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Research Article

Global Existence of Strong Solutions to a Class of Fully Nonlinear Wave Equations with Strongly Damped Terms

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We consider the global existence of strong solution u, corresponding to a class of fully nonlinear wave equations with strongly damped terms $u_{tt} - k\Delta u_t = f(x, \Delta u) + g(x, u, Du, D^2 u)$ in a bounded and smooth domain Ω in R^n, where $f(x, \Delta u)$ is a given monotone in Δu nonlinearity satisfying some dissipativity and growth restrictions and $g(x, u, Du, D^2 u)$ is in a sense subordinated to $f(x, \Delta u)$. By using spatial sequence techniques, the Galerkin approximation method, and some monotonicity arguments, we obtained the global existence of a solution $u \in L^\infty_{\text{loc}}((0,\infty), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$.

1. Introduction

We are concerned with the following mixed problem for a class of fully nonlinear wave equations with strongly damped terms in a bounded and C^∞ domain $\Omega \subset R^n$:

$$\begin{gathered}
u_{tt} - k\Delta u_t = f(x, \Delta u) + g\left(x, u, Du, D^2 u\right), \quad \text{in } [0,\infty) \times \Omega, \\
u(0, x) = \varphi, \quad u_t(0, x) = \psi, \quad \text{in } \Omega, \\
u(t, x) = 0, \quad \text{on } [0,\infty) \times \partial\Omega,
\end{gathered} \tag{1.1}$$

where

$$u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \quad D^2 = \frac{\partial^2}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$
$$\alpha_1 + \cdots + \alpha_n = 2, \quad x = (x_1, \dots, x_n), \quad k > 0. \tag{1.2}$$

Equations of type (1.1) are a class essential nonlinear wave equations describing the speed of strain waves in a viscoelastic configuration (e.g., a bar if the space dimension $N = 1$ and a plate if $N = 2$) made up of the material of the rate type [1, 2]. They can also be seen as field equations governing the longitudinal motion of a viscoelastic bar obeying the nonlinear Voigt model [3]. Concerning damped cases, there is much to the global existence of solutions for the problem:

$$u_{tt} + u_t - u_{xx} = f(u), \quad \text{in } [0, \infty) \times \Omega,$$
$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \text{in } \Omega; \tag{1.3}$$

they discussed the global existence of weak solutions and regularity in R^1 and R^n [4–8]. On the other hand, Ikehata and Inoue [9] considered the global existence of weak solutions for two-dimensional problem in an exterior domain $\Omega \subset R^2$ with a compact smooth boundary $\partial\Omega$ for a semilinear strongly damped wave equation with a power-type nonlinearity $|u|^q$ and $q > 6$:

$$u_{tt}(t, x) - \Delta u(t, x) - \Delta u_t(t, x) = |u(t, x)|^q \quad \text{in } [0, \infty) \times \Omega,$$
$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \quad \text{in } \Omega, \tag{1.4}$$
$$u(t, x) = 0, \quad \text{on } [0, \infty) \times \partial\Omega.$$

Cholewa and Dlotko [10] discussed the global solvability and asymptotic behavior of solutions to semilinear Cauchy problem for strongly damped wave equation in the whole of R^n. They assume the nonlinear term f grows like $|u|^q$ and $q < (n+2)/(n-2)$ if $n \geq 3$. Similar problems attracted attention of the researchers for many years [11–13]. Especially, Yang [14] studied the global existence of weak solutions to the more general equation including (1.4), but he did not discuss the regularity of weak solution for the quasilinear wave equation. We are interested in discussing the global existence and regularity of weak solutions for strongly damped wave equation with the dissipative terms g containing Du and the nonlinear terms f containing Δu. Here $f(x, \Delta u)$ is a given monotone in Δu nonlinearity satisfying some dissipativity and growth restrictions and $g(x, u, Du, D^2 u)$ is in a sense subordinated to $f(x, \Delta u)$.

In [15], we have investigated the existence of global solutions to a class of nonlinear damped wave operator equations. In this paper, our first aim is to study the global existence of strong solutions to the more general equation including (1.4), which is the motivation that we establish our abstract strongly damped wave equation model with. The second aim is to deal with the global existence of strong solutions to a class of fully nonlinear wave equations with strongly damped terms under some weakly growing conditions.

This paper is organized as follows:

(i) in Section 2, we recall some preliminary tools and definitions;

(ii) in Section 3, we put forward our abstract strongly damped wave equation model and proof the global existence of strong solution of it;

(iii) in Section 4, we provide the proof of the main results about the mixed problem (1.1).

2. Preliminaries

We introduce two spatial sequences:

$$\begin{gathered} X \subset H_3 \subset X_2 \subset X_1 \subset H, \\ X_2 \subset H_2 \subset H_1 \subset H, \end{gathered} \tag{2.1}$$

where H, H_1, H_2, and H_3 are Hilbert spaces, X is a linear space, and X_1, X_2 are Banach spaces.

All embeddings of (2.1) are dense. Let

$$\begin{gathered} L : X \longrightarrow X_1 \quad \text{be one-for-one dense linear operator}, \\ \langle Lu, v \rangle_H = \langle u, v \rangle_{H_1}, \quad \forall u, v \in X. \end{gathered} \tag{2.2}$$

Furthermore, L has eigenvectors $\{e_k\}$ satisfying

$$Le_k = \lambda_k e_k, \quad (k = 1, 2, \ldots), \tag{2.3}$$

and $\{e_k\}$ constitutes common orthogonal basis of H and H_3.

We consider the following abstract wave equation model:

$$\begin{gathered} \frac{d^2 u}{dt^2} + k \frac{d}{dt} \mathcal{L} u = G(u), \quad k > 0, \\ u(0) = \varphi, \qquad u_t(0) = \psi, \end{gathered} \tag{2.4}$$

where $G : X_2 \times R^+ \longrightarrow X_1^*$ is a map, $R^+ = [0, \infty)$, and $\mathcal{L} : X_2 \longrightarrow X_1$ is a bounded linear operator, satisfying

$$\langle \mathcal{L} u, Lv \rangle_H = \langle u, v \rangle_{H_2}, \quad \forall u, v \in X_2. \tag{2.5}$$

Definition 2.1. We say that $u \in W^{1,\infty}((0,T), H_1) \cap L^\infty((0,T), X_2)$ is a global weak solution of the (2.4) provided for $(\varphi, \psi) \in X_2 \times H_1$

$$\langle u_t, v \rangle_H + k \langle \mathcal{L} u, v \rangle_H = \int_0^t \langle G(u), v \rangle d\tau + \langle \psi, v \rangle_H + k \langle \mathcal{L} \varphi, v \rangle_H, \tag{2.6}$$

for each $v \in X_1$ and $0 \le t \le T < \infty$.

Definition 2.2. Let $u_n, u_0 \in L^p((0,T),X_2)$. We say that $u_n \rightharpoonup u_0$ in $L^p((0,T),X_2)$ is uniformly weakly convergent if $\{u_n\} \subset L^\infty((0,T),H)$ is bounded, and

$$\begin{gathered} u_n \rightharpoonup u_0, \quad \text{in } L^p((0,T),X_2), \\ \lim_{n\to\infty}\int_0^T |\langle u_n - u_0, v\rangle_H|^2 dt = 0, \quad \forall v \in H. \end{gathered} \tag{2.7}$$

Lemma 2.3 (see [16]). *Let $\{u_n\} \in L^p((0,T),W^{m,p}(\Omega))$ $(m \geq 1)$ be bounded sequences and $\{u_n\}$ uniformly weakly convergent to $u_0 \in L^p((0,T),W^{m,p}(\Omega))$. Then, for each $|\alpha| \leq m-1$, it follows that*

$$D^\alpha u_n \longrightarrow D^\alpha u_0, \quad \text{in } L^2((0,T)\times\Omega). \tag{2.8}$$

Lemma 2.4 (see [17]). *Let $\Omega \subset R^n$ be an open set and $f : \Omega \times R^N \longrightarrow R^1$ satisfy Caratheodory condition and*

$$\left|f(x,\xi)\right| \leq C\sum_{i=1}^N |\xi_i|^{p_i/p} + b(x). \tag{2.9}$$

If $\{u_{i_k}\} \subset L^{p_i}(\Omega)$ $(1 \leq i \leq N)$ is bounded and u_{i_k} convergent to u_i in Ω_0 for all bounded $\Omega_0 \subset \Omega$, then for each $v \in L^{p'}(\Omega)$, the following equality holds

$$\lim_{k\to\infty}\int_\Omega f(x,u_{1_k},\ldots,u_{N_k})v\,dx = \int_\Omega f(x,u_1,\ldots,u_N)v\,dx. \tag{2.10}$$

3. Model Results

Let $G = A + B : X_2 \times R^+ \longrightarrow X_1^*$. Assume

(A1) there is a C^1 functional $F : X_2 \longrightarrow R^1$ such that

$$\langle Au, Lv\rangle = \langle -DF(u), v\rangle, \quad \forall u,v \in X; \tag{3.1}$$

(A2) functional $F : X_2 \longrightarrow R^1$ is coercive, that is,

$$F(u) \longrightarrow \infty, \Longleftrightarrow \|u\|_{X_2} \longrightarrow \infty; \tag{3.2}$$

(A3) B satisfies

$$|\langle Bu, Lv\rangle| \leq C_1F(u) + \frac{k}{2}\|v\|_{H_1}^2 + C_2, \quad \forall u,v \in X, \tag{3.3}$$

for $g \in L^1_{\text{loc}}(0,\infty)$.

Theorem 3.1. *Set $G : X_2 \times R^+ \longrightarrow X_1^*$, for each $(\varphi, \psi) \in X_2 \times H_1$, then the following assertions hold.*

(1) *If $G = A$ satisfies (A1) and (A2), then* (2.4) *has a globally weak solution*

$$u \in W^{1,\infty}((0,\infty), H_1) \cap W^{1,2}((0,\infty), H_2) \cap L^{\infty}((0,\infty), X_2). \tag{3.4}$$

(2) *If $G = A + B$ satisfies (A1)–(A3), then* (2.4) *has a global weak solution*

$$u \in W^{1,\infty}_{\text{loc}}((0,\infty), H_1) \cap W^{1,2}_{\text{loc}}((0,\infty), H_2) \cap L^{\infty}_{\text{loc}}((0,\infty), X_2). \tag{3.5}$$

(3) *Furthermore, if $\mathcal{L} : X_2 \longrightarrow X_1$ is symmetric sectorial operator, that is, $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}v\rangle$, and $G = A + B$ satisfies*

$$|\langle Gu, v\rangle| \le C_1 F(u) + \frac{1}{2}\|v\|_H^2 + C_2, \tag{3.6}$$

then $u \in W^{2,2}_{\text{loc}}((0,\infty), H)$.

Proof. Let $\{e_k\} \subset X$ be a common orthogonal basis of H and H_3, satisfying (2.3). Set

$$X_n = \left\{ \sum_{i=1}^{n} \alpha_i e_i \mid \alpha_i \in R^1 \right\},$$
$$\widetilde{X}_n = \left\{ \sum_{j=1}^{n} \beta_j(t) e_j \mid \beta_j \in C^2[0,\infty) \right\}. \tag{3.7}$$

Clearly, $LX_n = X_n$, $L\widetilde{X}_n = \widetilde{X}_n$.

By using Galerkin method, there exits $u_n \in C^2([0,\infty), X_n)$ satisfying

$$\left\langle \frac{du_n}{dt}, v \right\rangle_H + k\langle \mathcal{L}u_n, v\rangle_H = \int_0^t \langle G(u_n), v\rangle d\tau + \langle \psi_n, v\rangle_H + k\langle \mathcal{L}\varphi_n, v\rangle_H,$$
$$u_n(0) = \varphi_n, \qquad u_n'(0) = \psi_n, \tag{3.8}$$

for $\forall v \in X_n$, and

$$\int_0^t \left[\left\langle \frac{d^2 u_n}{dt^2}, v \right\rangle_H + k\left\langle \mathcal{L}\frac{du_n}{dt}, v \right\rangle_H \right] d\tau = \int_0^t \langle Gu_n, v\rangle d\tau \tag{3.9}$$

for $\forall v \in \widetilde{X}_n$.

Firstly, we consider $G = A$. Let $v = (d/dt)Lu_n$ in (3.9). Taking into account (2.2) and (3.1), it follows that

$$0 = \int_0^t \left[\left\langle \frac{d^2 u_n}{dt^2}, \frac{d}{dt} Lu_n \right\rangle \right] + k \left\langle \mathcal{L} \frac{du_n}{dt}, \frac{d}{dt} Lu_n \right\rangle_H d\tau - \int_0^t \left\langle Au_n, \frac{d}{dt} Lu_n \right\rangle d\tau,$$
$$0 = \int_0^t \left[\frac{1}{2} \frac{d}{dt} \left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_1} + k \left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_2} + \left\langle DF(u_n), \frac{du_n}{dt} \right\rangle \right] d\tau$$
$$= \frac{1}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 - \frac{1}{2} \|\psi_n\|_{H_1}^2 + k \int_0^t \left\| \frac{du_n}{dt} \right\|_{H_2}^2 d\tau + F(u_n) - F(\varphi_n). \tag{3.10}$$

We get

$$\frac{1}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 + k \int_0^t \left\| \frac{du_n}{dt} \right\|_{H_2}^2 d\tau + F(u_n) = F(\varphi_n) + \frac{1}{2} \|\psi_n\|_{H_1}^2. \tag{3.11}$$

Let $\varphi \in H_3$. From (2.1) and (2.2), it is known that $\{e_n\}$ are orthogonal basis of H_1. We find that $\varphi_n \longrightarrow \varphi$ in H_3, and $\psi_n \longrightarrow \psi$ in H_1. Since $H_3 \subset X_2$ is imbedding, it follows that

$$\varphi_n \longrightarrow \varphi, \quad \text{in } X_2,$$
$$\psi_n \longrightarrow \psi, \quad \text{in } H_1. \tag{3.12}$$

From (3.2), (3.11), and (3.12), we obtain that,

$$\{u_n\} \subset W^{1,\infty}_{\text{loc}}((0,\infty), H_1) \cap W^{1,2}_{\text{loc}}((0,\infty), H_2) \cap L^{\infty}_{\text{loc}}((0,\infty), X_2) \text{ is bounded.} \tag{3.13}$$

Let

$$u_n * \rightharpoonup u_0, \quad \text{in } W^{1,\infty}_{\text{loc}}((0,\infty), H_1) \cap L^{\infty}_{\text{loc}}((0,\infty), X_2),$$
$$u_n \rightharpoonup u_0, \quad \text{in } W^{1,2}_{\text{loc}}((0,\infty), H_2), \tag{3.14}$$

which implies that $u_n \to u_0$ in $W^{1,2}_{\text{loc}}((0,\infty), H)$ is uniformly weakly convergent from that $H_2 \subset H$ is compact imbedding.

If we have the following equality:

$$\lim_{n\to\infty} \left[-\int_0^t |\langle Gu_n - Gu_0, Lu_n - Lu_0 \rangle| d\tau + \frac{k}{2} \|u_n - u_0\|_{H_2}^2 \right] = 0, \tag{3.15}$$

then u_0 is a weak solution of (2.4) in view of (3.8), (3.14).

We will show (3.15) as follows. It follows that from (2.5),

$$\int_0^t \left\langle \frac{d}{dt}\mathcal{L}u_n - \frac{d}{dt}\mathcal{L}u_0, Lu_n - Lu_0 \right\rangle_H d\tau = \frac{1}{2}\int_0^t \frac{d}{dt}\langle u_n - u_0, u_n - u_0\rangle_{H_2} d\tau$$
$$= \frac{1}{2}\|u_n(t) - u_0(t)\|_{H_2}^2 - \frac{1}{2}\|\varphi_n - \varphi\|_{H_2}^2. \tag{3.16}$$

Taking into account (2.2), (2.5), and (3.9), we get that

$$-\int_0^t \langle Gu_n - Gu_0, Lu_n - Lu_0\rangle d\tau + \frac{k}{2}\|u_n - u_0\|_{H_2}^2$$
$$= \int_0^t \left[\langle Gu_0 - Gu_n, Lu_n - Lu_0\rangle + k\left\langle \frac{d}{dt}\mathcal{L}u_n - \frac{d}{dt}\mathcal{L}u_0, Lu_n - Lu_0\right\rangle_H\right] d\tau + \frac{k}{2}\|\varphi_n - \varphi\|_{H_2}^2$$
$$= \int_0^t \left[\langle Gu_0, Lu_n - Lu_0\rangle + \langle Gu_n, Lu_0\rangle - \langle Gu_n, Lu_n\rangle - k\left\langle \frac{du_n}{dt}, u_0\right\rangle_{H_2} \right.$$
$$\left. -k\left\langle \frac{du_0}{dt}, u_n - u_0\right\rangle_{H_2} + k\left\langle \frac{d}{dt}\mathcal{L}u_n, Lu_n\right\rangle_H\right] d\tau + \frac{k}{2}\|\varphi_n - \varphi\|_{H_2}^2$$
$$= \int_0^t \left[\langle Gu_0, Lu_n - Lu_0\rangle + \langle Gu_n, Lu_0\rangle - k\left\langle \frac{du_n}{dt}, u_0\right\rangle_{H_2} - k\left\langle \frac{du_0}{dt}, u_n - u_0\right\rangle_{H_2} \right.$$
$$\left. -\left\langle \frac{d^2u_n}{dt^2} + k\frac{d}{dt}\mathcal{L}u_n, Lu_n\right\rangle_H + k\left\langle \frac{d}{dt}\mathcal{L}u_n, Lu_n\right\rangle_H\right] d\tau + \frac{k}{2}\|\varphi_n - \varphi\|_{H_2}^2$$
$$= \int_0^t \left[\langle Gu_0, Lu_n - Lu_0\rangle + \langle Gu_n, Lu_0\rangle \right.$$
$$\left. -k\left\langle \frac{du_n}{dt}, u_0\right\rangle_{H_2} - k\left\langle \frac{d}{dt}u_0, u_n - u_0\right\rangle_{H_2} + \left\langle \frac{du_n}{dt}, \frac{du_n}{dt}\right\rangle_{H_1}\right] d\tau$$
$$- \left\langle \frac{du_n}{dt}, u_n\right\rangle_{H_1} + \langle \psi_n, \varphi_n\rangle_{H_1} + \frac{k}{2}\|\varphi_n - \varphi\|_{H_2}^2. \tag{3.17}$$

From (2.1) and (3.14), we have

$$\lim_{n\to\infty}\|\varphi_n - \varphi\|_{H_2} = 0,$$
$$\lim_{n\to\infty}\int_0^t \langle Gu_0, Lu_n - Lu_0\rangle d\tau = 0, \tag{3.18}$$
$$\lim_{n\to\infty}\int_0^t \left\langle \frac{d}{dt}u_0, u_n - u_0\right\rangle_{H_2} d\tau = 0.$$

Then, we get

$$\begin{aligned}
&\lim_{n\to\infty} -\int_0^t \langle Gu_n - Gu_0, Lu_n - Lu_0\rangle d\tau + \frac{k}{2}\lim_{n\to\infty}\|u_n - u_0\|_{H_2}^2 \\
&\quad = \lim_{n\to\infty}\int_0^t \left[\langle Gu_n, Lu_0\rangle - k\left\langle \frac{du_n}{dt}, u_0\right\rangle_{H_2} + \left\|\frac{du_n}{dt}\right\|_{H_1}^2\right] d\tau \\
&\qquad - \lim_{n\to\infty}\left\langle \frac{du_n}{dt}, u_n\right\rangle_{H_1} + \langle \psi, \varphi\rangle_{H_1}.
\end{aligned} \tag{3.19}$$

In view of (3.9), (3.14), we obtain for all $v \in \cup_{n=1}^{\infty}\widetilde{X}_n$

$$\begin{aligned}
\lim_{n\to\infty}\int_0^t \langle Gu_n, Lv\rangle d\tau = &\int_0^t \left[k\left\langle \frac{du_0}{dt}, v\right\rangle_{H_2} - \left\langle \frac{du_0}{dt}, \frac{dv}{dt}\right\rangle_{H_1}\right] d\tau \\
&+ \left\langle \frac{du_0}{dt}, v\right\rangle_{H_1} - \langle \psi, v(0)\rangle_{H_1}.
\end{aligned} \tag{3.20}$$

Since $\cup_{n=1}^{\infty}\widetilde{X}_n$ is dense in $W^{1,2}((0,T), H_2) \cap L^p((0,T), X_2)$, for all $p < \infty$, (3.20) holds for all $v \in W^{1,2}((0,T), H_2) \cap L^p((0,T), X_2)$. Thus, we have

$$\begin{aligned}
\lim_{n\to\infty}\int_0^t \langle Gu_n, Lu_0\rangle d\tau = &\int_0^t \left[k\left\langle \frac{du_0}{dt}, u_0\right\rangle_{H_2} - \left\|\frac{du_0}{dt}\right\|_{H_1}^2\right] d\tau \\
&+ \left\langle \frac{du_0}{dt}, u_0\right\rangle_{H_1} - \langle \psi, \varphi\rangle_{H_1}.
\end{aligned} \tag{3.21}$$

From (3.14) and $H_2 \subset H_1$ being compact imbedding, it follows that

$$\begin{aligned}
\lim_{n\to\infty}\int_0^t \left\|\frac{du_n}{dt}\right\|_{H_1}^2 d\tau &= \int_0^t \left\|\frac{du_0}{dt}\right\|_{H_1}^2 d\tau, \\
\lim_{n\to\infty}\left\langle \frac{du_n}{dt}, u_n\right\rangle_{H_1} &= \left\langle \frac{du_0}{dt}, u_0\right\rangle_{H_1}, \quad \text{a.e. } t \geq 0.
\end{aligned} \tag{3.22}$$

Clearly,

$$\lim_{n\to\infty}\int_0^t \left\langle \frac{du_n}{dt}, u_n\right\rangle_{H_1} d\tau = \int_0^t \left\langle \frac{du_0}{dt}, u_0\right\rangle_{H_1} d\tau. \tag{3.23}$$

Then, (3.14) follows from (3.19)–(3.21), which implies assertion (1).

Secondly, we consider $G = A + B$. Let $v = (d/dt)Lu_n$ in (3.9). In view of (2.2) and (2.8), it follows that

$$\int_0^t \left[\left\langle \frac{d^2 u_n}{dt^2}, \frac{d}{dt} L u_n \right\rangle_H + k \left\langle \mathcal{L} \frac{du_n}{dt}, \frac{d}{dt} L u_n \right\rangle_H \right] d\tau = \int_0^t \left\langle (A+B) u_n, \frac{d}{dt} L u_n \right\rangle_H d\tau$$

$$\int_0^t \left[\frac{1}{2} \frac{d}{dt} \left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_1} + k \left\langle \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{H_2} \right] d\tau$$

$$= \int_0^t \left[\left\langle -DF(u_n), \frac{du_n}{dt} \right\rangle + \left\langle B(u_n), \frac{d}{dt} L u_n \right\rangle_H \right] d\tau,$$

$$\frac{1}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 - \frac{1}{2} \|\varphi_n\|_{H_1}^2 + k \int_0^t \left\| \frac{du_n}{dt} \right\|_{H_2}^2 d\tau + F(u_n) - F(\varphi_n) = \int_0^t \left\langle B(u_n), \frac{d}{dt} L u_n \right\rangle_H d\tau,$$

$$\frac{1}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 + k \int_0^t \left\| \frac{du_n}{dt} \right\|_{H_2}^2 dt + F(u_n) = \int_0^t \left\langle B(u_n), \frac{d}{dt} L u_n \right\rangle d\tau + F(\varphi_n) + \frac{1}{2} \|\varphi_n\|_{H_1}^2. \tag{3.24}$$

From (3.3), we have

$$\frac{1}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 + F(u_n) + k \int_0^t \left\| \frac{du_n}{dt} \right\|_{H_2}^2 d\tau \le C \int_0^t \left[C_1 F(u_n) + \frac{k}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 + C_2 \right] d\tau$$

$$+ F(\varphi_n) + \frac{1}{2} \|\varphi_n\|_{H_1}^2 \tag{3.25}$$

$$\le C \int_0^t \left[F(u_n) + \frac{1}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 \right] d\tau + f(t),$$

where $f(t) = (1/2)\|\varphi\|_{H_1}^2 + \sup_n F(\varphi_n)$.

By using Gronwall inequality, it follows that

$$\frac{1}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 + F(u_n) \le f(0) e^{Ct} + \int_0^t f(\tau) e^{C(t-\tau)} d\tau, \tag{3.26}$$

which implies that for all $0 < T < \infty$,

$$\{u_n\} \subset W^{1,\infty}((0,T), H_1) \cap L^\infty((0,T), X_2) \text{ is bounded.} \tag{3.27}$$

From (3.25) and (3.21), it follows that

$$\{u_n\} \subset W^{1,2}((0,T), H_2) \text{ is bounded.} \tag{3.28}$$

Let

$$u_n * \rightharpoonup u_0, \quad \text{in } W^{1,\infty}((0,T),H_1) \cap L^\infty((0,T),X_2),$$
$$u_n \rightharpoonup u_0, \quad \text{in } W^{1,2}((0,T),H_2), \tag{3.29}$$

which implies that $u_n \to u_0$ in $W^{1,2}((0,T),H)$ is uniformly weakly convergent from that $H_2 \subset H$ is compact imbedding.

The left proof is same as assertion (1).

Lastly, assume (3.6) hold. Let $v = d^2u_n/dt^2$ in (3.9). It follows that

$$\int_0^t \left[\left\langle \frac{d^2u_n}{dt^2}, \frac{d^2u_n}{dt^2} \right\rangle_H + k\left\langle \mathcal{L}\frac{du_n}{dt}, \frac{d^2u_n}{dt^2} \right\rangle_H \right] d\tau$$
$$= \int_0^t \left\langle (A+B)(u_n), \frac{d^2u_n}{dt^2} \right\rangle d\tau$$
$$\leq \int_0^t \left[CF(u_n) + \frac{1}{2}\left\| \frac{d^2u_n}{dt^2} \right\|_H^2 + g(t) \right] d\tau,$$
$$\int_0^t \left[\left\langle \frac{d^2u_n}{dt^2}, \frac{d^2u_n}{dt^2} \right\rangle_H + \frac{k}{2}\int_0^t \int_\Omega \frac{d}{dt}\left(u_n'(t)\right)^2 \right] dx d\tau$$
$$\leq \int_0^t \left[CF(u_n) + \frac{1}{2}\left\| \frac{d^2u_n}{dt^2} \right\|_H^2 + g(t) \right] d\tau, \tag{3.30}$$
$$\int_0^t \left\langle \frac{d^2u_n}{dt^2}, \frac{d^2u_n}{dt^2} \right\rangle_H dt + \frac{k}{2}\left\| \frac{du_n}{dt} \right\|_{H_1}^2$$
$$\leq \frac{k}{2}\|\varphi_n\|_H^2 + \int_0^t \left[\frac{1}{2}\left\| \frac{d^2u_n}{dt^2} \right\|_H^2 + CF(u_n) + g(\tau) \right] d\tau.$$

From (3.26), the above inequality implies

$$\int_0^t \left\| \frac{d^2u_n}{dt^2} \right\|_H^2 d\tau \leq C, \quad (C > 0 \text{ is constant}). \tag{3.31}$$

We see that for all $0 < T < \infty$, $\{u_n\} \subset W^{2,2}((0,T),H)$ is bounded. Thus $u \in W^{2,2}((0,T),H)$. □

4. Main Result

Now, we begin to consider the mixed problem (1.1). Set

$$F(x,y)=\int_0^y f(x,z)dz. \tag{4.1}$$

We assume

$$\begin{gathered} F(x,y)\geq C_1|y|^p-C_2, \quad p\geq 2, \\ |f(x,y)|\leq C\left(|y|^{p-1}+1\right), \end{gathered} \tag{4.2}$$

$$(f(x,y_1)-f(x,y_2))(y_1-y_2)\geq \lambda|y_1-y_2|^2, \quad \lambda>0, \tag{4.3}$$

$$|g(x,z,\xi,\eta)|\leq C\left(|z|^{p/2}+|\xi|^{p/2}+|\eta|^{p/2}+1\right), \tag{4.4}$$

$$|g(x,z,\xi,\eta_1)-g(x,z,\xi,\eta_2)|\leq K_1|\eta_1-\eta_2|, \tag{4.5}$$

where C, C_1, C_2 are constant and $K_1<\lambda K$, K is the best constant satisfying

$$K^2\|u\|_{H^2}^2\leq \int_\Omega |\Delta u|^2dx. \tag{4.6}$$

Theorem 4.1. *If the assumptions of* (4.1)–(4.5) *hold, for* $(\varphi,\psi)\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\times H_0^1(\Omega)$, *then* (1.1) *is a strong solution*

$$\begin{gathered} u\in L^\infty_{\text{loc}}\left((0,\infty), W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\right), \\ u_t\in L^\infty_{\text{loc}}\left((0,\infty),H_0^1(\Omega)\right)\cap L^2_{\text{loc}}\left((0,\infty),H^2(\Omega)\right), \\ u_{tt}\in L^{p'}((0,T)\times\Omega), \quad p'=\frac{p}{p-1}, \ \forall 0<T<\infty. \end{gathered} \tag{4.7}$$

Proof. We introduce spatial sequences

$$\begin{aligned} X &= \left\{u\in C^\infty(\Omega) \,\Big|\, \Delta^k u\Big|_{\partial\Omega}=0,\ k=0,1,2,\dots\right\}, \\ X_1 &= L^p(\Omega), \\ X_2 &= W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega), \\ H &= L^2(\Omega), \\ H_1 &= H_0^1(\Omega), \\ H_2 &= H^2(\Omega)\cap H_0^1(\Omega), \\ H_3 &= \left\{u\in H^{2m}(\Omega):\ u|_{\partial\Omega}=\cdots=\Delta^{m-1}u|_{\partial\Omega}=0\right\}, \end{aligned} \tag{4.8}$$

where the inner products of H_2 and H_3 are defined by

$$\langle u, v\rangle_{H_2} = \int_\Omega \Delta u \Delta v \, dx, \quad \langle u, v\rangle_{H_2} = \int_\Omega \Delta^m u \Delta^m v \, dx, \tag{4.9}$$

where $m \geq 1$ such that $H_3 \subset X_2$ is an embedding.

Linear operators $\mathcal{L} : X \longrightarrow X_1$ and $L : X \longrightarrow X_1$ are defined by

$$\mathcal{L}u = Lu = -\Delta u. \tag{4.10}$$

It is known that $\mathcal{L}$ and L satisfy (2.2), (2.3), and (2.5). Define $G = A + B : X_2 \rightarrow X_1^*$ by

$$\langle Au, v\rangle = \int_\Omega f(x, \Delta u) v \, dx, \quad \langle Bu, v\rangle = \int_\Omega g\left(x, u, Du, D^2 u\right) v \, dx, \quad \text{for } v \in X_1. \tag{4.11}$$

We show that $G = A + B : X_2 \longrightarrow X_1^*$ is T-coercively weakly continuous. Let $\{u_n\} \subset L^\infty((0,T), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ satisfying (2.7) and

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T |\langle Gu_n - Gu_0, Lu_n - Lu_0\rangle| dt \\ &\quad = \lim_{n\to\infty} \int_0^T \int_\Omega \left[(f(x, \Delta u_n) - f(x, \Delta u_0))(u_n - u_0) \right. \\ &\qquad \left. + \left(g\left(x, u_n, Du_n, D^2 u_n\right) - g\left(x, u_0, Du_0, D^2 u_0\right)\right)(u_n - u_0) \right] dx \, dt = 0. \end{aligned} \tag{4.12}$$

We need to prove that

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \int_\Omega \left[f(x, \Delta u_n) + g\left(x, u_n, Du_n, D^2 u_n\right)\right] v \, dx \, dt \\ &\quad = \int_0^T \int_\Omega \left[f(x, \Delta u_0) + g\left(x, u_0, Du_0, D^2 u_0\right)\right] v \, dx \, dt. \end{aligned} \tag{4.13}$$

From (2.7) and Lemma 2.3, we obtain

$$u_n \longrightarrow u_0, \quad Du_n \longrightarrow Du_0 \quad \text{in } L^2((0,T) \times \Omega). \tag{4.14}$$

From (4.3), we get

$$\int_0^T \int_\Omega [f(x, \Delta u_n) - f(x, \Delta u_0)](\Delta u_n - \Delta u_0) dx \, dt \geq \lambda \int_0^T \int_\Omega |\Delta u_n - \Delta u_0|^2 dx \, dt. \tag{4.15}$$

We have the deformation

$$\begin{aligned}
&\int_0^T \int_\Omega \left[g\left(x, u_n, Du_n, D^2 u_n\right) - g\left(x, u_0, Du_0, D^2 u_0\right) \right] (\Delta u_n - \Delta u_0) dx\, dt \\
&\quad = \int_0^T \int_\Omega \left[g\left(x, u_n, Du_n, D^2 u_0\right) - g\left(x, u_0, Du_0, D^2 u_0\right) \right] (\Delta u_n - \Delta u_0) dx\, dt \\
&\qquad + \int_0^T \int_\Omega \left[g\left(x, u_n, Du_n, D^2 u_n\right) - g\left(x, u_n, Du_n, D^2 u_0\right) \right] (\Delta u_n - \Delta u_0) dx\, dt.
\end{aligned} \tag{4.16}$$

From (4.14) and Lemma 2.4, we have

$$\lim_{n \to \infty} \int_0^T \int_\Omega \left[g\left(x, u_n, Du_n, D^2 u_0\right) - g\left(x, u_0, Du_0, D^2 u_0\right) \right] (\Delta u_n - \Delta u_0) dx\, dt = 0. \tag{4.17}$$

From (4.12), (4.15)–(4.17), it follows that

$$\begin{aligned}
0 &\geq \lambda \int_0^T \int_\Omega |\Delta u_n - \Delta u_0|^2 dx\, dt + \int_0^T \int_\Omega \left[g\left(x, u_n, Du_n, D^2 u_n\right) - g\left(x, u_n, Du_n, D^2 u_0\right) \right] \\
&\qquad \times (\Delta u_n - \Delta u_0) dx\, dt \\
&\geq \lambda \int_0^T \int_\Omega |\Delta u_n - \Delta u_0|^2 dx\, dt - K_1 \int_0^T \int_\Omega \left| D^2 u_n - D^2 u_0 \right| |\Delta u_n - \Delta u_0| dx\, dt \\
&\geq \frac{\lambda}{2} \int_0^T \int_\Omega |\Delta u_n - \Delta u_0|^2 dx\, dt - \frac{K_1^2}{2\lambda} \int_0^T \int_\Omega \left| D^2 u_n - D^2 u_0 \right|^2 dx\, dt \\
&\geq \frac{\lambda^2 K^2 - K_1^2}{2\lambda} \int_0^T \int_\Omega \left| D^2 u_n - D^2 u_0 \right|^2 dx\, dt.
\end{aligned} \tag{4.18}$$

Since $\lambda K > K_1$, we have

$$\lim_{n \to \infty} \int_0^T \int_\Omega \left| D^2 u_n - D^2 u_0 \right|^2 dx\, dt = 0. \tag{4.19}$$

From (4.14), (4.19), (4.1), (4.4), and Lemma 2.3, we get (4.13).
Let $F_1(u) = \int_\Omega F(x, \Delta u) dx$, where F is same as (4.1). We get

$$\begin{gathered}
\langle Au, Lu \rangle = -\langle DF_1(u), v \rangle, \\
F(u) \longrightarrow \infty \Longleftrightarrow \|u\|_{X_2} \longrightarrow \infty,
\end{gathered} \tag{4.20}$$

which implies Conditions (A1), (A2) of model results in Theorem 3.1.

We will show (3.3) as follows. It follows that

$$\begin{aligned}
|\langle Bu, Lv\rangle| &= \int_\Omega \left| g\left(x,u,Du,D^2u\right)\right| |\Delta v| dx \\
&\le \frac{k}{2}\int_\Omega |\Delta v|^2 dx + \frac{2}{k}\int_\Omega \left| g\left(x,u,Du,D^2u\right)\right|^2 dx \\
&\le \frac{k}{2}\|v\|_{H_2}^2 + C\int_\Omega \left[\left|D^2u\right|^p + |\nabla u|^p + |u|^p + 1\right] dx \\
&\le \frac{k}{2}\|v\|_{H_2}^2 + CF_1(u) + C,
\end{aligned} \tag{4.21}$$

which imply Conditions (A3) of Theorem 3.1. From Theorem 3.1, (1.1) has a solution

$$\begin{gathered}
u \in L^\infty_{\text{loc}}\left((0,\infty), W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\right), \\
u_t \in L^\infty_{\text{loc}}\left((0,\infty), H_0^1(\Omega)\right) \cap L^2_{\text{loc}}\left((0,\infty), H^2(\Omega)\right), \\
u_{tt} \in L^{p'}((0,T)\times\Omega), \quad p' = \frac{p}{p-1}, \ \forall 0 < T < \infty,
\end{gathered} \tag{4.22}$$

satisfying

$$\begin{aligned}
\int_\Omega \frac{\partial u}{\partial t} v\, dx - k\int_\Omega \Delta u v\, dx = \int_0^t\int_\Omega f(x,\Delta u) v\, dx\, d\tau + \int_0^t\int_\Omega g\left(x,u,Du,D^2u\right) v\, dx\, d\tau \\
+ \int_\Omega \varphi v\, dx - k\int_\Omega \Delta\varphi v\, dx.
\end{aligned} \tag{4.23}$$

□

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