of the mappings  $\{T_i\}_{i=1}^N$  is averaged, then so is the composite  $T_1 \cdots T_N$  [14]. In particular, an averaged mapping is a nonexpansive mapping [15]. As a result, the GPA can be rewritten as the composite of a projection and an averaged mapping which is again an averaged mapping.

Generally speaking, in infinite-dimensional Hilbert spaces, GPA has only weak convergence. Xu [11] provided a modification of GPA so that strong convergence is guaranteed. He considered the following hybrid gradient-projection algorithm:

$$x_{n+1} = \theta_n h(x_n) + (1 - \theta_n) \text{Proj}_C(x_n - \lambda_n \nabla f(x_n)). \quad (1.6)$$

It is proved that if the sequences  $\{\theta_n\}$  and  $\{\lambda_n\}$  satisfy appropriate conditions, the sequence  $\{x_n\}$  generated by (1.6) converges in norm to a minimizer of (1.1) which solves the variational inequality

$$x^* \in S, \quad \langle (I - h)x^*, x - x^* \rangle \geq 0, \quad x \in S. \quad (1.7)$$

On the other hand, Ming Tian [16] introduced the following general iterative algorithm for solving the variational inequality

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F) T x_n, \quad n \geq 0, \quad (1.8)$$

where  $F$  is a  $\kappa$ -Lipschitzian and  $\eta$ -strongly monotone operator with  $\kappa > 0$ ,  $\eta > 0$  and  $f$  is a contraction with coefficient  $0 < \alpha < 1$ . Then, he proved that if  $\{\alpha_n\}$  satisfying appropriate conditions, the  $\{x_n\}$  generated by (1.8) converges strongly to the unique solution of variational inequality

$$\langle (\mu F - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0, \quad z \in \text{Fix}(T). \quad (1.9)$$

In this paper, motivated and inspired by the research work in this direction, we will combine the iterative method (1.8) with the gradient-projection algorithm (1.3) and consider the following hybrid gradient-projection algorithm:

$$x_{n+1} = \theta_n \gamma h(x_n) + (I - \mu \theta_n F) \text{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \quad n \geq 0. \quad (1.10)$$

We will prove that if the sequence  $\{\theta_n\}$  of parameters and the sequence  $\{\lambda_n\}$  of parameters satisfy appropriate conditions, then the sequence  $\{x_n\}$  generated by (1.10) converges in norm to a minimizer of (1.1) which solves the variational inequality (VI)

$$x^* \in S, \quad \langle (\mu F - \gamma h)x^*, x - x^* \rangle \geq 0, \quad \forall x \in S, \quad (1.11)$$

where  $S$  is the solution set of the minimization problem (1.1).

## 2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Throughout this paper, we write  $x_n \rightharpoonup x$  to indicate that the sequence  $\{x_n\}$  converges weakly to  $x$ ,  $x_n \rightarrow x$  implies that  $\{x_n\}$  converges strongly to  $x$ .  $\omega_\omega(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$  is the weak  $\omega$ -limit set of the sequence  $\{x_n\}_{n=1}^\infty$ .

**Lemma 2.1** (see [17]). *Assume that  $\{a_n\}_{n=0}^\infty$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n + \beta_n, \quad n \geq 0, \quad (2.1)$$

where  $\{\gamma_n\}_{n=0}^\infty$  and  $\{\beta_n\}_{n=0}^\infty$  are sequences in  $[0, 1]$  and  $\{\delta_n\}_{n=0}^\infty$  is a sequence in  $\mathbb{R}$  such that

- (i)  $\sum_{n=0}^\infty \gamma_n = \infty$ ;
- (ii) either  $\limsup_{n \rightarrow \infty} \delta_n \leq 0$  or  $\sum_{n=0}^\infty \gamma_n |\delta_n| < \infty$ ;
- (iii)  $\sum_{n=0}^\infty \beta_n < \infty$ .

Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 2.2** (see [18]). *Let  $C$  be a closed and convex subset of a Hilbert space  $H$ , and let  $T : C \rightarrow C$  be a nonexpansive mapping with  $\text{Fix } T \neq \emptyset$ . If  $\{x_n\}_{n=1}^\infty$  is a sequence in  $C$  weakly converging to  $x$  and if  $\{(I - T)x_n\}_{n=1}^\infty$  converges strongly to  $y$ , then  $(I - T)x = y$ .*

**Lemma 2.3.** Let  $H$  be a Hilbert space, and let  $C$  be a nonempty closed and convex subset of  $H$ .  $h : C \rightarrow C$  a contraction with coefficient  $0 < \rho < 1$ , and  $F : C \rightarrow C$  a  $\kappa$ -Lipschitzian continuous operator and  $\eta$ -strongly monotone operator with  $\kappa, \eta > 0$ . Then, for  $0 < \gamma < \mu\eta/\rho$ ,

$$\langle x - y, (\mu F - \gamma h)x - (\mu F - \gamma h)y \rangle \geq (\mu\eta - \gamma\rho)\|x - y\|^2, \quad \forall x, y \in C. \quad (2.2)$$

That is,  $\mu F - \gamma h$  is strongly monotone with coefficient  $\mu\eta - \gamma\rho$ .

**Lemma 2.4.** Let  $C$  be a closed subset of a real Hilbert space  $H$ , given  $x \in H$  and  $y \in C$ . Then,  $y = P_C x$  if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C. \quad (2.3)$$

### 3. Main Results

Let  $H$  be a real Hilbert space, and let  $C$  be a nonempty closed and convex subset of  $H$  such that  $C \pm C \subset C$ . Assume that the minimization problem (1.1) is consistent, and let  $S$  denote its solution set. Assume that the gradient  $\nabla f$  satisfies the Lipschitz condition (1.2). Since  $S$  is a closed convex subset, the nearest point projection from  $H$  onto  $S$  is well defined. Recall also that a contraction on  $C$  is a self-mapping of  $C$  such that  $\|h(x) - h(y)\| \leq \rho\|x - y\|$ , for all  $x, y \in C$ , where  $\rho \in [0, 1)$  is a constant. Let  $F$  be a  $\kappa$ -Lipschitzian and  $\eta$ -strongly monotone operator on  $C$  with  $\kappa, \eta > 0$ . Denote by  $\Pi$  the collection of all contractions on  $C$ , namely,

$$\Pi = \{h : h \text{ is a contraction on } C\}. \quad (3.1)$$

Now given  $h \in \Pi$  with  $0 < \rho < 1$ ,  $s \in (0, 1)$ . Let  $0 < \mu < 2\eta/\kappa^2$ ,  $0 < \gamma < \mu(\eta - (\mu\kappa^2)/2)/\rho = \tau/\rho$ . Assume that  $\lambda_s$  with respect to  $s$  is continuous and, in addition,  $\lambda_s \in [a, b] \subset (0, 2/L)$ . Consider a mapping  $X_s$  on  $C$  defined by

$$X_s(x) = s\gamma h(x) + (I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x), \quad x \in C. \quad (3.2)$$

It is easy to see that  $X_s$  is a contraction. Setting  $V_s := \text{Proj}_C(I - \lambda_s \nabla f)$ . It is obvious that  $V_s$  is a nonexpansive mapping. We can rewrite  $X_s(x)$  as

$$X_s(x) = s\gamma h(x) + (I - s\mu F)V_s(x). \quad (3.3)$$

First observe that for  $s \in (0, 1)$ , we can get

$$\begin{aligned}
& \|(I - s\mu F)V_s(x) - (I - s\mu F)V_s(y)\|^2 \\
&= \|V_s(x) - V_s(y) - s\mu(FV_s(x) - FV_s(y))\|^2 \\
&= \|V_s(x) - V_s(y)\|^2 - 2s\mu\langle V_s(x) - V_s(y), FV_s(x) - FV_s(y) \rangle \\
&\quad + s^2\mu^2\|FV_s(x) - FV_s(y)\|^2 \\
&\leq \|x - y\|^2 - 2s\mu\eta\|V_s(x) - V_s(y)\|^2 + s^2\mu^2\kappa^2\|V_s(x) - V_s(y)\|^2 \\
&\leq \left(1 - s\mu\left(2\eta - s\mu\kappa^2\right)\right)\|x - y\|^2 \\
&\leq \left(1 - \frac{s\mu(2\eta - s\mu\kappa^2)}{2}\right)^2\|x - y\|^2 \\
&\leq \left(1 - s\mu\left(\eta - \frac{\mu\kappa^2}{2}\right)\right)^2\|x - y\|^2 \\
&= (1 - s\tau)^2\|x - y\|^2.
\end{aligned} \tag{3.4}$$

Indeed, we have

$$\begin{aligned}
\|X_s(x) - X_s(y)\| &= \|s\gamma h(x) + (I - s\mu F)V_s(x) - s\gamma h(y) - (I - s\mu F)V_s(y)\| \\
&\leq s\gamma\|h(x) - h(y)\| + \|(I - s\mu F)V_s(x) - (I - s\mu F)V_s(y)\| \\
&\leq s\gamma\rho\|x - y\| + (1 - s\tau)\|x - y\| \\
&= (1 - s(\tau - \gamma\rho))\|x - y\|.
\end{aligned} \tag{3.5}$$

Hence,  $X_s$  has a unique fixed point, denoted  $x_s$ , which uniquely solves the fixed-point equation

$$x_s = s\gamma h(x_s) + (I - s\mu F)V_s(x_s). \tag{3.6}$$

The next proposition summarizes the properties of  $\{x_s\}$ .

**Proposition 3.1.** *Let  $x_s$  be defined by (3.6).*

- (i)  $\{x_s\}$  is bounded for  $s \in (0, (1/\tau))$ .
- (ii)  $\lim_{s \rightarrow 0} \|x_s - \text{Proj}_C(I - \lambda_s \nabla f)(x_s)\| = 0$ .
- (iii)  $x_s$  defines a continuous curve from  $(0, 1/\tau)$  into  $H$ .

*Proof.* (i) Take a  $\bar{x} \in S$ , then we have

$$\begin{aligned}
\|x_s - \bar{x}\| &= \|s\gamma h(x_s) + (I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x_s) - \bar{x}\| \\
&= \|(I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x_s) - (I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(\bar{x}) \\
&\quad + s(\gamma h(x_s) - \mu F \text{Proj}_C(I - \lambda_s \nabla f)(\bar{x}))\| \\
&\leq (1 - s\tau)\|x_s - \bar{x}\| + s\|\gamma h(x_s) - \mu F(\bar{x})\| \\
&\leq (1 - s\tau)\|x_s - \bar{x}\| + s\gamma\rho\|x_s - \bar{x}\| + s\|\gamma h(\bar{x}) - \mu F(\bar{x})\|.
\end{aligned} \tag{3.7}$$

It follows that

$$\|x_s - \bar{x}\| \leq \frac{\|\gamma h(\bar{x}) - \mu F(\bar{x})\|}{\tau - \gamma\rho}. \tag{3.8}$$

Hence,  $\{x_s\}$  is bounded.

(ii) By the definition of  $\{x_s\}$ , we have

$$\begin{aligned}
\|x_s - \text{Proj}_C(I - \lambda_s \nabla f)(x_s)\| &= \|s\gamma h(x_s) + (I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x_s) \\
&\quad - \text{Proj}_C(I - \lambda_s \nabla f)(x_s)\| \\
&= s\|\gamma h(x_s) - \mu F \text{Proj}_C(I - \lambda_s \nabla f)(x_s)\| \longrightarrow 0,
\end{aligned} \tag{3.9}$$

$\{x_s\}$  is bounded, so are  $\{h(x_s)\}$  and  $\{F \text{Proj}_C(I - \lambda_s \nabla f)(x_s)\}$ .

(iii) Take  $s, s_0 \in (0, 1/\tau)$ , and we have

$$\begin{aligned}
&\|x_s - x_{s_0}\| \\
&= \|s\gamma h(x_s) + (I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x_s) - s_0\gamma h(x_{s_0}) \\
&\quad - (I - s_0\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_{s_0})\| \\
&\leq \|(s - s_0)\gamma h(x_s) + s_0\gamma(h(x_s) - h(x_{s_0}))\| \\
&\quad + \|(I - s_0\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - (I - s_0\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_{s_0})\| \\
&\quad + \|(I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x_s) - (I - s_0\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\| \\
&\leq \|(s - s_0)\gamma h(x_s) + s_0\gamma(h(x_s) - h(x_{s_0}))\| \\
&\quad + \|(I - s_0\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - (I - s_0\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_{s_0})\| \\
&\quad + \|(I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x_s) - (I - s\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\| \\
&\quad + \|(I - s\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - (I - s_0\mu F)\text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\| \\
&\leq |s - s_0|\gamma\|h(x_s)\| + s_0\gamma\rho\|x_s - x_{s_0}\| + (1 - s_0\tau)\|x_s - x_{s_0}\| \\
&\quad + |\lambda_s - \lambda_{s_0}|\|\nabla f(x_s)\| \\
&\quad + \|s\mu F \text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s) - s_0\mu F \text{Proj}_C(I - \lambda_{s_0} \nabla f)(x_s)\|
\end{aligned}$$

$$\begin{aligned}
&= |s - s_0|\gamma\|h(x_s)\| + s_0\gamma\rho\|x_s - x_{s_0}\| + (1 - s_0\tau)\|x_s - x_{s_0}\| \\
&\quad + |\lambda_s - \lambda_{s_0}|\|\nabla f(x_s)\| + |s - s_0|\|\mu F \text{Proj}_C(I - \lambda_{s_0}\nabla f)(x_s)\| \\
&= (\gamma\|h(x_s)\| + \mu\|F \text{Proj}_C(I - \lambda_{s_0}\nabla f)(x_s)\|)|s - s_0| \\
&\quad + s_0\gamma\rho\|x_s - x_{s_0}\| + (1 - s_0\tau)\|x_s - x_{s_0}\| + |\lambda_s - \lambda_{s_0}|\|\nabla f(x_s)\|.
\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}
\|x_s - x_{s_0}\| \leq & \frac{\gamma\|h(x_s)\| + \mu\|F \text{Proj}_C(I - \lambda_{s_0}\nabla f)(x_s)\|}{s_0(\tau - \gamma\rho)}|s - s_0| \\
& + \frac{\|\nabla f(x_s)\|}{s_0(\tau - \gamma\rho)}|\lambda_s - \lambda_{s_0}|.
\end{aligned} \tag{3.11}$$

Therefore,  $x_s \rightarrow x_{s_0}$  as  $s \rightarrow s_0$ . This means  $x_s$  is continuous.  $\square$

Our main result in the following shows that  $\{x_s\}$  converges in norm to a minimizer of (1.1) which solves some variational inequality.

**Theorem 3.2.** *Assume that  $\{x_s\}$  is defined by (3.6), then  $x_s$  converges in norm as  $s \rightarrow 0$  to a minimizer of (1.1) which solves the variational inequality*

$$\langle (\mu F - \gamma h)x^*, \tilde{x} - x^* \rangle \geq 0, \quad \forall \tilde{x} \in S. \tag{3.12}$$

Equivalently, we have  $\text{Proj}_S(I - (\mu F - \gamma h))x^* = x^*$ .

*Proof.* It is easy to see that the uniqueness of a solution of the variational inequality (3.12). By Lemma 2.3,  $\mu F - \gamma h$  is strongly monotone, so the variational inequality (3.12) has only one solution. Let  $x^* \in S$  denote the unique solution of (3.12).

To prove that  $x_s \rightarrow x^*$  ( $s \rightarrow 0$ ), we write, for a given  $\tilde{x} \in S$ ,

$$\begin{aligned}
x_s - \tilde{x} &= s\gamma h(x_s) + (I - s\mu F)\text{Proj}_C(I - \lambda_s\nabla f)(x_s) - \tilde{x} \\
&= s(\gamma h(x_s) - \mu F\tilde{x}) + (I - s\mu F)\text{Proj}_C(I - \lambda_s\nabla f)(x_s) \\
&\quad - (I - s\mu F)\text{Proj}_C(I - \lambda_s\nabla f)(\tilde{x}).
\end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}
\|x_s - \tilde{x}\|^2 &= s\langle \gamma h(x_s) - \mu F\tilde{x}, x_s - \tilde{x} \rangle \\
&\quad + \langle (I - s\mu F)\text{Proj}_C(I - \lambda_s\nabla f)(x_s) - (I - s\mu F)\text{Proj}_C(I - \lambda_s\nabla f)(\tilde{x}), x_s - \tilde{x} \rangle \\
&\leq (1 - s\tau)\|x_s - \tilde{x}\|^2 + s\langle \gamma h(x_s) - \mu F\tilde{x}, x_s - \tilde{x} \rangle.
\end{aligned} \tag{3.14}$$

Hence,

$$\begin{aligned}\|x_s - \tilde{x}\|^2 &\leq \frac{1}{\tau} \langle \gamma h(x_s) - \mu F \tilde{x}, x_s - \tilde{x} \rangle \\ &\leq \frac{1}{\tau} \left\{ \gamma \rho \|x_s - \tilde{x}\|^2 + \langle \gamma h(\tilde{x}) - \mu F \tilde{x}, x_s - \tilde{x} \rangle \right\}.\end{aligned}\quad (3.15)$$

To derive that

$$\|x_s - \tilde{x}\|^2 \leq \frac{1}{\tau - \gamma \rho} \langle \gamma h(\tilde{x}) - \mu F \tilde{x}, x_s - \tilde{x} \rangle. \quad (3.16)$$

Since  $\{x_s\}$  is bounded as  $s \rightarrow 0$ , we see that if  $\{s_n\}$  is a sequence in  $(0,1)$  such that  $s_n \rightarrow 0$  and  $x_{s_n} \rightarrow \bar{x}$ , then by (3.16),  $x_{s_n} \rightarrow \bar{x}$ . We may further assume that  $\lambda_{s_n} \rightarrow \lambda \in [0, 2/L]$  due to condition (1.4). Notice that  $\text{Proj}_C(I - \lambda \nabla f)$  is nonexpansive. It turns out that

$$\begin{aligned}\|x_{s_n} - \text{Proj}_C(I - \lambda \nabla f)x_{s_n}\| &\leq \|x_{s_n} - \text{Proj}_C(I - \lambda_{s_n} \nabla f)x_{s_n}\| + \|\text{Proj}_C(I - \lambda_{s_n} \nabla f)x_{s_n} - \text{Proj}_C(I - \lambda \nabla f)x_{s_n}\| \\ &\leq \|x_{s_n} - \text{Proj}_C(I - \lambda_{s_n} \nabla f)x_{s_n}\| + \|(\lambda - \lambda_{s_n}) \nabla f(x_{s_n})\| \\ &= \|x_{s_n} - \text{Proj}_C(I - \lambda_{s_n} \nabla f)x_{s_n}\| + |\lambda - \lambda_{s_n}| \|\nabla f(x_{s_n})\|.\end{aligned}\quad (3.17)$$

From the boundedness of  $\{x_s\}$  and  $\lim_{s \rightarrow 0} \|\text{Proj}_C(I - \lambda_s \nabla f)x_s - x_s\| = 0$ , we conclude that

$$\lim_{n \rightarrow \infty} \|x_{s_n} - \text{Proj}_C(I - \lambda \nabla f)x_{s_n}\| = 0. \quad (3.18)$$

Since  $x_{s_n} \rightarrow \bar{x}$ , by Lemma 2.2, we obtain

$$\bar{x} = \text{Proj}_C(I - \lambda \nabla f)\bar{x}. \quad (3.19)$$

This shows that  $\bar{x} \in S$ .

We next prove that  $\bar{x}$  is a solution of the variational inequality (3.12). Since

$$x_s = s\gamma h(x_s) + (I - s\mu F)\text{Proj}_C(I - \lambda_s \nabla f)(x_s), \quad (3.20)$$

we can derive that

$$\begin{aligned}(\mu F - \gamma h)(x_s) &= -\frac{1}{s}(I - \text{Proj}_C(I - \lambda_s \nabla f))(x_s) + \mu(F(x_s) - F\text{Proj}_C(I - \lambda_s \nabla f)(x_s)).\end{aligned}\quad (3.21)$$



Therefore, for  $\tilde{x} \in S$ ,

$$\begin{aligned}
& \langle (\mu F - \gamma h)(x_s), x_s - \tilde{x} \rangle \\
&= -\frac{1}{s} \langle (I - \text{Proj}_C(I - \lambda_s \nabla f))(x_s) - (I - \text{Proj}_C(I - \lambda_s \nabla f))(\tilde{x}), x_s - \tilde{x} \rangle \\
&\quad + \mu \langle F(x_s) - F \text{Proj}_C(I - \lambda_s \nabla f)(x_s), x_s - \tilde{x} \rangle \\
&\leq \mu \langle F(x_s) - F \text{Proj}_C(I - \lambda_s \nabla f)(x_s), x_s - \tilde{x} \rangle.
\end{aligned} \tag{3.22}$$

Since  $\text{Proj}_C(I - \lambda_s \nabla f)$  is nonexpansive, we obtain that  $I - \text{Proj}_C(I - \lambda_s \nabla f)$  is monotone, that is,

$$\langle (I - \text{Proj}_C(I - \lambda_s \nabla f))(x_s) - (I - \text{Proj}_C(I - \lambda_s \nabla f))(\tilde{x}), x_s - \tilde{x} \rangle \geq 0. \tag{3.23}$$

Taking the limit through  $s = s_n \rightarrow 0$  ensures that  $\bar{x}$  is a solution to (3.12). That is to say

$$\langle (\mu F - \gamma h)(\bar{x}), \bar{x} - \tilde{x} \rangle \leq 0. \tag{3.24}$$

Hence  $\bar{x} = x^*$  by uniqueness. Therefore,  $x_s \rightarrow x^*$  as  $s \rightarrow 0$ . The variational inequality (3.12) can be written as

$$\langle (I - \mu F + \gamma h)x^* - x^*, \tilde{x} - x^* \rangle \leq 0, \quad \forall \tilde{x} \in S. \tag{3.25}$$

So, by Lemma 2.4, it is equivalent to the fixed-point equation

$$P_S(I - \mu F + \gamma h)x^* = x^*. \tag{3.26}$$

□

Taking  $F = A$ ,  $\mu = 1$  in Theorem 3.2, we get the following

**Corollary 3.3.** *We have that  $\{x_s\}$  converges in norm as  $s \rightarrow 0$  to a minimizer of (1.1) which solves the variational inequality*

$$\langle (A - \gamma h)x^*, \tilde{x} - x^* \rangle \geq 0, \quad \forall \tilde{x} \in S. \tag{3.27}$$

Equivalently, we have  $\text{Proj}_S(I - (A - \gamma h))x^* = x^*$ .

Taking  $F = I$ ,  $\mu = 1$ ,  $\gamma = 1$  in Theorem 3.2, we get the following.

**Corollary 3.4.** *Let  $z_s \in H$  be the unique fixed point of the contraction  $z \mapsto sh(z) + (1 - s)\text{Proj}_C(I - \lambda_s \nabla f)(z)$ . Then,  $\{z_s\}$  converges in norm as  $s \rightarrow 0$  to the unique solution of the variational inequality*

$$\langle (I - h)x^*, \tilde{x} - x^* \rangle \geq 0, \quad \forall \tilde{x} \in S. \tag{3.28}$$

Finally, we consider the following hybrid gradient-projection algorithm,

$$\begin{cases} x_0 \in \text{Carbitrarily}, \\ x_{n+1} = \theta_n \gamma h(x_n) + (I - \mu \theta_n F) \text{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \forall n \geq 0. \end{cases} \quad (3.29)$$

Assume that the sequence  $\{\lambda_n\}_{n=0}^{\infty}$  satisfies the condition (1.4) and, in addition, that the following conditions are satisfied for  $\{\lambda_n\}_{n=0}^{\infty}$  and  $\{\theta_n\}_{n=0}^{\infty} \subset [0, 1]$ :

- (i)  $\theta_n \rightarrow 0$ ;
- (ii)  $\sum_{n=0}^{\infty} \theta_n = \infty$ ;
- (iii)  $\sum_{n=0}^{\infty} |\theta_{n+1} - \theta_n| < \infty$ ;
- (iv)  $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ .

**Theorem 3.5.** *Assume that the minimization problem (1.1) is consistent and the gradient  $\nabla f$  satisfies the Lipschitz condition (1.2). Let  $\{x_n\}$  be generated by algorithm (3.29) with the sequences  $\{\theta_n\}$  and  $\{\lambda_n\}$  satisfying the above conditions. Then, the sequence  $\{x_n\}$  converges in norm to  $x^*$  that is obtained in Theorem 3.2.*

*Proof.* (1) The sequence  $\{x_n\}_{n=0}^{\infty}$  is bounded. Setting

$$V_n := \text{Proj}_C(I - \lambda_n \nabla f). \quad (3.30)$$

Indeed, we have, for  $\bar{x} \in S$ ,

$$\begin{aligned} \|x_{n+1} - \bar{x}\| &= \|\theta_n \gamma h(x_n) + (I - \mu \theta_n F) V_n x_n - \bar{x}\| \\ &= \|\theta_n (\gamma h(x_n) - \mu F(\bar{x})) + (I - \mu \theta_n F) V_n x_n - (I - \mu \theta_n F) V_n \bar{x}\| \\ &\leq (1 - \theta_n \tau) \|x_n - \bar{x}\| + \theta_n \rho \gamma \|x_n - \bar{x}\| + \theta_n \|\gamma h(\bar{x}) - \mu F(\bar{x})\| \\ &= (1 - \theta_n (\tau - \gamma \rho)) \|x_n - \bar{x}\| + \theta_n \|\gamma h(\bar{x}) - \mu F(\bar{x})\| \\ &\leq \max \left\{ \|x_n - \bar{x}\|, \frac{1}{\tau - \gamma \rho} \|\gamma h(\bar{x}) - \mu F(\bar{x})\| \right\}, \quad \forall n \geq 0. \end{aligned} \quad (3.31)$$

By induction,

$$\|x_n - \bar{x}\| \leq \max \left\{ \|x_0 - \bar{x}\|, \frac{\|\gamma h(\bar{x}) - \mu F(\bar{x})\|}{\tau - \gamma \rho} \right\}. \quad (3.32)$$

In particular,  $\{x_n\}_{n=0}^{\infty}$  is bounded.

(2) We prove that  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $M$  be a constant such that

$$M > \max \left\{ \sup_{n \geq 0} \gamma \|h(x_n)\|, \sup_{\kappa, n \geq 0} \mu \|F V_{\kappa} x_n\|, \sup_{n \geq 0} \|\nabla f(x_n)\| \right\}. \quad (3.33)$$

We compute

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
&= \|\theta_n \gamma h(x_n) + (I - \mu\theta_n F)V_n x_n - \theta_{n-1} \gamma h(x_{n-1}) - (I - \mu\theta_{n-1} F)V_{n-1} x_{n-1}\| \\
&= \|\theta_n \gamma (h(x_n) - h(x_{n-1})) + \gamma(\theta_n - \theta_{n-1})h(x_{n-1}) + (I - \mu\theta_n F)V_n x_n \\
&\quad - (I - \mu\theta_n F)V_n x_{n-1} + (I - \mu\theta_n F)V_n x_{n-1} - (I - \mu\theta_{n-1} F)V_{n-1} x_{n-1}\| \\
&= \|\theta_n \gamma (h(x_n) - h(x_{n-1})) + \gamma(\theta_n - \theta_{n-1})h(x_{n-1}) + (I - \mu\theta_n F)V_n x_n \\
&\quad - (I - \mu\theta_n F)V_n x_{n-1} + (I - \mu\theta_n F)V_n x_{n-1} - (I - \mu\theta_n F)V_{n-1} x_{n-1} \\
&\quad + (I - \mu\theta_n F)V_{n-1} x_{n-1} - (I - \mu\theta_{n-1} F)V_{n-1} x_{n-1}\| \\
&\leq \theta_n \gamma \rho \|x_n - x_{n-1}\| + \gamma |\theta_n - \theta_{n-1}| \|h(x_{n-1})\| + (1 - \theta_n \tau) \|x_n - x_{n-1}\| \\
&\quad + \|V_n x_{n-1} - V_{n-1} x_{n-1}\| + \mu |\theta_n - \theta_{n-1}| \|F V_{n-1} x_{n-1}\| \\
&\leq \theta_n \gamma \rho \|x_n - x_{n-1}\| + M |\theta_n - \theta_{n-1}| + (1 - \theta_n \tau) \|x_n - x_{n-1}\| \\
&\quad + \|V_n x_{n-1} - V_{n-1} x_{n-1}\| + M |\theta_n - \theta_{n-1}| \\
&= (1 - \theta_n (\tau - \gamma \rho)) \|x_n - x_{n-1}\| + 2M |\theta_n - \theta_{n-1}| + \|V_n x_{n-1} - V_{n-1} x_{n-1}\|, \tag{3.34}
\end{aligned}$$

$$\begin{aligned}
\|V_n x_{n-1} - V_{n-1} x_{n-1}\| &= \|\text{Proj}_C(I - \lambda_n \nabla f)x_{n-1} - \text{Proj}_C(I - \lambda_{n-1} \nabla f)x_{n-1}\| \\
&\leq \|(I - \lambda_n \nabla f)x_{n-1} - (I - \lambda_{n-1} \nabla f)x_{n-1}\| \\
&= |\lambda_n - \lambda_{n-1}| \|\nabla f(x_{n-1})\| \\
&\leq M |\lambda_n - \lambda_{n-1}|. \tag{3.35}
\end{aligned}$$

Combining (3.34) and (3.35), we can obtain

$$\|x_{n+1} - x_n\| \leq (1 - (\tau - \gamma \rho)\theta_n) \|x_n - x_{n-1}\| + 2M(|\theta_n - \theta_{n-1}| + |\lambda_n - \lambda_{n-1}|). \tag{3.36}$$

Apply Lemma 2.1 to (3.36) to conclude that  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

(3) We prove that  $\omega_w(x_n) \subset S$ . Let  $\hat{x} \in \omega_w(x_n)$ , and assume that  $x_{n_j} \rightarrow \hat{x}$  for some subsequence  $\{x_{n_j}\}_{j=1}^\infty$  of  $\{x_n\}_{n=0}^\infty$ . We may further assume that  $\lambda_{n_j} \rightarrow \lambda \in [0, 2/L]$  due to condition (1.4). Set  $V := \text{Proj}_C(I - \lambda \nabla f)$ . Notice that  $V$  is nonexpansive and  $\text{Fix } V = S$ . It turns out that

$$\begin{aligned}
\|x_{n_j} - V x_{n_j}\| &\leq \|x_{n_j} - V_{n_j} x_{n_j}\| + \|V_{n_j} x_{n_j} - V x_{n_j}\| \\
&\leq \|x_{n_j} - x_{n_j+1}\| + \|x_{n_j+1} - V_{n_j} x_{n_j}\| + \|V_{n_j} x_{n_j} - V x_{n_j}\| \\
&\leq \|x_{n_j} - x_{n_j+1}\| + \theta_{n_j} \|\gamma h(x_{n_j}) - \mu F V_{n_j} x_{n_j}\| \\
&\quad + \|\text{Proj}_C(I - \lambda_{n_j} \nabla f)x_{n_j} - \text{Proj}_C(I - \lambda \nabla f)x_{n_j}\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_{n_j} - x_{n_j+1}\| + \theta_{n_j} \|\gamma h(x_{n_j}) - \mu F V_{n_j} x_{n_j}\| + |\lambda - \lambda_{n_j}| \|\nabla f(x_{n_j})\| \\
&\leq \|x_{n_j} - x_{n_j+1}\| + 2M(\theta_{n_j} + |\lambda - \lambda_{n_j}|) \rightarrow 0 \quad \text{as } j \rightarrow \infty.
\end{aligned} \tag{3.37}$$

So Lemma 2.2 guarantees that  $\omega_w(x_n) \subset \text{Fix } V = S$ .

(4) We prove that  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ , where  $x^*$  is the unique solution of the VI (3.12). First observe that there is some  $\hat{x} \in \omega_w(x_n) \subset S$  Such that

$$\limsup_{n \rightarrow \infty} \langle (\mu F - \gamma h)x^*, x_n - x^* \rangle = \langle (\mu F - \gamma h)x^*, \hat{x} - x^* \rangle \geq 0. \tag{3.38}$$

We now compute

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\theta_n \gamma h(x_n) + (I - \mu \theta_n F) \text{Proj}_C(I - \lambda_n \nabla f)(x_n) - x^*\|^2 \\
&= \|\theta_n (\gamma h(x_n) - \mu F x^*) + (I - \mu \theta_n F) V_n(x_n) - (I - \mu \theta_n F) V_n x^*\|^2 \\
&= \|\theta_n \gamma (h(x_n) - h(x^*)) + (I - \mu \theta_n F) V_n(x_n) - (I - \mu \theta_n F) V_n x^* + \theta_n (\gamma h(x^*) - \mu F x^*)\|^2 \\
&\leq \|\theta_n \gamma (h(x_n) - h(x^*)) + (I - \mu \theta_n F) V_n(x_n) - (I - \mu \theta_n F) V_n x^*\|^2 \\
&\quad + 2\theta_n \langle (\gamma h - \mu F)x^*, x_{n+1} - x^* \rangle \\
&= \|\theta_n \gamma (h(x_n) - h(x^*))\|^2 + \|(I - \mu \theta_n F) V_n(x_n) - (I - \mu \theta_n F) V_n x^*\|^2 \\
&\quad + 2\theta_n \gamma \langle h(x_n) - h(x^*), (I - \mu \theta_n F) V_n(x_n) - (I - \mu \theta_n F) V_n x^* \rangle \\
&\quad + 2\theta_n \langle (\gamma h - \mu F)x^*, x_{n+1} - x^* \rangle \\
&\leq \theta_n^2 \gamma^2 \rho^2 \|x_n - x^*\|^2 + (1 - \theta_n \tau)^2 \|x_n - x^*\|^2 + 2\theta_n \gamma \rho (1 - \theta_n \tau) \|x_n - x^*\|^2 \\
&\quad + 2\theta_n \langle (\gamma h - \mu F)x^*, x_{n+1} - x^* \rangle \\
&= \left( \theta_n^2 \gamma^2 \rho^2 + (1 - \theta_n \tau)^2 + 2\theta_n \gamma \rho (1 - \theta_n \tau) \right) \|x_n - x^*\|^2 \\
&\quad + 2\theta_n \langle (\gamma h - \mu F)x^*, x_{n+1} - x^* \rangle \\
&\leq \left( \theta_n \gamma^2 \rho^2 + 1 - 2\theta_n \tau + \theta_n \tau^2 + 2\theta_n \gamma \rho \right) \|x_n - x^*\|^2 \\
&\quad + 2\theta_n \langle (\gamma h - \mu F)x^*, x_{n+1} - x^* \rangle \\
&= \left( 1 - \theta_n (2\tau - \gamma^2 \rho^2 - \tau^2 - 2\gamma \rho) \right) \|x_n - x^*\|^2 + 2\theta_n \langle (\gamma h - \mu F)x^*, x_{n+1} - x^* \rangle.
\end{aligned} \tag{3.39}$$

Applying Lemma 2.1 to the inequality (3.39), together with (3.38), we get  $\|x_n - x^*\| \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**Corollary 3.6** (see [11]). Let  $\{x_n\}$  be generated by the following algorithm:

$$x_{n+1} = \theta_n h(x_n) + (1 - \theta_n) \text{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0. \quad (3.40)$$

Assume that the sequence  $\{\lambda_n\}_{n=0}^{\infty}$  satisfies the conditions (1.4) and (iv) and that  $\{\theta_n\} \subset [0, 1]$  satisfies the conditions (i)–(iii). Then  $\{x_n\}$  converges in norm to  $x^*$  obtained in Corollary 3.4.

**Corollary 3.7.** Let  $\{x_n\}$  be generated by the following algorithm:

$$x_{n+1} = \theta_n \gamma h(x_n) + (I - \theta_n A) \text{Proj}_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0. \quad (3.41)$$

Assume that the sequences  $\{\theta_n\}$  and  $\{\lambda_n\}$  satisfy the conditions contained in Theorem 3.5, then  $\{x_n\}$  converges in norm to  $x^*$  obtained in Corollary 3.3.

## Acknowledgments

Ming Tian is Supported in part by The Fundamental Research Funds for the Central Universities (the Special Fund of Science in Civil Aviation University of China: No. ZXH2012 K001) and by the Science Research Foundation of Civil Aviation University of China (No. 2012KYM03).

## References

- [1] E. S. Levitin and B. T. Poljak, "Minimization methods in the presence of constraints," *Žurnal Vychislitel' noi Matematiki i Matematičeskoj Fiziki*, vol. 6, pp. 787–823, 1966.
- [2] P. H. Calamai and J. J. Moré, "Projected gradient methods for linearly constrained problems," *Mathematical Programming*, vol. 39, no. 1, pp. 93–116, 1987.
- [3] B. T. Polyak, *Introduction to Optimization*, Translations Series in Mathematics and Engineering, Optimization Software, New York, NY, USA, 1987.
- [4] M. Su and H. K. Xu, "Remarks on the gradient-projection algorithm," *Journal of Nonlinear Analysis and Optimization*, vol. 1, pp. 35–43, 2010.
- [5] Y. Yao and H.-K. Xu, "Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications," *Optimization*, vol. 60, no. 6, pp. 645–658, 2011.
- [6] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2–4, pp. 221–239, 1994.
- [7] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.
- [8] J. S. Jung, "Strong convergence of composite iterative methods for equilibrium problems and fixed point problems," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 498–505, 2009.
- [9] G. Lopez, V. Martin, and H. K. Xu, "Iterative algorithms for the multiple-sets split feasibility problem," in *Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems*, Y. Censor, M. Jiang, and G. Wang, Eds., pp. 243–279, Medical Physics, Madison, Wis, USA, 2009.
- [10] P. Kumam, "A hybrid approximation method for equilibrium and fixed point problems for a monotone mapping and a nonexpansive mapping," *Nonlinear Analysis*, vol. 2, no. 4, pp. 1245–1255, 2008.
- [11] H.-K. Xu, "Averaged mappings and the gradient-projection algorithm," *Journal of Optimization Theory and Applications*, vol. 150, no. 2, pp. 360–378, 2011.
- [12] P. Kumam, "A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an strongly monotone operator and a nonexpansive mapping," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 263–280, 2009.

- [13] H. Brezis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, The Netherlands, 1973.
- [14] P. L. Combettes, "Solving monotone inclusions via compositions of nonexpansive averaged operators," *Optimization*, vol. 53, no. 5-6, pp. 475–504, 2004.
- [15] Y. Yao, Y.-C. Liou, and R. Chen, "A general iterative method for an infinite family of nonexpansive mappings," *Nonlinear Analysis*, vol. 69, no. 5-6, pp. 1644–1654, 2008.
- [16] M. Tian, "A general iterative algorithm for nonexpansive mappings in Hilbert spaces," *Nonlinear Analysis*, vol. 73, no. 3, pp. 689–694, 2010.
- [17] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society. Second Series*, vol. 66, no. 1, pp. 240–256, 2002.
- [18] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.



# Hindawi

Submit your manuscripts at  
<http://www.hindawi.com>

