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Research Article

Well-Posedness by Perturbations for Variational-Hemivariational Inequalities

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We generalize the concept of well-posedness by perturbations for optimization problem to a class of variational-hemivariational inequalities. We establish some metric characterizations of the well-posedness by perturbations for the variational-hemivariational inequality and prove their equivalence between the well-posedness by perturbations for the variational-hemivariational inequality and the well-posedness by perturbations for the corresponding inclusion problem.

1. Introduction and Preliminaries

The concept well-posedness is important in both theory and methodology for optimization problems. An initial, already classical concept of well-posedness for unconstrained optimization problem is due to Tykhonov in [1]. Let $f : V \to \mathbb{R} \cup \{+\infty\}$ be a real-valued functional on Banach space V. The problem of minimizing f on V is said to be well-posed if there exists a unique minimizer, and every minimizing sequence converges to the unique minimizer. Soon after, Levitin and Polyak [2] generalized the Tykhonov well-posedness to the constrained optimization problem, which has been known as the Levitin-Polyak well-posedness. It is clear that the concept of well-posedness is motivated by the numerical methods producing optimizing sequences for optimization problems. Unfortunately, these concepts generally cannot establish appropriate continuous dependence of the solution on the data. In turn, they are not suitable for the numerical methods when the objective functional f is approximated by a family or a sequence of functionals. For this reason, another important concept of well-posedness for optimization problem, which is called the well-posedness by perturbations or extended well-posedness, has been introduced and studied by [3–6]. Also, many other notions of well-posedness have been introduced and studied for optimization problem. For details, we refer to [7] and the reference therein.

The concept well-posedness also has been generalized to other related problems, especially to the variational inequality problem. Lucchetti and Patrone [8] first introduced the well-posedness for a variational inequality, which can be regarded as an extension of the Tykhonov well-posedness of optimization problem. Since then, many authors were devoted to generalizing the concept of well-posedness for the optimization problem to various variational inequalities. In [9], Huang et al. introduced several types of (generalized) Levitin-Polyak well-posednesses for a variational inequality problem with abstract and functional constraint and gave some criteria, characterizations, and their relations for these types of well-posednesses. Recently, Fang et al. [10] generalized the concept of well-posedness by perturbations, introduced by Zelezzi for a minimization problem, to a generalized mixed variational inequality problem in Banach space. They established some metric characterizations of well-posedness by perturbations and discussed its links with well-posedness by perturbations of corresponding inclusion problem and the well-posedness by perturbations of corresponding fixed point problem. Also they derived some conditions under which the well-posedness by perturbations of the mixed variational inequality is equivalent to the existence and uniqueness of its solution. For further more results on the well-posedness of variational inequalities, we refer to [8–15] and the references therein.

When the corresponding energy functions are not convex, the mathematical model describing many important phenomena arising in mechanics and engineering is no longer variational inequality but a new type of inequality problem that is called hemivariational inequality, which was first introduced by Panagiotopoulos [16] as a generalization of variational inequality. A more generalized variational formulation which is called variational-hemivariational inequality is presented to model the problems subject to constraints because the setting of hemivariational inequalities cannot incorporate the indicator function of a convex closed subset. Due to the fact that the potential is neither convex nor smooth generally, the hemivariational inequalities have been proved very efficient to describe a variety of mechanical problems using the generalized gradient of Clarke for nonconvex and nondifferentiable functions [17], such as unilateral contact problems in nonlinear elasticity, obstacles problems, and adhesive grasping in robotics (see, e.g., [18–20]). So, in recent years all kinds of hemivariational inequalities have been studied [21–30] and the study of hemivariational inequalities has emerged as a new and interesting branch of applied mathematics. However, there are very few researchers extending the well-posedness to hemivariational inequality. In 1995, Goeleven and Mentagui [23] first defined the well-posedness for hemivariational inequalities. Recently, Xiao et al. [31] generalized the concept of well-posedness to hemivariational inequalities. They established some metric characterizations of the well-posed hemivariational inequality, derived some conditions under which the hemivariational inequality is strongly well-posed in the generalized sense, and proved the equivalence between the well-posedness of hemivariational inequality and the well-posedness of a corresponding inclusion problem. Moreover, Xiao and Huang [32] studied the well-posedness of variational-hemivariational inequalities and generalized some related results.

In the present paper, we generalize the well-posedness by perturbations for optimization problem to a class of variational-hemivariational inequality. We establish some metric characterizations of the well-posedness by perturbations for variational-hemivariational inequality and prove the equivalence between the well-posedness by perturbations for the variational-hemivariational inequality and the well-posedness by perturbations for the corresponding inclusion problem.

We suppose in what follows that V is a real reflexive Banach space with its dual V^*, and $\langle \cdot, \cdot \rangle$ is the duality between V and V^*. We denote the norms of Banach space V and V^* by $\| \cdot \|_V$ and $\| \cdot \|_{V^*}$, respectively. Let $A : V \to V^*$ be a mapping, let $J : V \to \mathbb{R}$ be a locally Lipschitz functional, let $G : V \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional, and let $f \in V^*$ be some given element. Denote by dom G the domain of functional G, that is,

$$\operatorname{dom} G = \{u \in V : G(u) < +\infty\}. \tag{1.1}$$

The functional G is called proper if its domain is nonempty. The variational-hemivariational inequality associated with (A, f, J, G) is specified as follows:

$$\begin{aligned}&\mathrm{VHVI}(A, f, J, G): \\ &\text{find } u \in \operatorname{dom} G \text{ such that } \langle A(u), v-u \rangle + J^{\circ}(u, v-u) + G(v) - G(u) \geq \langle f, v-u \rangle, \quad \forall v \in V,\end{aligned} \tag{1.2}$$

where $J^{\circ}(u, v)$ denotes the generalized directional derivative in the sense of Clarke of a locally Lipschitz functional J at u in the direction v (see [17]) given by

$$J^{\circ}(u, v) = \limsup_{w \to u \;\; \lambda \downarrow 0} \frac{J(w + \lambda v) - J(w)}{\lambda}. \tag{1.3}$$

The variational-hemivariational inequality which includes many problems as special cases has been studied intensively. Some special cases of $\mathrm{VHVI}(A, f, J, G)$ are as follows:

(i) if $G = 0$, then $\mathrm{VHVI}(A, f, J, G)$ reduces to hemivariational inequality:

$$\begin{aligned}&\mathrm{HVI}(A, f, J): \\ &\text{find } u \in V \text{such that } \langle A(u), v-u \rangle + J^{\circ}(u, v-u) \geq \langle f, v-u \rangle, \quad \forall v \in V,\end{aligned} \tag{1.4}$$

(ii) if $J = 0$, then $\mathrm{VHVI}(A, f, J, G)$ is equivalent to the following mixed variational inequality:

$$\begin{aligned}&\mathrm{MVI}(A, f, G): \\ &\text{find } u \in \operatorname{dom} G \text{ such that } \langle A(u), v-u \rangle + G(v) - G(u) \geq \langle f, v-u \rangle, \quad \forall v \in V,\end{aligned} \tag{1.5}$$

(iii) if $A = 0$, $J = 0$ and $f = 0$, then $\mathrm{VHVI}(A, f, J, G)$ reduces to the global minimization problem:

$$\mathrm{MP}(G) : \min_{u \in V} G(u). \tag{1.6}$$

Let $\partial G(u) : V \to 2^{V^*} \setminus \{\emptyset\}$ and $\overline{\partial} J(u) : V \to 2^{V^*} \setminus \{\emptyset\}$ denote the subgradient of convex functional G in the sense of convex analysis (see [33]) and the Clarke's generalized gradient of locally Lipschitz functional J (see [17]), respectively, that is,

$$\begin{aligned} \partial G(u) &= \{u^* \in V^* : G(v) - G(u) \geq \langle u^*, v-u\rangle,\ \forall v \in V\}, \\ \overline{\partial} J(u) &= \{\omega \in V^* : J^\circ(u,v) \geq \langle \omega, v\rangle,\ \forall v \in V\}. \end{aligned} \tag{1.7}$$

About the subgradient in the sense of convex analysis, the Clarke's generalized directional derivative and the Clarke's generalized gradient, we have the following basic properties (see, e.g., [17, 19, 33, 34]).

Proposition 1.1. *Let V be a Banach space and $G : V \to \mathbb{R} \cup \{+\infty\}$ a convex and proper functional. Then one has the following properties of ∂G:*

(i) *$\partial G(u)$ is convex and weak*-closed;*

(ii) *if G is continuous at $u \in \operatorname{dom} G$, then $\partial G(u)$ is nonempty, convex, bounded, and weak*-compact;*

(iii) *if G is Gâteaux differentiable at $u \in \operatorname{dom} G$, then $\partial G(u) = \{DG(u)\}$, where $DG(u)$ is the Gâteaux derivative of G at u.*

Proposition 1.2. *Let V be a Banach space, and let G_1, $G_2 : V \to \mathbb{R} \cup \{+\infty\}$ be two convex functionals. If there is a point $u_0 \in \operatorname{dom} G_1 \cap \operatorname{dom} G_2$ at which G_1 is continuous, then the following equation holds:*

$$\partial(G_1 + G_2)(u) = \partial G_1(u) + \partial G_2(u), \quad \forall u \in V. \tag{1.8}$$

Proposition 1.3. *Let V be a Banach space, $u, v \in V$, and let J be a locally Lipschitz functional defined on V. Then*

(1) *the function $v \mapsto J^\circ(u,v)$ is finite, positively homogeneous, subadditive, and then convex on V,*

(2) *$J^\circ(u,v)$ is upper semicontinuous as a function of (u,v), as a function of v alone, is Lipschitz continuous on V,*

(3) *$J^\circ(u,-v) = (-J)^\circ(u,v)$,*

(4) *$\overline{\partial} J(u)$ is a nonempty, convex, bounded, and weak*-compact subset of V^*,*

(5) *for every $v \in V$, one has*

$$J^\circ(u,v) = \max\left\{\langle \xi, v\rangle : \xi \in \overline{\partial} J(u)\right\}. \tag{1.9}$$

Suppose that L is a parametric normed space with norm $\|\cdot\|_L$, $P \subset L$ is a closed ball with positive radius, and $p^* \in P$ is a given point. We denote the perturbed mappings of A, J, G as $\widehat{A} : P \times V \to V^*$ and $\widehat{J}$, $\widehat{G} : P \times V \to \mathbb{R}$, respectively, which have the property that for any $p \in P$, $\widehat{J}(p,\cdot)$ is a locally Lipschitz functional in V, $\widehat{G}(p,\cdot)$ is proper, convex, and lower semicontinuous in V, and

$$\widehat{A}(p^*,\cdot) = A(\cdot), \qquad \widehat{J}(p^*,\cdot) = J(\cdot), \qquad \widehat{G}(p^*,\cdot) = G(\cdot). \tag{1.10}$$

Then the perturbed Clarke's generalized directional derivative $\widehat{J}_2^{\circ}(p,\cdot) : V \times V \to \mathbb{R}$ and the perturbed Clarke's generalized gradient $\overline{\partial}_2\widehat{J}(p,\cdot) : V \to 2^{V^*}$ corresponding to the perturbed locally Lipschitz functional $\widehat{J}$ are, respectively, specified as

$$\begin{aligned} \widehat{J}_2^{\circ}(p,\cdot)(u,v) &= \limsup_{w\to u \ \lambda\downarrow 0} \frac{\widehat{J}(p,w+\lambda v) - \widehat{J}(p,w)}{\lambda}, \\ \overline{\partial}_2\widehat{J}(p,u) &= \left\{\omega \in V^* : \widehat{J}_2^{\circ}(p,\cdot)(u,v) \geq \langle \omega, v\rangle,\ \forall v \in V\right\}. \end{aligned} \tag{1.11}$$

The perturbed subgradient $\partial_2\widehat{G}(p,\cdot) : \operatorname{dom} G \to 2^{V^*}$ corresponding to the perturbed convex functional $\widehat{G}$ is

$$\partial_2\widehat{G}(p,u) = \left\{u^* \in V^* : \widehat{G}(p,v) - \widehat{G}(p,u) \geq \langle u^*, v-u\rangle,\ \forall v \in V\right\}. \tag{1.12}$$

Based on the above-perturbed mappings, the perturbed problem of VHVI(A,f,J,G) is given by

$$\begin{aligned} &\mathrm{VHVI}_p(A,f,J,G) : \\ &\text{find } u \in \operatorname{dom} \widehat{G}(p,\cdot) \text{ such that } \left\langle \widehat{A}(p,u) - f, v-u\right\rangle + \widehat{J}_2^{\circ}(p,\cdot)(u,v-u) + \widehat{G}(p,v) \\ &- \widehat{G}(p,u) \geq 0, \quad \forall v \in V. \end{aligned} \tag{1.13}$$

In the sequel, we recall some important definitions and useful results.

Definition 1.4 (see [35]). Let S be a nonempty subsets of V. The measure of noncompactness μ of the set S is defined by

$$\mu(S) = \inf\left\{\epsilon > 0 : S \subset \bigcup_{i=1}^{n} S_i, \operatorname{diam}(S_i) < \epsilon,\ i = 1,2,\ldots,n\right\}, \tag{1.14}$$

where $\operatorname{diam}(S_i)$ means the diameter of set S_i.

Definition 1.5 (see [35]). Let A and B be two given subsets of V. The excess of A over B is defined by

$$e(A,B) = \sup_{a\in A} d(a,B), \tag{1.15}$$

where $d(\cdot, B)$ is the distance function generated by B, that is,

$$d(x, B) = \inf_{b \in B} \|a - b\|_V, \quad x \in V. \tag{1.16}$$

The Hausdorff metric $\mathscr{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathscr{H}(A, B) = \max\{e(A, B), e(B, A)\}. \tag{1.17}$$

Let $\{A_n\}$ be a sequence of nonempty subset of V. One says that A_n converges to A in the sense of Hausdorff metric if $\mathscr{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, the reader should refer to [35]. The following theorem is crucial to our main results.

Theorem 1.6 (see [36]). *Let $C \subset V$ be nonempty, closed, and convex, let $C^* \subset V^*$ be nonempty, closed, convex, and bounded, let $\varphi : V \to \mathbb{R}$ be proper, convex, and lower semicontinuous, and let $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that*

$$\langle x^*(x), x - y \rangle \geq \varphi(y) - \varphi(x). \tag{1.18}$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in C. \tag{1.19}$$

2. Well-Posedness by Perturbations of VHVI(A, f, J, G) with Metric Characterizations

In this section, we generalize the concept of well-posedness by perturbations to the variational-hemivariational inequality VHVI(A, f, J, G) and establish its metric characterizations.

Definition 2.1. Let $\{p_n\} \subset P$ with $p_n \to p^*$. A sequence $\{u_n\} \subset V$ is said to be an approximating sequence corresponding to $\{p_n\}$ for VHVI(A, f, J, G) if there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that $u_n \in \operatorname{dom} \widehat{G}(p_n, \cdot)$ and

$$\begin{aligned} &\left\langle \widehat{A}(p_n, u_n) - f, v - u_n \right\rangle + \widehat{J}_2^\circ(p_n, \cdot)(u_n, v - u_n) + \widehat{G}(p_n, v) - \widehat{G}(p_n, u_n) \\ &\quad \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in V. \end{aligned} \tag{2.1}$$

Definition 2.2. VHVI(A, f, J, G) is said to be strongly (resp., weakly) well-posed by perturbations if VHVI(A, f, J, G) has a unique solution in V, and for any $\{p_n\} \subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution.

Remark 2.3. Strong well-posedness by perturbations implies weak well-posedness by perturbations, but the converse is not true in general.

Definition 2.4. VHVI(A,f,J,G) is said to be strongly (resp., weakly) well-posed by perturbations in the generalized sense if VHVI(A,f,J,G) has a nonempty solution set S in V, and for any $\{p_n\}\subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ has some subsequence which converges strongly (resp., weakly) to some point of solution set S.

Remark 2.5. Strong well-posedness by perturbations in the generalized sense implies weak well-posedness by perturbations in the generalized sense, but the converse is not true in general.

To derive the metric characterizations of well-posedness by perturbations for VHVI(A,f,J,G), we define the following approximating solution set of VHVI(A,f,J,G): For any $\epsilon>0$,

$$\begin{aligned}\Omega(\epsilon) = \bigcup_{p\in B(p^*,\epsilon)} \Big\{ u\in \operatorname{dom}\widehat{G}(p,\cdot) : \left\langle \widehat{A}(p,u)-f, v-u\right\rangle + \widehat{J}_2^\circ(p,\cdot)(u,v-u) \\ + \widehat{G}(p,v)-\widehat{G}(p,u) \geq -\epsilon\|v-u\|_V, \forall v\in V\Big\},\end{aligned} \tag{2.2}$$

where $B(p^*,\epsilon)$ denotes the closed ball centered at p^* with radius ϵ. For any $\epsilon>0, u\in\Omega(\epsilon)$ and any set $K\subset\Omega(\epsilon)$, we define the following two functions which are specified as follows:

$$\begin{aligned} p(\epsilon,u) &= \sup\{\|v-u\|_V : v\in\Omega(\epsilon)\}, \\ q(\epsilon,K) &= e(\Omega(\epsilon),K). \end{aligned} \tag{2.3}$$

It is easy to see that $p(\epsilon,u)$ is the smallest radius of the closed ball centered at u containing $\Omega(\epsilon)$, and $q(\epsilon,K)$ is the excess of approximating solution set $\Omega(\epsilon)$ over K.

Based on the two functions $p(\epsilon,u)$ and $q(\epsilon,K)$, we now give some metric characterizations of well-posedness by perturbations for the VHVI(A,f,J,G).

Theorem 2.6. *VHVI(A,f,J,G) is strongly well-posed by perturbations if and only if there exists a solution u^* for VHVI(A,f,J,G) and $p(\epsilon,u^*)\to 0$ as $\epsilon\to 0$.*

Proof. "Necessity": suppose that VHVI(A,f,J,G) is strongly well-posed by perturbations. Then $\Omega(\epsilon)\neq\emptyset$ for all $\epsilon>0$ since there is a unique solution u^* belonging to $\Omega(\epsilon)$ by the strong well-posedness by perturbations for VHVI(A,f,J,G). We now need to prove $p(\epsilon,u^*)\to 0$ as $\epsilon\to 0$. Assume by contradiction that $p(\epsilon,u^*)$ does not converge to 0 as $\epsilon\to 0$, then there exist a constant $l>0$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n\to 0$ such that

$$p(\epsilon_n,u^*) > l > 0, \quad \forall n\in\mathbb{N}. \tag{2.4}$$

By the definition of function $p(\epsilon,u)$, there exists $u_n\in\Omega(\epsilon_n)$ such that

$$\|u_n-u^*\|_V > l, \quad \forall n\in\mathbb{N}. \tag{2.5}$$

Since $u_n \in \Omega(\epsilon_n)$, there exists some $p_n \in B(p^*, \epsilon_n)$ such that

$$\begin{aligned} &\langle \widehat{A}(p_n, u_n) - f, v - u_n \rangle + \widehat{J}_2^{\circ}(p_n, \cdot)(u_n, v - u_n) + \widehat{G}(p_n, v) - \widehat{G}(p_n, u_n) \\ &\quad \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in V, n \in \mathbb{N}. \end{aligned} \tag{2.6}$$

It is obvious that $p_n \to p^*$ as $n \to \infty$ and so $\{u_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for VHVI(A, f, J, G). Therefore, by the strong well-posedness by perturbations for VHVI(A, f, J, G), we can get $u_n \to u^*$ which is a contradiction to (2.4).

"Sufficiency": suppose that VHVI(A, f, J, G) has a solution u^* and $p(\epsilon, u^*) \to 0$ as $\epsilon \to 0$. First, we claim that u^* is a unique solution for VHVI(A, f, J, G). In fact, if VHVI(A, f, J, G) has another solution $\widehat{u}$ with $u^* \neq \widehat{u}$, it follows from the definition of $\Omega(\epsilon)$ that u^* and $\widehat{u}$ belong to $\Omega(\epsilon)$ for all $\epsilon > 0$, which together with the definition of $p(\epsilon, u)$ implies that

$$p(\epsilon, u^*) \geq \|\widehat{u} - u^*\|_V > 0, \quad \forall \epsilon > 0, \tag{2.7}$$

which is a contradiction to the assumption $p(\epsilon, u^*) \to 0$ as $\epsilon \to 0$. Now, let $\{p_n\} \subset P$ with $p_n \to p^*$ and $\{u_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for VHVI(A, f, J, G). Then there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that

$$\begin{aligned} &\langle \widehat{A}(p_n, u_n) - f, v - u_n \rangle + \widehat{J}_2^{\circ}(p_n, \cdot)(u_n, v - u_n) + \widehat{G}(p_n, v) - \widehat{G}(p_n, u_n) \\ &\quad \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in V, n \in \mathbb{N}. \end{aligned} \tag{2.8}$$

Taking $\delta_n = \|p_n - p^*\|_L$ and $\epsilon'_n = \max\{\delta_n, \epsilon_n\}$, it easy to see that $\epsilon'_n \to 0$ as $n \to \infty$ and $u_n \in \Omega(\epsilon'_n)$. Since u^* is the unique solution for VHVI(A, f, J, G), u^* also belongs to $\Omega(\epsilon'_n)$. And so, it follows from the definition of $p(\epsilon, u)$ that

$$\|u_n - u^*\|_V \leq p(\epsilon'_n, u^*) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.9}$$

which implies that VHVI(A, f, J, G) is strongly well-posed by perturbations. This completes the proof of Theorem 2.6. □

Theorem 2.7. *VHVI(A, f, J, G) is strongly well-posed by perturbations in the generalized sense if and only if the solution set S of VHVI(A, f, J, G) is nonempty and compact, and $q(\epsilon, S) \to 0$ as $\epsilon \to 0$.*

Proof. "Necessity": suppose that VHVI(A, f, J, G) is strongly well-posed by perturbations in the generalized sense. Then VHVI(A, f, J, G) has nonempty solution set S by the definition of strong well-posedness by perturbations in the generalized sense of VHVI(A, f, J, G). Let $\{u_n\}$ be any sequence in S. It is obvious that $\{u_n\}$ is an approximating sequence corresponding to constant sequence $\{p^*\}$ for VHVI(A, f, J, G). Again by the strong well-posedness by perturbations in the generalized sense of VHVI(A, f, J, G), $\{u_n\}$ has a subsequence which converges strongly to some point of S, which implies that the solution set S of VHVI(A, f, J, G) is compact. Now we show that $q(\epsilon, S) \to 0$ as $\epsilon \to 0$. Assume by contradiction that $q(\epsilon, S) \nrightarrow 0$ as $\epsilon \to 0$, then there exist a constant $l > 0$ and a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and $x_n \in \Omega(\epsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \in \mathbb{N}. \tag{2.10}$$

Since $x_n \in \Omega(\epsilon_n)$, there exists $p_n \in B(p^*, \epsilon_n)$ such that

$$\begin{aligned} &\left\langle \widehat{A}(p_n, x_n) - f, v - x_n \right\rangle + \widehat{J}_2^\circ(p_n, \cdot)(x_n, v - x_n) + \widehat{G}(p_n, v) - \widehat{G}(p_n, x_n) \\ &\quad \geq -\epsilon_n \|v - x_n\|_V, \quad \forall v \in V. \end{aligned} \tag{2.11}$$

Clearly, $p_n \to p^*$ as $n \to \infty$. This together with the above inequality implies that $\{x_n\}$ is an approximating consequence corresponding to $\{p_n\}$ for VHVI(A, f, J, G). It follows from the strongly well-posedness by perturbations in the generalized sense for VHVI(A, f, J, G) that there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which converges to some point of S. This is contradiction to (2.10) and so $q(\epsilon, S) \to 0$ as $\epsilon \to 0$.

"Sufficiency": we suppose that the solution set S of VHVI(A, f, J, G) is nonempty compact and $q(\epsilon, S) \to 0$ as $\epsilon \to 0$. Let $\{p_n\} \subset P$ be any sequence with $p_n \to p^*$ and $\{u_n\}$ an approximating sequence corresponding to p_n for VHVI(A, f, J, G), which implies that

$$\begin{aligned} &\left\langle \widehat{A}(p_n, u_n) - f, v - u_n \right\rangle + \widehat{J}_2^\circ(p_n, \cdot)(u_n, v - u_n) + \widehat{G}(p_n, v) - \widehat{G}(p_n, u_n) \\ &\quad \geq -\epsilon_n \|v - u_n\|_V, \quad \forall v \in V. \end{aligned} \tag{2.12}$$

Taking $\epsilon'_n = \max\{\epsilon_n, \|p_n - p^*\|_L\}$, it is easy to see that $\epsilon'_n \to 0$ and $u_n \in \Omega(\epsilon'_n)$. It follows that

$$d(u_n, S) \leq e\big(\Omega(\epsilon'_n, S)\big) = q(\epsilon'_n, S) \longrightarrow 0. \tag{2.13}$$

Since the solution set S of VHVI(A, f, J, G) is compact, there exists $\overline{u}_n \in S$ such that

$$\|u_n - \overline{u}_n\|_V = d(u_n, S) \longrightarrow 0. \tag{2.14}$$

Again from the compactness of solution set S, $\overline{u}_n$ has a subsequence $\{\overline{u}_{n_k}\}$ converging strongly to some point $\overline{u} \in S$. It follows from (2.14) that

$$\|u_{n_k} - \overline{u}\|_V \leq \|u_{n_k} - \overline{u}_{n_k}\|_V + \|\overline{u}_{n_k} - \overline{u}\|_V \longrightarrow 0, \tag{2.15}$$

which implies that $\{u_{n_k}\}$ converges strongly to $\overline{u}$. Thus, VHVI(A, f, J, G) is strongly well-posed by perturbations in the generalized sense. This completes the proof of Theorem 2.7. □

The strong well-posedness by perturbations in the generalized sense for VHVI(A, f, J, G) can also be characterized by the behavior of noncompactness measure of its approximating solution set.

Theorem 2.8. *Let L be a finite-dimensional space. Suppose that*

(i) $\widehat{A}(\cdot,\cdot) : P \times V \to V^*$, *the perturbed mapping of A, is continuous with respect to (p,v),*

(ii) $\widehat{G} : P \times V \to \mathbb{R} \cup \{+\infty\}$, *the perturbed functional of G, is lower semicontinuous with respect to (p,v) and continuous with respect to p for any given $v \in V$,*

(iii) $\widehat{J} : P \times V \to \mathbb{R}$, *the perturbed functional of J, is locally Lipschitz with respect to v for any $p \in P$, and its Clarke's generalized directional derivative $\widehat{J}_2^{\circ}(p,\cdot) : P \to \mathcal{L}(V \times V, \mathbb{R})$ is continuous with respect to p.*

Then, VHVI(A,f,J,G) is strongly well-posed by perturbations in the generalized sense if and only if

$$\Omega(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \qquad \mu(\Omega(\epsilon)) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{2.16}$$

Proof. From the metric characterization of strongly well-posedness by perturbations in the generalized sense for VHVI(A,f,J,G) in Theorem 2.7, we can easily prove the necessity. In fact, since VHVI(A,f,J,G) is strongly well-posed by perturbations in the generalized sense, it follows from Theorem 2.7 that the solution set S of VHVI(A,f,J,G) is nonempty compact and $q(\epsilon,S) \to 0$ as $\epsilon \to 0$. Then, we can easily get from the compactness of S and the fact $S \subset \Omega(\epsilon)$ for all $\epsilon > 0$ that $\Omega(\epsilon) \neq \emptyset$ for all $\epsilon > 0$ and

$$\mu(\Omega(\epsilon)) \leq 2\mathcal{H}(\Omega(\epsilon),S) + \mu(S) = 2e(\Omega(\epsilon),S) = 2q(\epsilon,S) \longrightarrow 0. \tag{2.17}$$

Now we prove the sufficiency. First, we claim that $\Omega(\epsilon)$ is closed for all $\epsilon > 0$. In fact, let $\{u_n\} \subset \Omega(\epsilon)$ and $u_n \to u$. Then there exists $p_n \in B(p^*,\epsilon)$ such that

$$\begin{aligned}&\left\langle \widehat{A}(p_n,u_n) - f, v - u_n \right\rangle + \widehat{J}_2^{\circ}(p_n,\cdot)(u_n, v-u_n) + \widehat{G}(p_n,v) - \widehat{G}(p_n,u_n)\\ &\quad \geq -\epsilon\|v-u_n\|_V, \quad \forall v \in V.\end{aligned} \tag{2.18}$$

Without loss of generality, we can suppose that $p_n \to \overline{p} \in B(p^*,\epsilon)$ since L is finite dimensional. By taking lim sup at both sides of above inequality, it follows from the assumptions (i)–(iii) and the upper semicontinuity of $\widehat{J}_2^{\circ}(p,\cdot)(u,v)$ with respect to (u,v) that

$$\begin{aligned}&\left\langle \widehat{A}(\overline{p},u) - f, v - u \right\rangle + \widehat{J}_2^{\circ}(\overline{p},\cdot)(u, v-u) + \widehat{G}(\overline{p},v) - \widehat{G}(\overline{p},u)\\ &\quad \geq \limsup\left\{ \left\langle \widehat{A}(p_n,u_n) - f, v - u_n \right\rangle + \widehat{J}_2^{\circ}(p_n,\cdot)(u_n, v-u_n) + \widehat{G}(p_n,v) - \widehat{G}(p_n,u_n)\right\}\\ &\quad \geq \limsup\{-\epsilon\|v-u_n\|_V\}\\ &\quad = -\epsilon\|v-u\|_V.\end{aligned} \tag{2.19}$$

Thus, $u \in \Omega(\epsilon)$ and so $\Omega(\epsilon)$ is closed.

Second, we prove that

$$S = \bigcap_{\epsilon>0}\Omega(\epsilon). \tag{2.20}$$

It is obvious that $S \subset \cap_{\epsilon>0}\Omega(\epsilon)$ since the solution set $S \subset \Omega(\epsilon)$ for all $\epsilon > 0$. Conversely, let $u \in \cap_{\epsilon>0}\Omega(\epsilon)$, and let $\{\epsilon_n\}$ be a nonnegative sequence with $\epsilon_n \to 0$ as $n \to +\infty$. Then for any $n \in \mathbb{N}$, $u \in \Omega(\epsilon_n)$, and so there exists $p_n \in B(p^*, \epsilon_n)$ such that

$$\begin{aligned}&\left\langle \widehat{A}(p_n, u) - f, v - u \right\rangle + \widehat{J}_2^\circ(p_n, \cdot)(u, v-u) + \widehat{G}(p_n, v) - \widehat{G}(p_n, u)\\ &\quad \geq -\epsilon_n \|v-u\|_V, \quad \forall v \in V.\end{aligned} \tag{2.21}$$

Since $p_n \in B(p^*, \epsilon_n)$ and $\epsilon_n \to 0$, it is clear that $p_n \to p^*$. By letting $n \to +\infty$ in the above inequality, we get from the continuity of $\widehat{A}$, $\widehat{J}_2^\circ(p, \cdot)$, and $\widehat{G}$ in assumptions that

$$\begin{aligned}&\langle A(u) - f, v-u\rangle + J^\circ(u, v-u) + G(v) - G(u)\\ &\quad = \left\langle \widehat{A}(p^*, u) - f, v-u \right\rangle + \widehat{J}_2^\circ(p^*, \cdot)(u, v-u) + \widehat{G}(p^*, v) - \widehat{G}(p^*, u)\\ &\quad \geq \lim\{-\epsilon_n \|v-u\|_V\} = 0.\end{aligned} \tag{2.22}$$

Thus, $u \in S$ and so $\cap_{\epsilon>0}\Omega(\epsilon) \subset S$.

Now, we suppose that

$$\Omega(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \qquad \mu(\Omega(\epsilon)) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{2.23}$$

From the definition of approximating solution set $\Omega(\epsilon)$, $\Omega(\epsilon)$ is increasing with respect to ϵ. Then by applying the Kuratowski theorem on page 318 in [35], we have from (2.20) that S is nonempty compact and

$$q(\epsilon, S) = e(\Omega(\epsilon), S) = \mathscr{H}(\Omega(\epsilon), S) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{2.24}$$

Therefore, by Theorem 2.7, VHVI(A, f, J, G) is strongly well-posed by perturbations in the generalized sense. □

Example 2.9. Let L be a finite-dimensional space with norm $\|\cdot\|_L$, let $P \subset L$ be a closed ball in L, and let p^* be a given point in P. We supposed that the perturbed mappings $\widehat{A} : P \times V \to V^*$, $\widehat{G}$, $\widehat{J} : P \times V \to \mathbb{R}$ of the mapping A, G, J are, respectively, specified as follows:

$$\begin{gathered}\widehat{A}(p, v) = \exp^{\alpha\|p-p^*\|_L} A(v), \qquad \widehat{G}(p, v) = G(v) + \beta\left\|p - p^*\right\|_L,\\ \widehat{J}(p, v) = J(v) + \gamma\left\|p - p^*\right\|_L,\end{gathered} \tag{2.25}$$

where α, β, γ are three positive numbers. It is obvious that $\widehat{A}$ is continuous with respect to (p, v) due to the continuity of the mapping $A : V \to V^*$, and $\widehat{G}$ is lower semicontinuous with respect to (p, v) and continuous with respect to p for any given $v \in V$ because the functional $G : V \to \mathbb{R}$ is proper convex and lower semicontinuous. Also, the perturbed functional $\widehat{J}$ is locally Lipschitz with respect to v since $J : V \to \mathbb{R}$ is locally Lipschitz. Furthermore, it is easy to check that the perturbed Clarke's generalized directional derivative corresponding to the perturbed function $\widehat{J}$ can be specified as

$$\widehat{J}_2^{\circ}(p,\cdot)(u,v) = J^{\circ}(u,v) = \limsup_{w\to u\ \lambda\downarrow 0} \frac{J(w+\lambda v) - J(w)}{\lambda}, \tag{2.26}$$

which implies that $\widehat{J}_2^{\circ}(p,\cdot)$ is continuous with respect to p. Thus, the assumptions in Theorem 2.8 are satisfied, and so the VHVI(A, f, J, G) is strongly well-posed by perturbations in the generalized sense if and only if (2.16) holds.

3. Links with Well-Posedness by Perturbations for Corresponding Inclusion Problem

In this section, we recall some concepts of well-posedness by perturbations for inclusion problems, which are introduced by Lemaire et al. [4], and investigate the relations between the well-posedness by perturbations for VHVI(A, f, J, G) and the well-posedness by perturbations for the corresponding inclusion problem.

In what follows, we always let F be a set-valued mapping from real reflexive Banach space V to its dual space V^*. The inclusion problem associated with mapping F is defined by

$$\text{IP}(F): \text{ find } x \in V \text{ such that } 0 \in F(x), \tag{3.1}$$

whose corresponding perturbed problem is specified as

$$\text{IP}_p(F): \text{ find } x \in V \text{ such that } 0 \in \widehat{F}(p,x), \tag{3.2}$$

where $\widehat{F} : P \times V \to 2^{V^*}$ is the perturbed set-valued mapping such that $\widehat{F}(p^*,\cdot) = F$.

Definition 3.1 (see [4]). Let $\{p_n\} \subset P$ be a sequence in P with $p_n \to p^*$. A sequence $\{u_n\} \subset V$ is said to be an approximating sequence corresponding to $\{p_n\}$ for inclusion problem IP(F) if $u_n \in \operatorname{dom} \widehat{F}(p_n,\cdot)$ for all $n \in \mathbb{N}$ and $d(0, \widehat{F}(p_n, u_n)) \to 0$, or equivalently, there exists a sequence $w_n \in \widehat{F}(p_n, u_n)$ such that $\|w_n\|_{V^*} \to 0$ as $n \to \infty$.

Definition 3.2 (see [4]). One says that inclusion problem IP(F) is strongly (resp., weakly) well-posed by perturbations if it has a unique solution, and for any $\{p_n\} \subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution of IP(F).

Definition 3.3 (see [4]). One says that inclusion problem IP(F) is strongly (resp., weakly) well-posed by perturbations in the generalized sense if the solution set S of IP(F) is nonempty, and for any $\{p_n\} \subset P$ with $p_n \to p^*$, every approximating sequence corresponding to $\{p_n\}$ has a subsequence converging strongly (resp., weakly) to some point of solution set S for IP(F).

In order to obtain the relations between the strong (resp., weak) well-posedness by perturbations for variational-hemivariational inequality VHVI(A, f, J, G) and the strong (resp., weak) well-posedness by perturbations for the corresponding inclusion problem, we first give the following important lemma which establishes the equivalence between the variational-hemivariational inequality VHVI(A, f, J, G) and the corresponding inclusion problem. Although the lemma is a corollary of Lemma 4.1 in [32] with $T = 0$, we also give proof here for its importance and the completeness of our paper.

Lemma 3.4. *Let A be a mapping from Banach space V to its dual V^*, let $J : V \to \mathbb{R}$ be a locally Lipschitz functional, let $G : V \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex, and lower semicontinuous functional, and let f be a given element in dual space V^*. Then $u \in \operatorname{dom} G$ is a solution of VHVI(A, f, J, G) if and only if u is a solution of the following inclusion problem:*

$$IP\Big(A - f + \overline{\partial} J + \partial G\Big) : \text{ find } u \in \operatorname{dom} G \text{ such that } Au - f + \overline{\partial} J(u) + \partial G(u) \ni 0. \tag{3.3}$$

Proof. “Sufficiency”: assume that $u \in \operatorname{dom} G$ is a solution of inclusion problem IP$(A - f + \overline{\partial} J + \partial G)$. Then there exist $\omega_1 \in \overline{\partial} J(u)$ and $\omega_2 \in \partial G(u)$ such that

$$Au - f + \omega_1 + \omega_2 = 0. \tag{3.4}$$

By multiplying $v - u$ at both sides of above equation (3.4), we obtain from the definitions of the Clarke’s generalized gradient for locally Lipschitz functional and the subgradient for convex functional that

$$\begin{aligned} 0 = \langle Au - f + \omega_1 + \omega_2, v - u\rangle \le \langle Au - f, v - u\rangle + J^\circ(u, v - u) \\ + G(v) - G(u), \quad \forall v \in V, \end{aligned} \tag{3.5}$$

which implies that $u \in \operatorname{dom} G$ is a solution of VHVI(A, f, J, G).

“Necessity”: conversely, suppose that $u \in \operatorname{dom} G$ is a solution of VHVI(A, f, J, G). Then,

$$\langle Au - f, v - u\rangle + J^\circ(u, v - u) + G(v) - G(u) \ge 0, \quad \forall v \in V. \tag{3.6}$$

From the fact that

$$J^\circ(u, v - u) = \max\Big\{\langle \omega, v - u\rangle : \omega \in \overline{\partial} J(u)\Big\}, \tag{3.7}$$

we get that there exists a $\omega(u, v) \in \overline{\partial} J(u)$ such that

$$\langle Au - f, v - u\rangle + \langle \omega(u, v), v - u\rangle + G(v) - G(u) \ge 0, \quad \forall v \in V. \tag{3.8}$$

By virtue of Proposition 1.3, $\overline{\partial} J(u)$ is a nonempty, convex, and bounded subset in V^* which implies that $\{Au - f + \omega : \omega \in \overline{\partial} J(u)\}$ is nonempty, convex, and bounded in V^*. Since $G : V \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous functional, it follows from

(3.8) and Theorem 1.6 with $\varphi(u) = G(u)$ that there exists $\omega(u) \in \overline{\partial} J(u)$, which is independent on v, such that

$$\langle Au - f, v - u\rangle + \langle \omega(u), v - u\rangle + G(v) - G(u) \geq 0, \quad \forall v \in V. \tag{3.9}$$

For the sake of simplicity in writing, we denote $\omega = \omega(u)$. Then by (3.9), we have

$$G(v) - G(u) \geq \langle -Au + f - \omega, v - u\rangle, \quad \forall v \in V, \tag{3.10}$$

that is, $-Au + f - \omega \in \partial G(u)$. Thus, it follows from $\omega \in \overline{\partial} J(u)$ that

$$Au - f + \overline{\partial} J(u) + \partial G(u) \ni 0, \tag{3.11}$$

which implies that $u \in \operatorname{dom} G$ is a solution of the inclusion problem $\mathrm{IP}(A - f + \overline{\partial} J + \partial G)$. This completes the proof of Lemma 3.4. □

Remark 3.5. The corresponding perturbed problem of inclusion problem $\mathrm{IP}(A - f + \overline{\partial} J + \partial G)$ is specified as

$$\begin{aligned} &\mathrm{IP}_p\Big(Au - f + \overline{\partial} J(u) + \partial G(u)\Big): \\ &\text{find } u \in \operatorname{dom} \widehat{G}(p,u) \text{ such that } \widehat{A}(p,u) - f + \overline{\partial}_2 \widehat{J}(p,u) + \partial_2 \widehat{G}(p,u) \ni 0. \end{aligned} \tag{3.12}$$

Now we prove the following two theorems which establish the relations between the strong (resp., weak) well-posedness by perturbations for variational-hemivariational inequality $\mathrm{VHVI}(A, f, J, G)$ and the strong (resp., weak) well-posedness by perturbations for the corresponding inclusion problem $\mathrm{IP}(A - f + \overline{\partial} J + \partial G)$.

Theorem 3.6. *Let A be a mapping from Banach space V to its dual V^*, let $J : V \to \mathbb{R}$ be a locally Lipschitz functional, let $G : V \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional, and let f be a given element in dual space V^*. The variational-hemivariational inequality $VHVI(A, f, J, G)$ is strongly (resp., weakly) well-posed by perturbations if and only if the corresponding inclusion problem $IP(A - f + \overline{\partial} J + \partial G)$ is strongly (resp., weakly) well-posed by perturbations.*

Proof. "Necessity": assume that $\mathrm{VHVI}(A, f, J, G)$ is strongly (resp., weakly) well-posed by perturbations, which implies that there is a unique solution u^* of $\mathrm{VHVI}(A, f, J, G)$. Clearly, the existence and uniqueness of solution for inclusion problem $\mathrm{IP}(A - f + \overline{\partial} J + \partial G)$ is obtained easily by Lemma 3.4. Let $\{p_n\} \subset P$ be a sequence with $p_n \to p^*$ and $\{u_n\}$ an approximating sequence corresponding to $\{p_n\}$ for $\mathrm{IP}(A - f + \overline{\partial} J + \partial G)$. Then there exists a sequence $\omega_n \in \widehat{A}(p_n, u_n) - f + \overline{\partial}_2 \widehat{J}(p_n, u_n) + \partial_2 \widehat{G}(p_n, u_n)$ such that $\|\omega_n\|_{V^*} \to 0$ as $n \to \infty$. And so, there exist $\xi_n \in \overline{\partial}_2 \widehat{J}(p_n, u_n)$ and $\eta_n \in \partial_2 \widehat{G}(p_n, u_n)$ such that

$$\omega_n = \widehat{A}(p_n, u_n) - f + \xi_n + \eta_n. \tag{3.13}$$

From the definition of the perturbed Clarke's generalized gradient $\overline{\partial}_2\widehat{J}(p,\cdot)$ corresponding to the perturbed locally Lipschitz functional $\widehat{J}$ and the definition of the perturbed subgradient $\partial_2\widehat{G}(p,\cdot)$ corresponding to the perturbed convex functional $\widehat{G}$, we obtain by multiplying $v-u_n$ at both sides of above equation (3.13) that

$$\begin{aligned}
&\left\langle \widehat{A}(p_n,u_n)-f,v-u_n\right\rangle+\widehat{J}_2^{\circ}(p_n,\cdot)(u_n,v-u_n)+\widehat{G}(p_n,v)-\widehat{G}(p_n,u_n)\\
&\quad\geq\left\langle \widehat{A}(p_n,u_n)-f,v-u_n\right\rangle+\langle\xi_n,v-u_n\rangle+\langle\eta_n,v-u_n\rangle\\
&\quad=\langle\omega_n,v-u_n\rangle\\
&\quad\geq-\|\omega_n\|_{V^*}\|v-u_n\|_V,\quad\forall v\in V.
\end{aligned}\tag{3.14}$$

Letting $\epsilon_n=\|\omega_n\|_{V^*}$, we obtain from (3.14) and the fact $\|\omega_n\|_{V^*}\to 0$ as $n\to\infty$ that $\{u_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for VHVI(A,f,J,G). Therefore, it follows from the strong (resp., weak) well-posedness by perturbations for VHVI(A,f,J,G) that u_n converges strongly (resp., weakly) to the unique solution u^*. Thus, the inclusion problem IP$(A-f+\overline{\partial}J+\partial G)$ is strongly (resp., weakly) well-posed.

"Sufficiency": conversely, suppose that inclusion problem IP$(A-f+\overline{\partial}J+\partial G)$ is strongly (resp., weakly) well-posed by perturbations. Then IP$(A-f+\overline{\partial}J+\partial G)$ has a unique solution u^*, which implies that u^* is the unique solution of VHVI(A,f,J,G) by Lemma 3.4. Let $\{p_n\}\subset P$ be a sequence with $p_n\to p^*$ and $\{u_n\}$ an approximating sequence corresponding to $\{p_n\}$ for VHVI(A,f,J,G). Then there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n\to 0$ as $n\to 0$ such that

$$\begin{aligned}
&\left\langle \widehat{A}(p_n,u_n)-f,v-u_n\right\rangle+\widehat{J}_2^{\circ}(p_n,\cdot)(u_n,v-u_n)+\widehat{G}(p_n,v)-\widehat{G}(p_n,u_n)\\
&\quad\geq-\epsilon_n\|v-u_n\|_V,\quad\forall v\in V.
\end{aligned}\tag{3.15}$$

By the same arguments in proof of Lemma 3.4, there exists a $\omega(p_n,u_n,v)\in\overline{\partial}_2\widehat{J}(p_n,u_n)$ such that

$$\begin{aligned}
&\left\langle \widehat{A}(p_n,u_n)-f,v-u_n\right\rangle+\langle\omega(p_n,u_n,v),v-u_n\rangle+\widehat{G}(p_n,v)-\widehat{G}(p_n,u)\\
&\quad\geq-\epsilon_n\|v-u_n\|_V,\quad\forall v\in V,
\end{aligned}\tag{3.16}$$

and the set $\{\widehat{A}(p_n,u_n)-f+\omega:\omega\in\overline{\partial}_2\widehat{J}(p_n,u_n)\}$ is nonempty, convex, and bounded in V^*. Then, it follows from (3.16) and Theorem 1.6 with $\varphi(u)=\widehat{G}(p_n,u)+\epsilon_n\|u-u_n\|$, which is proper convex and lower semicontinuous, that there exists $\omega(p_n,u_n)\in\overline{\partial}_2\widehat{J}(p_n,u_n)$ such that

$$\begin{aligned}
&\left\langle \widehat{A}(p_n,u_n)-f,v-u_n\right\rangle+\langle\omega(p_n,u_n),v-u_n\rangle+\widehat{G}(p_n,v)-\widehat{G}(p_n,u_n)\\
&\quad\geq-\epsilon_n\|v-u_n\|_V,\quad\forall v\in V.
\end{aligned}\tag{3.17}$$

For the sake of simplicity in writing, we denote $\omega_n = \omega(p_n, u_n)$. Then it follows from (3.17) that

$$\widehat{G}(p_n, u_n) \le \widehat{G}(p_n, v) + \left\langle \widehat{A}(p_n, u_n) - f + \omega_n, v - u_n \right\rangle + \epsilon_n \|v - u_n\|_V, \quad \forall v \in V. \tag{3.18}$$

Define functional $T_n : V \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$T_n(v) = \widehat{G}(p_n, v) + R_n(v) + \epsilon_n Q_n(v), \tag{3.19}$$

where $R_n(v)$, $Q_n(v)$ are two functional on V defined by

$$R_n(v) = \left\langle \widehat{A}(p_n, u_n) - f + \omega_n, v - u_n \right\rangle, \qquad Q_n(v) = \|v - u_n\|_V. \tag{3.20}$$

Clearly, the functionals R_n and Q_n are convex and continuous on V, and so T_n is proper, convex, and lower semicontinuous because $\widehat{G}(p_n, v)$ is proper, convex, and lower semicontinuous with respect to v. Furthermore, it follows from (3.18) that u_n is a global minimizer of T_n on V. Thus, the zero element in V^*, we also denote to be 0, belongs to the subgradient $\partial T_n(u_n)$ which is specified as follows due to Proposition 1.2:

$$\partial T_n(v) = \partial_2 \widehat{G}(p_n, v) + \widehat{A}(p_n, u_n) - f + \omega_n + \epsilon_n \partial Q_n(v). \tag{3.21}$$

It is easy to calculate that

$$\partial Q_n(v) = \{v^* \in V^* : \|v^*\|_{V^*} = 1, \langle v^*, v - u_n \rangle = \|v - u_n\|_V\}, \tag{3.22}$$

and so there exists a $\xi_n \in \partial Q_n(u_n)$ with $\|\xi_n\|_{V*} = 1$ such that

$$0 \in \partial_2 \widehat{G}(p_n, v) + \widehat{A}(p_n, u_n) - f + \omega_n + \epsilon_n \xi_n. \tag{3.23}$$

Let $u_n^* = -\epsilon_n \xi_n$, then $\|u_n^*\|_{V^*} \to 0$ due to $\epsilon_n \to 0$ as $n \to \infty$. This together with (3.23) and $\omega_n \in \overline{\partial}_2 \widehat{J}(p_n, u_n)$ implies that

$$u_n^* \in \widehat{A}(p_n, u_n) - f + \overline{\partial}_2 \widehat{J}(p_n, u_n) + \partial_2 \widehat{G}(p_n, v). \tag{3.24}$$

Therefore, $\{u_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for IP$(A - f + \overline{\partial} J + \partial G)$. Since inclusion problem IP$(A - f + \overline{\partial} J + \partial G)$ is strongly (resp., weakly) well-posed by perturbations, u_n converges strongly (resp., weakly) to the unique solution u^*. Therefore, variational-hemivariational inequality VHVI(A, f, J, G) is strongly (resp., weakly) well-posed. This completes the proof of Theorem 3.6. □

Theorem 3.7. *Let A be a mapping from Banach space V to its dual V^*, let $J : V \to \mathbb{R}$ be a locally Lipschitz functional, let $G : V \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional, and let f be a given element in dual space V^*. The variational-hemivariational inequality*

$VHVI(A, f, J, G)$ is strongly (resp., weakly) well-posed by perturbations in the generalized sense if and only if the corresponding inclusion problem $IP(A - f + \overline{\partial} J + \partial G)$ is strongly (resp., weakly) well-posed by perturbations in the generalized sense.

Proof. The proof of Theorem 3.7 is similar to Theorem 3.6, and so we omit it here. □

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