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Research Article

Some Fixed Point Theorems for Nonlinear Set-Valued Contractive Mappings

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Four fixed point theorems for nonlinear set-valued contractive mappings in complete metric spaces are proved. The results presented in this paper are extensions of a few well-known fixed point theorems. Two examples are also provided to illustrate our results.

1. Introduction and Preliminaries

The existence of fixed points for various set-valued contractive mappings had been researched by many authors under different conditions, see, for example, [1–9] and the references cited therein. In 1969, Nadler [7] proved a well-known fixed point theorem for the set-valued contraction mapping (1.1) below.

Theorem 1.1 (see [7]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a set-valued mapping such that*

$$H(Tx, Ty) \le rd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where $r \in (0, 1)$ is a constant. Then T has a fixed point.

In 1972, Reich [8] extended Nadler's result and established an interesting fixed point theorem for the set-valued contraction mapping (1.2) below.

Theorem 1.2 (see [8]). *Let (X, d) be a complete metric space and $T : X \to C(X)$ satisfy that*

$$H(Tx, Ty) \le \varphi(d(x, y))d(x, y), \quad \forall x, y \in X, \tag{1.2}$$

where

$$\varphi : (0,+\infty) \longrightarrow [0,1) \text{ with } \limsup_{r\to t^+} \varphi(r) < 1, \quad \forall t \in (0,+\infty). \tag{1.3}$$

Then T has a fixed point.

In [8] Reich posed the question whether Theorem 1.2 is also true for the set-valued contractive mapping $T : X \to CB(X)$ with (1.2). The affirmative answer under the hypothesis of $\limsup_{r\to t^+}\varphi(r) < 1$, for all $t \in [0,+\infty)$ was given by Mizoguchi and Takahashi in [6]. They deduced the following fixed point theorem which is a generalization of the Nadler fixed point theorem.

Theorem 1.3 (see [6]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ satisfy* (1.2)*, where*

$$\varphi : (0,+\infty) \longrightarrow [0,1) \text{ with } \limsup_{r\to t^+} \varphi(r) < 1, \quad \forall t \in [0,+\infty). \tag{1.4}$$

Then T has a fixed point.

Remark 1.4. It is clear that the mappings T in Theorems 1.1–1.3 are continuous on X.

Remark 1.5. Each of Theorems 1.2 and 1.3 ensures that T has a fixed point $a \in Ta \subseteq X$, which together with (1.2) implies that $\varphi(0) = \varphi(d(a,a))$, that is, φ is defined at 0. Thus the domain of φ in each of (1.3) and (1.4) should be $[0,+\infty)$ but not $(0,+\infty)$.

The aim of this paper is to present four fixed point theorems for some nonlinear set-valued contractive mappings. Our results extend, improve, and unify the corresponding results in [6–8]. Two nontrivial examples are given to show that our results are genuine generalizations or different from these results in [6–8].

Throughout this paper, we assume that $\mathbb{R} = (-\infty,+\infty)$, $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of all positive integers and nonnegative integers, respectively, and

$$\Theta = \{\theta : \theta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ satisfies (a)–(d)}\}, \tag{1.5}$$

where

(a) θ is nondecreasing on $\mathbb{R}^+$;

(b) $\theta(t) > 0$, for all $t \in (0,+\infty)$;

(c) θ is subadditive in $(0,+\infty)$, that is,

$$\theta(t_1 + t_2) \le \theta(t_1) + \theta(t_2), \quad \forall t_1, t_2 \in (0,+\infty); \tag{1.6}$$

(d) $\theta(\mathbb{R}^+) = \mathbb{R}^+$.
Clearly (a)–(d) imply that

(e) θ is strictly inverse on $\mathbb{R}^+$, that is, if there exist $t, s \in \mathbb{R}^+$ satisfying $\theta(t) < \theta(s)$, then $t < s$.

Let (X, d) be a metric space, $CL(X)$, $CB(X)$, and $C(X)$ denote the families of all nonempty closed, all nonempty bounded closed, and all nonempty compact subsets of X. For $x \in X$ and $A, B \in CL(X)$, put $d(x, A) = \inf\{d(x, y) : y \in A\}$ and

$$H(A,B) = \begin{cases} \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, & \text{if the maximum exists} \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.7}$$

Such a mapping H is called a *generalized Hausdorff metric induced by* d in $CL(X)$. It is well known that H is a metric on $CB(X)$. Let $T : X \to CL(X)$ be a set-valued mapping, $x_0 \in X$ and $f : X \to \mathbb{R}^+$ be defined by

$$f(x) = d(x, Tx), \quad \forall x \in X. \tag{1.8}$$

A sequence $\{x_n\}_{n\in\mathbb{N}_0}$ is said to be an *orbit of* T if it satisfies that $\{x_n\}_{n\in\mathbb{N}_0} \subset X$ and $x_n \in Tx_{n-1}$ for each $n \in \mathbb{N}_0$. The function $f : X \to \mathbb{R}^+$ is said to be T*-orbitally lower semicontinuous at* $z \in X$ if for each orbit $\{x_n\}_{n\in\mathbb{N}_0} \subset X$ of T with $\lim_{n\to\infty} x_n = z$, we have that $f(z) \le \liminf_{n\to\infty} f(x_n)$.

2. Main Results

The following lemmas play important roles in this paper.

Lemma 2.1. *Let (X, d) be a metric space and $B \in CL(X)$. Then for each $x \in X$ and $\varepsilon > 0$ there exists $b \in B$ satisfying $d(x, b) \le d(x, B) + \varepsilon$.*

Proof. Suppose that there exist $x_0 \in X$ and $\varepsilon_0 > 0$ such that

$$d(x_0, b) > d(x_0, B) + \varepsilon_0, \quad \forall b \in B, \tag{2.1}$$

which yields that

$$d(x_0, B) = \inf_{b\in B} d(x_0, b) \ge d(x_0, B) + \varepsilon_0 > d(x_0, B), \tag{2.2}$$

which is a contradiction. This completes the proof. □

Lemma 2.2. *Let (X, d) be a metric space, $B \in CL(X)$ and $\theta \in \Theta$. Then for each $x \in X$ and $q > 1$ there exists $b \in B$ such that*

$$\theta(d(x, b)) \le q\theta(d(x, B)). \tag{2.3}$$

Proof. Let $x \in X$ and $q > 1$. Now we consider two possible cases as follows.

Case 1. Suppose that $\theta(d(x, B)) = 0$. It follows from (b) and (d) that $d(x, B) = 0$. Since B is a closed subset of X, it follows that $x \in B$. Put $b = x$. Clearly (2.3) holds.

Case 2. Suppose that $\theta(d(x,B)) > 0$. Note that (b) and (d) mean that

$$(q-1)\theta(d(x,B)) \in \mathbb{R}^+ \setminus \{0\} = \theta(\mathbb{R}^+ \setminus \{0\}). \tag{2.4}$$

Choose $p \in \theta^{-1}((q-1)\theta(d(x,B)))$ and $\varepsilon = p/2 > 0$. Lemma 2.1 ensures that there exists $b \in B$ satisfying $d(x,b) \le d(x,B) + \varepsilon$, which together with (a) and (c) gives that

$$\begin{aligned}\theta(d(x,b)) &\le \theta(d(x,B)+\varepsilon) \le \theta(d(x,B)) + \theta(\varepsilon)\\ &\le \theta(d(x,B)) + \theta\left(\theta^{-1}((q-1)\theta(d(x,B)))\right) = q\theta(d(x,B)).\end{aligned} \tag{2.5}$$

That is, (2.3) holds. This completes the proof. □

Now we prove four fixed point theorems for the nonlinear set-valued contractive mappings (2.6), (2.25), (2.26), and (2.36) below in complete metric spaces.

Theorem 2.3. *Let (X,d) be a complete metric space and $T : X \to CL(X)$ satisfy that*

$$\theta(d(y,Ty)) \le \varphi(d(x,y))\theta(d(x,y)), \quad \forall (x,y) \in X \times Tx, \tag{2.6}$$

where $\theta \in \Theta$ and

$$\varphi : \mathbb{R}^+ \longrightarrow [0,1) \text{ with } \limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+. \tag{2.7}$$

Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by (1.8) *is T orbitally lower semicontinuous at z.*

Proof. Let $x_0 \in X$ be any initial point and choose $x_1 \in Tx_0$. It follows from (2.6), (2.7) and Lemma 2.2 that for $q_1 = 1/\max\{\sqrt{\varphi(d(x_0,x_1))}, 1/2\} > 1$ there exists $x_2 \in Tx_1$ satisfying

$$\begin{aligned}\theta(d(x_1,x_2)) &\le \frac{\theta(d(x_1,Tx_1))}{\max\left\{\sqrt{\varphi(d(x_0,x_1))},1/2\right\}} \le \frac{\varphi(d(x_0,x_1))\theta(d(x_0,x_1))}{\max\left\{\sqrt{\varphi(d(x_0,x_1))},1/2\right\}}\\ &\le \sqrt{\varphi(d(x_0,x_1))}\theta(d(x_0,x_1)),\end{aligned} \tag{2.8}$$

and for $q_2 = 1/\max\{\sqrt{\varphi(d(x_1,x_2))}, 1/3\} > 1$ there exists $x_3 \in Tx_2$ satisfying

$$\begin{aligned}\theta(d(x_2,x_3)) &\le \frac{\theta(d(x_2,Tx_2))}{\max\left\{\sqrt{\varphi(d(x_1,x_2))},1/3\right\}} \le \frac{\varphi(d(x_1,x_2))\theta(d(x_1,x_2))}{\max\left\{\sqrt{\varphi(d(x_1,x_2))},1/3\right\}}\\ &\le \sqrt{\varphi(d(x_1,x_2))}\theta(d(x_1,x_2)).\end{aligned} \tag{2.9}$$

Repeating the above argument we obtain a sequence $\{x_n\}_{n\in\mathbb{N}_0} \subset X$ such that $x_k \in Tx_{k-1}$ for $1 \le k \le n$ and for $q_n = 1/\max\{\sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1)\} > 1$, there exists $x_{n+1} \in Tx_n$ satisfying

$$\begin{aligned}
\theta(d(x_n, x_{n+1})) &\le \frac{\theta(d(x_n, Tx_n))}{\max\left\{\sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1)\right\}} \\
&\le \frac{\varphi(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n))}{\max\left\{\sqrt{\varphi(d(x_{n-1}, x_n))}, 1/(n+1)\right\}} \\
&\le \sqrt{\varphi(d(x_{n-1}, x_n))}\theta(d(x_{n-1}, x_n)), \quad \forall n \ge 1.
\end{aligned} \tag{2.10}$$

Suppose that there exists some $n_0 \in \mathbb{N}_0$ satisfying $x_{n_0} = x_{n_0+1} \in Tx_{n_0}$. It follows from (a), (b), and (2.10) that $x_n = x_{n_0}$ for all $n \ge n_0 + 1$. It is clear the conclusion of Theorem 2.3 holds.

Suppose that $x_{n+1} \in Tx_n \setminus \{x_n\}$ for any $n \in \mathbb{N}_0$. It follows that $d(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}_0$. Note that (b), (2.7), and (2.10) give that $\{\theta(d(x_n, x_{n+1}))\}_{n\in\mathbb{N}_0}$ is a positive and decreasing sequence. It follows from (e) that $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is decreasing. Therefore, there exist constants p and q satisfying

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = p \ge 0, \qquad \lim_{n\to\infty} d(x_n, x_{n+1}) = q \ge 0. \tag{2.11}$$

Notice that (2.7) implies that there exists a constant r satisfying

$$\limsup_{n\to\infty} \varphi(d(x_n, x_{n-1})) \le \limsup_{t\to q^+} \varphi(t) = r \in [0, 1). \tag{2.12}$$

Taking upper limits in (2.10) and by (2.11) and (2.12) we get that

$$p \le \sqrt{\limsup_{n\to\infty} \varphi(d(x_{n-1}, x_n))}\limsup_{n\to\infty} \theta(d(x_{n-1}, x_n)) \le \sqrt{r}p, \tag{2.13}$$

which implies that $p = 0$.

Next we assert that $q = 0$. Since $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is a decreasing sequence, it follows from (a) and (2.11) that

$$0 \le \theta(q) < \theta(d(x_n, x_{n+1})) \longrightarrow p = 0 \quad \text{as } n \longrightarrow \infty, \tag{2.14}$$

that is, $\theta(q) = 0$, which together with (b) and (d) yields that $q = 0$.

Put $c = (1 + r)/2$. It follows from (2.12) that $c \in (r, 1) \subset [0, 1)$, which gives that $c^2 \in (r, 1)$. Notice that (2.11), (2.12), and $q = 0$ ensure that there exist $\delta > 0$ and $N \in \mathbb{N}$ satisfying

$$\varphi(t) < c^2, \quad \forall t \in (0, \delta), \qquad d(x_n, x_{n+1}) < \delta, \quad \forall n \ge N, \tag{2.15}$$

which implies that

$$\varphi(d(x_n, x_{n+1})) < c^2, \quad \forall n \geq N. \tag{2.16}$$

Note that (2.10) and (2.16) mean that

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &\leq \prod_{k=N}^{n-1} \sqrt{\varphi(d(x_k, x_{k+1}))}\theta(d(x_N, x_{N+1})) \\ &\leq c^{n-N}\theta(d(x_N, x_{N+1})), \quad \forall n \geq N.\end{aligned} \tag{2.17}$$

Given $\varepsilon > 0$. Since $\lim_{n\to\infty} c^{n-N}\theta(d(x_N, x_{N+1})) = 0$, it follows from (b) that there exists $N_1 > N$ satisfying

$$\frac{c^{n-N}}{1-c}\theta(d(x_N, x_{N+1})) < \theta(\varepsilon), \quad \forall n \geq N_1, \tag{2.18}$$

which together with (2.17), (a), and (c) gives that

$$\begin{aligned}\theta(d(x_n, x_m)) &\leq \theta\left(\sum_{k=n}^{m-1} d(x_k, x_{k+1})\right) \leq \sum_{k=n}^{m-1}\theta(d(x_k, x_{k+1})) \\ &\leq \sum_{k=n}^{m-1} c^{k-N}\theta(d(x_N, x_{N+1})) \\ &\leq \frac{c^{n-N}}{1-c}\theta(d(x_N, x_{N+1})) < \theta(\varepsilon), \quad \forall m > n \geq N_1.\end{aligned} \tag{2.19}$$

In view of (e) and (2.19), we deduce that $d(x_n, x_m) < \varepsilon$, for all $m > n \geq N_1$, which means that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Hence there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$ by completeness of X.

Suppose that f is T orbitally lower semicontinuous at z. Since $\{x_n\}_{n\geq 0}$ is an orbit of T with $\lim_{n\to\infty} x_n = z$, it follows that

$$f(z) \leq \liminf_{n\to\infty} f(x_n). \tag{2.20}$$

Using (2.6) and (2.7), we infer that

$$\theta(d(x_n, Tx_n)) \leq \varphi(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n)) < \theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \tag{2.21}$$

which together with (e), (2.11), and $q = 0$ implies that

$$0 < d(x_n, Tx_n) < d(x_{n-1}, x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.22}$$

that is, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, which together with (2.20) yields that

$$0 \le d(z, Tz) = f(z) \le \liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0, \tag{2.23}$$

which gives that $d(z, Tz) = 0$, that is, $z \in Tz$.

Conversely, suppose that $z \in X$ is a fixed point of T. Let $\{y_n\}_{n\in\mathbb{N}_0} \subset X$ be an arbitrarily orbit of T with $\lim_{n\to\infty} y_n = z$. It is clear that

$$f(z) = d(z, Tz) = 0 \le \liminf_{n\to\infty} f(y_n), \tag{2.24}$$

which implies that f is T orbitally lower semicontinuous at z. This completes the proof. □

Notice that $d(y, Ty) \le H(Tx, Ty)$ for each $y \in Tx$. In light of Theorem 2.3, we have

Theorem 2.4. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ satisfy that*

$$\theta(H(Tx, Ty)) \le \varphi(d(x, y))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx, \tag{2.25}$$

where $\theta \in \Theta$ and φ satisfies (2.7). *Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by* (1.8) *is T orbitally lower semicontinuous at z.*

If $\varphi(d(x, y))$ in (2.6) is replaced by $\varphi(d(x, Tx))$, one has

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ satisfy that*

$$\theta(d(y, Ty)) \le \varphi(d(x, Tx))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx, \tag{2.26}$$

where $\theta \in \Theta$ and φ satisfies (2.7). *Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by* (1.8) *is T orbitally lower semicontinuous at z.*

Proof. Let $x_0 \in X$ be any initial point and choose $x_1 \in Tx_0$. It follows from (2.7), (2.26), and Lemma 2.2 that for $q = 1/\max\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\} > 1$ there exists $x_2 \in Tx_1$ such that

$$\begin{aligned}
\theta(d(x_1, x_2)) &\leq \frac{\theta(d(x_1, Tx_1))}{\max\left\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\right\}} \\
&\leq \frac{\varphi(d(x_0, Tx_0))\theta(d(x_0, x_1))}{\max\left\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\right\}} \\
&\leq \sqrt{\varphi(d(x_0, Tx_0))}\theta(d(x_0, x_1)), \\
\theta(d(x_2, Tx_2)) &\leq \varphi(d(x_1, Tx_1))\theta(d(x_1, x_2)) \\
&\leq \frac{\varphi(d(x_1, Tx_1))\theta(d(x_1, Tx_1))}{\max\left\{\sqrt{\varphi(d(x_0, Tx_0))}, \sqrt{\varphi(d(x_1, Tx_1))}, 1/2\right\}} \\
&\leq \sqrt{\varphi(d(x_1, Tx_1))}\theta(d(x_1, Tx_1)).
\end{aligned} \tag{2.27}$$

Repeating the above argument we obtain a sequence $\{x_n\}_{n\in\mathbb{N}_0} \subset X$ satisfying $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$,

$$\begin{aligned}
\theta(d(x_n, x_{n+1})) &\leq \frac{\theta(d(x_n, Tx_n))}{\max\left\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}, 1/(n+1)\right\}} \\
&\leq \frac{\varphi(d(x_{n-1}, Tx_{n-1}))\theta(d(x_{n-1}, x_n))}{\max\left\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}, 1/(n+1)\right\}} \\
&\leq \sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}\theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N},
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
\theta(d(x_{n+1}, Tx_{n+1})) &\leq \varphi(d(x_n, Tx_n))\theta(d(x_n, x_{n+1})) \\
&\leq \frac{\varphi(d(x_n, Tx_n))\theta(d(x_n, Tx_n))}{\max\left\{\sqrt{\varphi(d(x_{n-1}, Tx_{n-1}))}, \sqrt{\varphi(d(x_n, Tx_n))}, 1/(n+1)\right\}} \\
&\leq \sqrt{\varphi(d(x_n, Tx_n))}\theta(d(x_n, Tx_n)), \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{2.29}$$

Suppose that $x_{n_0} \in Tx_{n_0}$ for some $n_0 \in \mathbb{N}_0$. It is easy to verify that $x_n = x_{n_0}$ for all $n \geq n_0$ and the conclusion of Theorem 2.5 holds.

Suppose that $x_n \notin Tx_n$ for each $n \in \mathbb{N}_0$. It follows that $\{d(x_n, Tx_n)\}_{n\in\mathbb{N}_0}$ and $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ are positive sequences. Combining (2.7), (2.28), (2.29), (b) and (e), we infer that $\{\theta(d(x_n, x_{n+1}))\}_{n\in\mathbb{N}_0}$ and $\{\theta(d(x_n, Tx_n))\}_{n\in\mathbb{N}_0}$ are both positive and decreasing, so do $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ and $\{d(x_n, Tx_n)\}_{n\in\mathbb{N}_0}$. It follows that there exist constants α, β, s and t satisfying

$$\begin{aligned} &\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = \alpha \geq 0, \qquad \lim_{n\to\infty} d(x_n, x_{n+1}) = \beta \geq 0, \\ &\lim_{n\to\infty} \theta(d(x_n, Tx_n)) = s \geq 0, \qquad \lim_{n\to\infty} d(x_n, Tx_n) = t \geq 0. \end{aligned} \tag{2.30}$$

Notice that (2.7) implies that there exists a constant r such that

$$\limsup_{n\to\infty} \varphi(d(x_n, Tx_n)) \leq \limsup_{l\to t^+} \varphi(l) = r \in [0, 1). \tag{2.31}$$

Taking upper limits in (2.29) and by (2.30) and (2.31) we get that

$$s \leq \sqrt{\limsup_{n\to\infty} \varphi(d(x_n, Tx_n))} \limsup_{n\to\infty} \theta(d(x_n, Tx_n)) \leq \sqrt{r} s, \tag{2.32}$$

which implies that $s = 0$, which together with (2.30) and (a) ensures that

$$0 \leq \theta(t) < \theta(d(x_n, Tx_n)) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{2.33}$$

that is, $\theta(t) = 0$, which gives that $t = 0$ by (b) and (d). It follows from (2.28), (2.30), and (2.31) that

$$\alpha \leq \sqrt{\limsup_{n\to\infty} \varphi(d(x_n, Tx_n))} \limsup_{n\to\infty} \theta(d(x_{n-1}, x_n)) \leq \sqrt{r} \alpha, \tag{2.34}$$

which yields that $\alpha = 0$. Notice that (2.30) and (a) guarantee that

$$0 \leq \theta(\beta) < \theta(d(x_n, x_{n+1})) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{2.35}$$

which together with (b) and (d) yields that $\beta = 0$. The rest of the proof is similar to that of Theorem 2.3 and is omitted. This completes the proof. □

The result below follows from Theorem 2.5.

Theorem 2.6. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ satisfy that*

$$\theta(H(Tx, Ty)) \leq \varphi(d(x, Tx))\theta(d(x, y)), \quad \forall (x, y) \in X \times Tx, \tag{2.36}$$

where $\theta \in \Theta$ and φ satisfies (2.7). Then for each $x_0 \in X$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z \in X$ is fixed point of T if and only if the function f defined by (1.8) is T orbitally lower semicontinuous at z.

3. Comparisons and Examples

Now we construct two examples to compare the results in Section 2 with the corresponding results in [6–8].

Remark 3.1. Theorems 2.3 and 2.4 extend Theorems 1.1–1.3, and Theorems 2.5 and 2.6 are different from Theorems 1.1–1.3, respectively, in the following ways:

(1) the ranges $CL(X)$ of the nonlinear set-valued contractive mappings T in Theorems 2.3–2.6 are more general than the ranges $C(X)$ and $CB(X)$ of the set-valued contraction mappings T in Theorems 1.1–1.3, respectively;

(2) the T orbit lower semicontinuity at some $z \in X$ of the functions $f(x) = d(x, Tx)$ in Theorems 2.3 and 2.4 is weaker than the continuity of the set-valued contraction mappings T in X in Theorems 1.1–1.3, respectively;

(3) the set-valued contraction mappings (1.1) and (1.2) are special cases of the nonlinear set-valued contractive mapping (2.6) with $\theta \equiv 1$ because

$$d(y, Ty) \le H(Tx, Ty), \quad \forall (x, y) \in X \times Tx. \tag{3.1}$$

Example 3.2 below shows that Theorems 2.3 and 2.4 extend substantively Theorems 1.1–1.3, respectively.

Example 3.2. Let $X = (-\infty, 3/10]$ and d be the standard metric in X. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \to [0, 1)$ and $T : X \to CL(X)$ be defined by

$$\theta(t) = t^{1/2}, \quad \varphi(t) = \frac{2\sqrt{6}}{5}, \quad \forall t \in \mathbb{R}^+, \qquad Tx = \begin{cases} \left(-\infty, \dfrac{1}{4}x\right], & \forall x \in (-\infty, 0), \\ [0, 2x^2], & \forall x \in \left[0, \dfrac{3}{10}\right], \end{cases} \tag{3.2}$$

respectively. It is clear that $\theta \in \Theta$, φ satisfies (2.7) and

$$f(x) = d(x, Tx) = \begin{cases} 0, & \forall x \in (-\infty, 0) \\ x - 2x^2, & \forall x \in \left[0, \dfrac{3}{10}\right] \end{cases} \tag{3.3}$$

is T orbitally lower semicontinuous in X. In order to prove (2.6) holds, we consider two possible cases.

Case 1. Let $x \in (-\infty, 0)$ and $y \in Tx = (-\infty, (1/4)x]$. It is clear that

$$\theta(d(y, Ty)) \le \theta(H(Tx, Ty)) = \frac{1}{2}\theta(d(x, y)) \le \varphi(d(x, y))\theta(d(x, y)). \tag{3.4}$$

Case 2. Let $x \in [0, 3/10]$ and $y \in Tx = [0, 2x^2]$. It follows that

$$\begin{aligned}\theta(d(y,Ty)) &\le \theta(H(Tx,Ty)) = \sqrt{2}|x+y|^{1/2}\theta(d(x,y)) \\ &\le \sqrt{2}\left(\frac{3}{10}+\frac{9}{50}\right)^{1/2}\theta(d(x,y)) = \varphi(d(x,y))\theta(d(x,y)),\end{aligned} \tag{3.5}$$

that is, (2.6) holds. Therefore all assumptions of Theorems 2.3 and 2.4 are satisfied. It follows from each of Theorems 2.3 and 2.4 that T has a fixed point in X. However, we cannot invoke any one of Theorems 1.1–1.3 to show the existence of fixed points for the mapping T in X. Indeed, taking $x_0 = 3/10$ and $y_0 = 1/5$, we get that

$$H(Tx_0, Ty_0) = d\left(2\left(\frac{3}{10}\right)^2, 2\left(\frac{1}{5}\right)^2\right) = \frac{1}{10} \not\le \frac{r}{10} = rd(x_0, y_0), \tag{3.6}$$

for any $r \in (0,1)$ and

$$H(Tx_0, Ty_0) = d\left(2\left(\frac{3}{10}\right)^2, 2\left(\frac{1}{5}\right)^2\right) = \frac{1}{10} \not\le \frac{1}{10}\varphi\left(\frac{1}{10}\right) = \varphi(d(x_0, y_0))d(x_0, y_0), \tag{3.7}$$

for any mapping $\varphi : \mathbb{R}^+ \to [0,1)$ with each of (1.3) and (1.4).

Next we construct an example to explain Theorems 2.5 and 2.6.

Example 3.3. Let $X = [-3/10, +\infty)$ and d be the standard metric in X. Define $\theta : \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \to [0,1)$ and $T : X \to CL(X)$ by

$$\begin{gathered}\theta(t) = t^{1/2}, \quad \forall t \in \mathbb{R}^+, \qquad \varphi(t) = \begin{cases} 2\sqrt{2}t^{1/2}, & \forall t \in \left(0, \dfrac{1}{8}\right), \\ \dfrac{2\sqrt{6}}{5}, & \forall t \in \{0\} \cup \left[\dfrac{1}{8}, +\infty\right), \end{cases} \\ Tx = \begin{cases} \left[\dfrac{x}{4(1+x)}, +\infty\right), & \forall x \in (0, +\infty), \\ [-2x^2, 0], & \forall x \in \left[-\dfrac{3}{10}, 0\right], \end{cases}\end{gathered} \tag{3.8}$$

respectively. It is easy to see that (2.7) holds and

$$f(x) = d(x, Tx) = \begin{cases} 0, & \forall x \in (0, +\infty), \\ -2x^2 - x, & \forall x \in \left[-\dfrac{3}{10}, 0\right] \end{cases} \tag{3.9}$$

is T orbitally lower semicontinuous in X. In order to check (2.26), we have to consider two cases as follows.

Case 1. Let $x \in (0, +\infty)$ and $y \in Tx = [x/4(1+x), +\infty)$. It is clear that

$$\begin{aligned}
\theta(d(y,Ty)) = 0 \le \theta(H(Tx,Ty)) &= \left|\frac{x}{4(1+x)} - \frac{y}{4(1+y)}\right|^{1/2} \\
&= \frac{\theta(d(x,y))}{2(1+x)^{1/2}(1+y)^{1/2}} \le \frac{\theta(d(x,y))}{2(1+x)^{1/2}(1+x/4(1+x))^{1/2}} \\
&= \frac{\theta(d(x,y))}{(5x+4)^{1/2}} \le \frac{\theta(d(x,y))}{2} \le \frac{2\sqrt{6}}{5}\theta(d(x,y)) \\
&= \varphi(0)\theta(d(x,y)) = \varphi(d(x,Tx))\theta(d(x,y)).
\end{aligned} \tag{3.10}$$

Case 2. Let $x \in [-3/10, 0]$ and $y \in Tx = [-2x^2, 0]$. It follows that

$$\theta(d(y,Ty)) \le \theta(H(Tx,Ty)) = \sqrt{2}|x+y|^{1/2}\theta(d(x,y)) \le \sqrt{2}\left|x-2x^2\right|^{1/2}\theta(d(x,y)). \tag{3.11}$$

For $x = 0$, we have

$$\sqrt{2}\left|x-2x^2\right|^{1/2}\theta(d(x,y)) = 0 \le \varphi(d(x,Tx))\theta(d(x,y)). \tag{3.12}$$

For $x \in [-3/10, -1/4) \cup (-1/4, 0)$, we infer that

$$\sqrt{2}\left|x-2x^2\right|^{1/2}\theta(d(x,y)) \le 2\sqrt{2}\left(-2x^2-x\right)^{1/2}\theta(d(x,y)) = \varphi(d(x,Tx))\theta(d(x,y)). \tag{3.13}$$

For $x = -1/4$, we get that

$$\sqrt{2}\left|x-2x^2\right|^{1/2}\theta(d(x,y)) = \frac{\sqrt{3}}{2}\theta(d(x,y)) \le \varphi\left(\frac{1}{8}\right)\theta(d(x,y)) = \varphi(d(x,Tx))\theta(d(x,y)). \tag{3.14}$$

Hence (2.26) holds. Thus all assumptions of Theorems 2.5 and 2.6 are satisfied. It follows from each of Theorems 2.5 and 2.6 that T has a fixed point in X.

Taking $x_0 = 1$ and $y_0 = -3/10$, we deduce that

$$H(Tx_0, Ty_0) = H\left(\left[\frac{1}{8}, +\infty\right), \left[-\frac{9}{50}, 0\right]\right) = +\infty \not\le \frac{13r}{10} = rd(x_0, y_0), \tag{3.15}$$

for any $r \in (0,1)$, and

$$H(Tx_0, Ty_0) = +\infty \not\le \frac{2\sqrt{6}}{5}\cdot\frac{13}{10} = \varphi(d(x_0,y_0))d(x_0,y_0), \tag{3.16}$$

for any mapping $\varphi : \mathbb{R}^+ \to [0,1)$ with each of (1.3) and (1.4). That is, Theorems 1.1–1.3 are inapplicable in proving the existence of fixed points for the nonlinear set-valued contractive mapping T.

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