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Research Article

The Spectral Method for the Cahn-Hilliard Equation with Concentration-Dependent Mobility

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This paper is concerned with the numerical approximations of the Cahn-Hilliard-type equation with concentration-dependent mobility. Convergence analysis and error estimates are presented for the numerical solutions based on the spectral method for the space and the implicit Euler method for the time. Numerical experiments are carried out to illustrate the theoretical analysis.

1. Introduction

In this paper, we apply the spectral method to approximate the solutions of Cahn-Hilliard equation, which is a typical class of nonlinear fourth-order diffusion equations. Diffusion phenomena is widespread in the nature. Therefore, the study of the diffusion equation caught wide concern. Cahn-Hilliard equation was proposed by Cahn and Hilliard in 1958 as a mathematical model describing the diffusion phenomena of phase transition in thermodynamics. Later, such equations were suggested as mathematical models of physical problems in many fields such as competition and exclusion of biological groups [1], moving process of river basin [2], and diffusion of oil film over a solid surface [3]. Due to the important application in chemistry, material science, and other fields, there were many investigations on the Cahn-Hilliard equations, and abundant results are already brought about.

The systematic study of Cahn-Hilliard equations started from the 1980s. It was Elliott and Zheng [4] who first study the following so-called standard Cahn-Hilliard equation with constant mobility:

$$\frac{\partial u}{\partial t} + \operatorname{div}[k\nabla\Delta u - \nabla A(u)] = 0. \tag{1.1}$$

Basing on global energy estimates, they proved the global existence and uniqueness of classical solution of the initial boundary problem. They also discussed the blow-up property of classical solutions. Since then, there were many remarkable studies on the Cahn-Hilliard equations, for example, the asymptotic behavior of solutions [5–8], perturbation of solutions [9, 10], stability of solutions [11, 12], and the properties of the solutions for the Cahn-Hilliard equations with dynamic boundary conditions [13–16]. In the mean time, a number of the numerical techniques for Cahn-Hilliard equations were produced and developed. These techniques include the finite element method [4, 17–24], the finite difference method [25–29], the spectral method, and the pseudospectral method [30–35]. The finite element method for the Cahn-Hilliard equation is well investigated by many researchers. For example, in [23, 36], (1.1) was discreted by conforming finite element method with an implicit time discretization. In [22], semidiscrete schemes which can define a Lyapunov functional and remain mass constant were used for a mixed formulation of the governing equation. In [37], a mixed finite element formulation with an implicit time discretization was presented for the Cahn-Hilliard equation (1.1) with Dirichlet boundary conditions. The conventional strategy to obtain numerical solutions by the finite differece method is to choose appropriate mesh size based on the linear stability analysis for different schemes. However, this conventional strategy does not work well for the Cahn-Hilliard equation due to the bad numerical stability. Therefore, an alternative strategy is proposed for general problems, for example, in [26, 38] the strategy was to design such that schemes inherit the energy dissipation property and the mass conservation by Furihata. In [27, 28], a conservative multigrid method was developed by Kim.

The advantage of the spectral method is the infinite order convergence; that is, if the exact solution of the Cahn-Hilliard equation is C^k smooth, the approximate solution will be convergent to the exact solution with power for exponent N^{-k}, where N is the number of the basis function. This method is superior to the finite element methods and finite difference methods, and a lot of practice and experiments convince the validity of the spectral method [39]. Many authors have studied the solution of the Cahn-Hilliard equation which has constant mobility by using spectral method. For example, in [33–35], Ye studied the solution of the Cahn-Hilliard equation by Fourier collocation spectral method and Legendre collocation spectral method under different boundary conditions. In [30], the author studied a class of the Cahn-Hilliard equation with pseudospectral method. However, the Cahn-Hilliard equation with varying mobility can depict the physical phenomena more accurately; therefore, there is practical meaning to study the numerical solution for the Cahn-Hilliard equation with varying mobility. Yin [40, 41] studied the Cahn-Hilliard equation with concentration-dependent mobility in one dimension and obtained the existence and uniqueness of the classical solution. Recently, Yin and Liu [42, 43] investigate the regularity for the solution in two dimensions. Some numerical techniques for the Cahn-Hilliard equation with concentration-dependent mobility are already studied with the finite element method [28] and with finite difference method [44].

In this paper, we consider an initial-boundary value problem for Cahn-Hilliard equation of the following form:

$$\frac{\partial u}{\partial t} + D\left[m(u)\left(D^3 u - DA(u)\right)\right] = 0, \quad (x,t) \in (0,1)\times(0,T), \tag{1.2}$$

$$Du|_{x=0,1} = D^3 u\Big|_{x=0,1} = 0, \quad t \in (0,T), \tag{1.3}$$

$$u(x,0) = u_0, \quad x \in (0,1), \tag{1.4}$$

where $D = \partial/\partial x$ and

$$A(s) = -s + \gamma_1 s^2 + \gamma_2 s^3, \quad \gamma_2 > 0. \tag{1.5}$$

Here, $u(x,t)$ represents a relative concentration of one component in binary mixture. The function $m(u)$ is the mobility which depends on the unknown function u, which restricts diffusion of both components to the interfacial region only. Denote $Q_T = (0,1) \times (0,T)$. Throughout this paper, we assume that

$$0 < m_0 \le m(s) \le M_0, \quad \left|m'(s)\right| \le M_1, \quad \forall s \in \mathbb{R}, \tag{1.6}$$

where m_0, M_0, and M_1 are positive constants. The existence and uniqueness of the classical solution of the problems (1.2)–(1.4) were proved by Yin [41].

In this paper, we will apply the spectral method to discretize the spatial variables of (1.2) to construct a semidiscrete system. We prove the existence and boundedness of the solutions of this semidiscrete system. Then, we apply implicit midpoint Euler scheme to discretize the time variable and obtain a full-discrete scheme, which inherits the energy dissipation property. The property of the mobility depending on the solution of (1.2) causes much troubles for the numerical analysis. Furthermore, with the aid of Nirenberg inequality we investigate the boundedness and convergence of the numerical solutions of the full-discrete equations. We also obtain the error estimation for the numerical solutions to the exact ones.

This paper is organized as follows. In Section 2, we study the spectral method for (1.2)–(1.4) and obtain the error estimate between the exact solution u and the spectral approximate solution u_N. In Section 3, we use the implicit Euler method to discretize the time variable and obtain the error estimate between the exact solution u and the full-discrete approximate solution U_N^j. Finally in Section 4, we present a numerical computation to illustrate the theoretical analysis.

2. Semidiscretization with Spectral Method

In this section, we apply the spectral method to discretize (1.2)–(1.4) and study the error estimate between the exact solution and the semidiscretization solution.

Denote by $\|\cdot\|_k$ and $|\cdot|_k$ the norm and seminorm of the Sobolev spaces $H^k(0,1)(k \in \mathbb{N})$, respectively. Let $(\cdot,\cdot)$ be the standard L^2 inner product over $(0,1)$. Define

$$H_E^2(0,1) = \left\{ v \in H^2(0,1);\, Dv|_{x=0,1} = 0 \right\}. \tag{2.1}$$

A function u is said to be a weak solution of the problems (1.2)–(1.4), if $u \in L^\infty(0,T;H_E^2(0,1))$ and satisfies the following equations:

$$\left(\frac{\partial u}{\partial t}, v\right) + \left(D^2 u - A(u), D(m(u)Dv)\right) = 0, \quad \forall v \in H_E^2, \tag{2.2}$$

$$(u(\cdot,0), v) - (u_0, v) = 0, \quad \forall v \in H_E^2. \tag{2.3}$$

For any integer $N > 0$, let $S_N = \text{span}\{\cos k\pi x, k = 0, 1, 2, \dots, N\}$. Define a projection operator $P_N : H_E^2 \mapsto S_N$ by

$$\int_0^1 (P_N) u(x) v(x) dx = \int_0^1 u(x) v(x) dx, \quad \forall v \in S_N. \tag{2.4}$$

We collect some properties of this projection P_N in the following lemma (see [39]).

Lemma 2.1. (i) *P_N commutes with the second derivation on $H_E^2(I)$, that is,*

$$P_N D^2 v = D^2 P_N v, \quad \forall v \in H_E^2(I). \tag{2.5}$$

(ii) *For any $0 \leq \mu \leq \sigma$, there exists a positive constant C such that*

$$\|v - P_N v\|_\mu \leq C N^{\mu-\sigma} |v|_\sigma, \quad v \in H^\sigma(I). \tag{2.6}$$

The following Nirenberg inequality is a key tool for our theoretical estimates.

Lemma 2.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u \in W^{m,r}(\Omega)$, then we have*

$$\left\| D^j u \right\|_{L^p} \leqslant C_1 \|D^m u\|_{L^r}^a \|u\|_{L^q}^{1-a} + C_2 \|u\|_{L^q}, \tag{2.7}$$

where

$$\frac{j}{m} \leqslant a < 1, \quad \frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}. \tag{2.8}$$

By [41], we have the following.

Lemma 2.3. *Assume that $m(s) \in C^{1+\alpha}(\mathcal{R})$, $u_0 \in C^{4+\alpha}(\overline{I})$, $D^i u_0(0) = D^i u_0(1) = 0$ $(i = 0, 1, 2, 3, 4)$, $m(s) > 0$, then there exists a unique solution of the problems* (1.2)–(1.4) *such that*

$$u \in C^{1+\alpha/4, 4+\alpha}\left(\overline{Q}_T\right). \tag{2.9}$$

The spectral approximation to (2.2) is to find an element $u_N(\cdot, t) \in S_N$ such that

$$\left(\frac{\partial u_N}{\partial t}, v_N\right) + \left(D^2 u_N - P_N A(u_N), D(m(u_N) D v_N)\right) = 0, \quad \forall v_N \in S_N, \tag{2.10}$$

$$(u_N(\cdot, 0), v_N) - (u_0, v_N) = 0, \quad \forall v_N \in S_N. \tag{2.11}$$

Now we study the L^∞ norm estimates of the function $u_N(\cdot, t)$ and $D u_N(\cdot, t)$ for $0 \leq t \leq T$.

Theorem 2.4. *Assume* (1.6) *and* $u_0 \in H_E^2$. *Then there is a unique solution of* (2.10) *and* (2.11) *such that*

$$\|u_N\|_\infty \leq C, \quad \|Du_N\| \leq C, \quad 0 \leq t \leq T, \tag{2.12}$$

where $C = C(u_0, T, \gamma_1, \gamma_2)$ *is a positive constant.*

Proof. From (2.11) it follows that $u_N(\cdot, 0) = P_N u_0(\cdot)$. The existence and uniqueness of the initial problem follow from the standard ODE theory. Now we study the estimate.

Define an energy function:

$$F(t) = (H(u_N), 1) + \frac{1}{2}\|Du_N\|^2, \tag{2.13}$$

where $H(s) = \int_0^s A(t)dt = (1/4)\gamma_2 s^4 + (1/3)\gamma_1 s^3 - (1/2)s^2$. Direct computation gives

$$\begin{aligned} \frac{dF}{dt} &= \left(A(u_N), \frac{\partial u_N}{\partial t}\right) + \left(Du_N, D\frac{\partial u_N}{\partial t}\right) = \left(P_N A(u_N), \frac{\partial u_N}{\partial t}\right) - \left(D^2 u_N, \frac{\partial u_N}{\partial t}\right) \\ &= \left(\frac{\partial u_N}{\partial t}, P_N A(u_N) - D^2 u_N\right). \end{aligned} \tag{2.14}$$

Noticing that $P_N A(u_N) - D^2 u_N \in S_N$ and setting $v_N = P_N A(u_N) - D^2 u_N$ in (2.10), applying integration by part, we obtain

$$\begin{aligned} \frac{dF}{dt} &= \left(m(u_N)\left(D^3 u_N - DP_N A(u_N)\right), D\left(P_N A(u_N) - D^2 u_N\right)\right) \\ &= -\left(m(u_N)\left(D^3 u_N - DP_N A(u_N)\right), D^3 u_N - DP_N A(u_N)\right) \\ &\leq -m_0 \left\|D\left(P_N A(u_N) - D^2 u_N\right)\right\|^2 \leq 0. \end{aligned} \tag{2.15}$$

Hence,

$$F(t) \leq F(0) = \int_0^1 H(P_N u_0)ds + \frac{1}{2}\|DP_N u_0\|^2, \quad \forall 0 \leq t \leq T. \tag{2.16}$$

Applying Young inequality, we obtain

$$u_N^2 \leq \varepsilon u_N^4 + C_{1\varepsilon}, \qquad u_N^3 \leq \varepsilon u_N^4 + C_{2\varepsilon}, \tag{2.17}$$

where $C_{1\varepsilon}$ and $C_{2\varepsilon}$ are positive constants. Letting $\varepsilon = 3\gamma_2/(8|\gamma_1| + 12)$, then for all $0 \leq t \leq T$,

$$\int_0^1 H(u_N)dx \geq \int_0^1 \left(\frac{1}{4}\gamma_2 u_N^4 - \frac{1}{3}\left|\gamma_1 u_N^3\right| - \frac{1}{2}u_N^2\right)dx \geq \frac{\gamma_2}{8}\int_0^1 (u_N)^4 dx - K_1, \tag{2.18}$$

where K_1 is a positive constant depending only on γ_1 and γ_2. Therefore, we get

$$\begin{aligned}\frac{1}{2}\|Du_N(\cdot,t)\|^2 + \frac{\gamma_2}{8}\int_0^1 (u_N(\cdot,t))^4 dx - K_1 &\le \frac{1}{2}\|Du_N(\cdot,t)\|^2 + \int_0^1 H(u_N(\cdot,t))dx \\ &= F(t) \le F(0) = \int_0^1 H(P_N u_0)dx + \|DP_N u_0\|^2.\end{aligned} \tag{2.19}$$

Thus,

$$\begin{aligned}\|Du_N(\cdot,t)\|^2 &\le 2K_1 + 2F(0) = 2K_1 + 2\int_0^1 H(P_N u_0)dx + \|DP_N u_0\|^2 \triangleq C, \\ \int_0^1 (u_N(x,t))^4 dx &\le \frac{8}{\gamma_2}(K_1 + F(0)) = \frac{8}{\gamma_2}\left(K_1 + \int_0^1 H(P_N u_0)dx + \frac{1}{2}\|DP_N u_0\|^2\right) \triangleq C,\end{aligned} \tag{2.20}$$

where $C = C(u_0, \gamma_1, \gamma_2)$ is a constant. By Hölder inequality, we obtain

$$\|u_N\|^2 = \int_0^1 u_N^2 dx \le \left(\int_0^1 u_N^4 dx\right)^{1/2} \left(\int_0^1 1^2 dx\right)^{1/2} = \left(\int_0^1 u_N^4 dx\right)^{1/2}. \tag{2.21}$$

Therefore,

$$\|u_N\| \le C, \qquad \|Du_N\| \le C. \tag{2.22}$$

From the embedding theorem it follows that

$$\|u_N\|_\infty \le C, \quad \forall 0 \le t \le T. \tag{2.23}$$

□

Theorem 2.5. *Assume* (1.6) *and let* $u_N(\cdot,t)$ *be the solution of* (2.10) *and* (2.11)*. Then there is a positive constant* $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ *such that*

$$\|Du_N\|_\infty \le C, \qquad \left\|D^2 u_N\right\| \le C. \tag{2.24}$$

Proof. Setting $v_N = D^4 u_N$ in (2.10) and integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|D^2 u_N\right\|^2 + \left(m(u_N)\left(D^4 u_N - D^2 P_N A(u_N)\right), D^4 u_N\right) \\ &\quad + \left(m'(u_N)Du_N\left(D^3 u_N - DP_N A(u_N)\right), D^4 u_N\right) = 0.\end{aligned} \tag{2.25}$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\left\|D^2u_N\right\|^2+\left(m(u_N)D^4u_N,D^4u_N\right)=I_1+I_2+I_3, \tag{2.26}$$

where

$$\begin{aligned}I_1&=\left(m(u_N)D^2P_NA(u_N),D^4u_N\right),\\I_2&=-\left(m'(u_N)Du_ND^3u_N,D^4u_N\right),\\I_3&=\left(m'(u_N)Du_NDP_NA(u_N),D^4u_N\right).\end{aligned} \tag{2.27}$$

Noticing (1.6), we have

$$\frac{1}{2}\frac{d}{dt}\left\|D^2u_N\right\|^2+m_0\left\|D^4u_N\right\|^2\leq\frac{1}{2}\frac{d}{dt}\left\|D^2u_N\right\|^2+\left(m(u_N)D^4u_N,D^4u_N\right). \tag{2.28}$$

In terms of the Nirenberg inequality (2.7), there is a constant $C>0$ such that

$$\begin{aligned}\|Du_N\|_\infty&\leq C\left(\left\|D^4u_N\right\|^{3/8}\|u\|^{5/8}+\|u\|\right),\\\|Du_N\|_\infty&\leq C\left(\left\|D^2u_N\right\|^{3/4}\|u\|^{1/4}+\|u\|\right),\\\left\|D^3u_N\right\|_\infty&\leq C\left(\left\|D^4u_N\right\|^{7/8}\|u\|^{1/8}+\|u\|\right).\end{aligned} \tag{2.29}$$

Noticing the definition of the function A and the estimates in (2.12), we have

$$\|A(u_N)\|_\infty\leq C,\qquad\left\|A'(u_N)\right\|_\infty\leq C,\qquad\left\|A''(u_N)\right\|_\infty\leq C, \tag{2.30}$$

for some constant $C = C(u_0, T, \gamma_1, \gamma_2) > 0$. Applying Hölder inequality and Young inequality, for any $\varepsilon > 0$,

$$
\begin{aligned}
|I_1| &\le M_0\left(\left\|A'(u_N)D^2u_N\right\| + \left\|A''(u_N)(Du_N)^2\right\|\right)\cdot\left\|D^4u_N\right\| \\
&\le M_0\|A'(u_N)\|_\infty\cdot\left\|D^2u_N\right\|\left\|D^4u_N\right\| + M_0\|A''(u_N)\|_\infty\cdot\|Du_N\|\|Du_N\|_\infty\left\|D^4u_N\right\| \\
&\le \varepsilon\left\|D^4u_N\right\|^2 + C\left\|D^2u_N\right\|^2 + C\left(\left\|D^4u_N\right\|^{3/8} + 1\right)\left\|D^4u_N\right\| \\
&\le \varepsilon\left\|D^4u_N\right\|^2 + C\left\|D^2u_N\right\|^2 + C\left\|D^4u_N\right\|^{11/8} + C\left\|D^4u_N\right\| \\
&\le \varepsilon\left\|D^4u_N\right\|^2 + C\left\|D^2u_N\right\|^2 + \frac{\varepsilon}{2}\left\|D^4u_N\right\|^2 + C + \frac{\varepsilon}{2}\left\|D^4u_N\right\|^2 + C \\
&\le 2\varepsilon\left\|D^4u_N\right\|^2 + C\left\|D^2u_N\right\|^2 + C,
\end{aligned}
\tag{2.31}
$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant. Similarly, we obtain

$$
\begin{aligned}
|I_2| &\le M_1\|Du_N\|\left\|D^3u_N\right\|_\infty\left\|D^4u_N\right\| \\
&\le C\left(\left\|D^4u_N\right\|^{7/8} + 1\right)\left\|D^4u_N\right\| \\
&\le C\left(\left\|D^4u_N\right\|^{15/8} + \left\|D^4u_N\right\|\right) \\
&\le \frac{\varepsilon}{2}\left\|D^4u_N\right\|^2 + C + \frac{\varepsilon}{2}\left\|D^4u_N\right\|^2 + C \\
&\le \varepsilon\left\|D^4u_N\right\|^2 + C, \\
|I_3| &\le M_1\|DP_NA(u_N)\|_\infty\cdot\|Du_N\|\cdot\left\|D^4u_N\right\| \\
&\le C\|DP_NA(u_N)\|_\infty\left\|D^4u_N\right\| \\
&\le C\left(\left\|D^2P_NA(u_N)\right\|^{3/4}\|P_NA(u_N)\|^{1/4} + \|P_NA(u_N)\|\right)\left\|D^4u_N\right\| \\
&\le C\left(\left\|D^2P_NA(u_N)\right\|^{3/4}\|A(u_N)\|^{1/4} + \|A(u_N)\|\right)\left\|D^4u_N\right\| \\
&\le C\left(\left\|D^2P_NA(u_N)\right\|^{3/4}\|A(u_N)\|_\infty^{1/4} + \|A(u_N)\|_\infty\right)\left\|D^4u_N\right\| \\
&\le C\left(\left\|D^2P_NA(u_N)\right\|^{3/4} + 1\right)\left\|D^4u_N\right\|
\end{aligned}
$$

$$\begin{aligned}
&\le C\left(\left\|D^2P_NA(u_N)\right\|+1\right)\left\|D^4u_N\right\| \\
&\le C\left(\left\|D^2A(u_N)\right\|+1\right)\left\|D^4u_N\right\| \\
&\le C\left(\left\|A'(u_N)D^2u_N\right\|+\left\|A''(u_N)(Du_N)^2\right\|\right)\cdot\left\|D^4u_N\right\| \\
&\le \varepsilon\left\|D^4u_N\right\|^2+C.
\end{aligned} \tag{2.32}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\left\|D^2u_N\right\|^2+m_0\left\|D^4u_N\right\|^2\le 4\varepsilon\left\|D^4u_N\right\|^2+C\left\|D^2u_N\right\|^2+C. \tag{2.33}$$

Taking $\varepsilon = m_0/8$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|D^2u_N\right\|^2+\frac{m_0}{2}\left\|D^4u_N\right\|^2\le C\left\|D^2u_N\right\|^2+C, \tag{2.34}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ is a positive constant. Therefore,

$$\frac{1}{2}\frac{d}{dt}\left\|D^2u_N\right\|^2\le C\left\|D^2u_N\right\|^2+C. \tag{2.35}$$

From Gronwall inequality it follows that

$$\left\|D^2u_N\right\|^2\le \exp^{Ct}\left(\left\|D^2u_0\right\|^2+1\right)\le \exp^{CT}\left(\left\|D^2u_0\right\|^2+1\right)\le C, \tag{2.36}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ is a positive constant. According to the embedding theorem, we have

$$\|Du_N\|_\infty \le C, \quad 0 \le t \le T, \tag{2.37}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ is a positive constant. □

Now, we study the error estimates between the exact solution u and the semidiscrete spectral approximation solution u_N. Set the following decomposition:

$$u-u_N=\eta+e, \qquad \eta=u-P_Nu, \qquad e=P_Nu-u_N. \tag{2.38}$$

From the inequality (2.6) it follows that

$$\|\eta\|\le CN^{-4}. \tag{2.39}$$

Hence, it remains to obtain the approximate bounds of e.

Theorem 2.6. *Assume that u is the solution of* (1.2)–(1.4), *$u_N \in S_N$ is the solution of* (2.10) *and* (2.11), *and m is smooth and satisfies* (1.6), *then there exists a constant $C = C(u_0, m, \gamma_1, \gamma_2)$ such that*

$$\|e\| \le C\left(N^{-2} + \|e(0)\|\right). \tag{2.40}$$

Before we prove this theorem, we study some useful approximation properties.

Lemma 2.7. *For any $\varepsilon > 0$, we have*

$$-\left(D^2\eta + D^2 e, D(m(u_N)De)\right) \le -m_0\left\|D^2 e\right\|^2 + 3\varepsilon\left\|D^2 e\right\|^2 + C\left(\|e\|^2 + N^{-4}\right), \tag{2.41}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant.

Proof. Direct computation gives

$$\begin{aligned}
&-\left(D^2\eta + D^2 e, D(m(u_N)De)\right)\\
&\quad = -\left(D^2 e, m(u_N)D^2 e\right) - \left(D^2\eta, m(u_N)D^2 e\right)\\
&\qquad - \left(D^2\eta, m'(u_N)Du_N De\right) - \left(D^2 e, m'(u_N)Du_N De\right)\\
&\quad \le -m_0\left\|D^2 e\right\|^2 + M_0\left\|D^2 e\right\|\left\|D^2\eta\right\|^2\\
&\qquad + M_1\left\|D^2\eta\right\|\|Du_N\|_\infty\|De\| + M_1\left\|D^2 e\right\|\|Du_N\|_\infty\|De\|\\
&\quad \le -m_0\left\|D^2 e\right\|^2 + \varepsilon\left\|D^2 e\right\|^2 + \frac{M_0^2}{4\varepsilon}\left\|D^2\eta\right\|^2\\
&\qquad + M_1\|Du_N\|_\infty\left(\|De\|^2 + \left\|D^2\eta\right\|^2\right) + \varepsilon\left\|D^2 e\right\|^2 + \frac{M_1^2\|Du_N\|_\infty^2}{4\varepsilon}\|De\|^2\\
&\quad \le -m_0\left\|D^2 e\right\|^2 + 2\varepsilon\left\|D^2 e\right\|^2 + \left(\frac{M_0^2}{4\varepsilon} + M_1\|Du_N\|_\infty\right)\left\|D^2\eta\right\|^2\\
&\qquad + \left(M_1\|Du_N\|_\infty + \frac{M_1^2\|Du_N\|_\infty^2}{4\varepsilon}\right)\|De\|^2\\
&\quad \le -m_0\left\|D^2 e\right\|^2 + 2\varepsilon\left\|D^2 e\right\|^2 + \left(\frac{M_0^2}{4\varepsilon} + M_1\|Du_N\|_\infty\right)\left\|D^2\eta\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\left(M_1\|Du_N\|_\infty+\frac{M_1^2\|Du_N\|_\infty^2}{4\varepsilon}\right)\left\|D^2e\right\|\cdot\|e\| \\
&\le -m_0\left\|D^2e\right\|^2+2\varepsilon\left\|D^2e\right\|^2+C\left\|D^2\eta\right\|^2+\varepsilon\left\|D^2e\right\|^2+C\|e\|^2 \\
&\le -m_0\left\|D^2e\right\|^2+3\varepsilon\left\|D^2e\right\|^2+C\left(\|e\|^2+\left\|D^2\eta\right\|^2\right) \\
&\le -m_0\left\|D^2e\right\|^2+3\varepsilon\left\|D^2e\right\|^2+C\left(\|e\|^2+N^{-4}\right),
\end{aligned} \tag{2.42}
$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant. □

Lemma 2.8. *For any $\varepsilon > 0$, we have*

$$
(A(u) - P_N A(u_N), D(m(u_N)De)) \le 3\varepsilon M_0^2\left\|D^2e\right\|^2 + C\left(\|e\|^2 + N^{-4}\right), \tag{2.43}
$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant.

Proof. Noticing that

$$
\begin{aligned}
&(A(u) - P_N A(u_N), D(m(u_N)De)) \\
&\quad= (A(u) - P_N A(u), D(m(u_N)De)) + (P_N A(u) - P_N A(u_N), D(m(u_N)De)) \\
&\quad\le \|A(u) - P_N A(u)\|\cdot\|D(m(u_N)De)\| + \left\|P_N\left(\overline{A}(u,u_N)(u-u_N)\right)\right\|\cdot\|D(m(u_N)De)\| \\
&\quad\le \|A(u) - P_N A(u)\|\cdot\|D(m(u_N)De)\| + \left\|\overline{A}(u,u_N)(\eta+e)\right\|\cdot\|D(m(u_N)De)\|,
\end{aligned} \tag{2.44}
$$

where

$$
\overline{A}(s,\tau) = \gamma_2\left(s^2 + s\tau + \tau^2\right) + \gamma_1(s+\tau) - 1. \tag{2.45}
$$

From the boundedness of u_N in Theorem 2.4 and the property of u in Lemma 2.3, it follows that

$$
\left\|\overline{A}(u,u_N)\right\|_\infty \le C. \tag{2.46}
$$

Then we obtain

$$\begin{aligned}
\|A(u) - P_N A(u)\|^2 &\le C\left(\left\|u^3 - P_N u^3\right\|^2 + \left\|u^2 - P_N u^2\right\|^2 + \|u - P_N u\|^2\right) \\
&\le CN^{-8}\left(\left|u^3\right|_4 + \left|u^2\right|_4 + |u|_4\right)^2 \le CN^{-8}, \\
\left\|\overline{A}(u, u_N)(\eta + e)\right\|^2 &\le \left\|\overline{A}(u, u_N)\right\|_\infty^2\left(\|e\|^2 + \|\eta\|^2\right) \le C\left(N^{-8} + \|e\|^2\right), \\
\|D(m(u_N)De)\|^2 &\le 2\left\|m(u_N)D^2 e\right\|^2 + 2\|m'(u_N)Du_N De\|^2 \\
&\le 2M_0^2\left\|D^2 e\right\|^2 + 2M_1^2\|Du_N\|_\infty^2\left(\left\|D^2 e\right\|^2\|e\|^2\right) \\
&\le 2M_0^2\left\|D^2 e\right\|^2 + M_0^2\left\|D^2 e\right\|^2 + \frac{M_1^4\|Du_N\|_\infty^4}{M_0^2}\|e\|^2 \\
&\le 3M_0^2\left\|D^2 e\right\|^2 + C\|e\|^2,
\end{aligned} \tag{2.47}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ is a positive constant. By Cauchy inequality, for any $\varepsilon > 0$, we have

$$\begin{aligned}
&(A(u) - P_N A(u_N), D(m(u_N)De)) \\
&\quad\le \|A(u) - P_N A(u)\| \cdot \|D(m(u_N)De)\| + \left\|\overline{A}(u, u_N)(\eta + e)\right\| \cdot \|D(m(u_N)De)\| \\
&\quad\le \frac{\varepsilon}{2}\|D(m(u)De)\|^2 + \frac{1}{2\varepsilon}\|A(u) - P_N A(u)\|^2 + \frac{\varepsilon}{2}\|D(m(u)De)\|^2 + \frac{1}{2\varepsilon}\left\|\overline{A}(u, u_N)(\eta + e)\right\|^2 \\
&\quad\le 3\varepsilon M_0^2\left\|D^2 e\right\|^2 + C\left(\|e\|^2 + N^{-8}\right),
\end{aligned} \tag{2.48}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2, \varepsilon) > 0$ is a positive constant. □

Lemma 2.9. *Assume that u is the solution of* (1.2)–(1.4)*, there exists a positive constant $C = C(u_0, m, \gamma_1, \gamma_2)$ such that*

$$\left(D^2 u - A(u), D(m(u) - m(u_N))De\right) \le \frac{m_0}{2}\left\|D^2 e\right\|^2 + C\left(\|e\|^2 + N^{-4}\right), \tag{2.49}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ is a positive constant.

Proof. By Lemma 2.3, we have

$$\left\|D^3 u - DA(u)\right\|_\infty \le \left\|D^3 u\right\|_\infty + \left\|A'(u)Du\right\|_\infty \le C. \tag{2.50}$$

In the other hand, it follows that

$$\|m(u) - m(u_N)\|^2 \le M_1^2 \|u - u_N\|^2 \le M_1^2 \left(\|e\|^2 + \|\eta\|^2\right). \tag{2.51}$$

By the Young inequality,

$$\begin{aligned}
\left(D^2 u - A(u), D(m(u) - m(u_N)De)\right) &= -\left(D^3 u - DA(u), (m(u) - m(u_N))De\right) \\
&\le \left\|D^3 u - DA(u)\right\|_\infty \|m(u) - m(u_N)\| \cdot \|De\| \\
&\le C\|m(u) - m(u_N)\|\|De\| \\
&\le \varepsilon\|De\|^2 + C_\varepsilon\left(\|e\|^2 + \|\eta\|^2\right) \\
&\le \varepsilon\left\|D^2 e\right\|^2 + C_\varepsilon\left(\|e\|^2 + \|\eta\|^2\right).
\end{aligned} \tag{2.52}$$

Choosing $\varepsilon = m_0/2$ in the previous inequality, we obtain (2.49). □

Proof of Theorem 2.6. Setting $v = e$ in (2.2), we obtain

$$\left(\frac{\partial u}{\partial t}, e\right) + \left(D^2 u - A(u), D(m(u)De)\right) = 0. \tag{2.53}$$

Setting $v_N = e$ in (2.10), we get

$$\left(\frac{\partial u_N}{\partial t}, e\right) + \left(D^2 u_N - P_N A(u_N), D(m(u_N)De)\right) = 0. \tag{2.54}$$

(2.53) minus (2.54) gives

$$\begin{aligned}
\left(\frac{\partial e}{\partial t}, e\right) = &-\left(Du^2 - A(u), D(m(u) - m(u_N)De)\right) - \left(D^2 u - D^2 u_N, D(m(u_N)De)\right) \\
&+ (A(u) - P_N A(u_N), D(m(u_N)De)).
\end{aligned} \tag{2.55}$$

According to Lemmas 2.7, 2.8 and 2.9, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e\|^2 \le &-m_0\left\|D^2 e\right\|^2 + 3\varepsilon\left\|D^2 e\right\|^2 + C\left(\|e\|^2 + N^{-4}\right) \\
&+ 3\varepsilon M_0^2\left\|D^2 e\right\|^2 + C\left(N^{-8} + \|e\|^2\right) \\
\le &\left(-m_0 + \left(4 + 3M_0^2\right)\varepsilon\right)\left\|D^2 e\right\|^2 + C\left(N^{-4} + \|e\|^2\right).
\end{aligned} \tag{2.56}$$

Set $\varepsilon = m_0/(4+3M_0^2)$, then there exists a positive constant $C = C(u_0, m, \gamma_1, \gamma_2)$ such that

$$\frac{d}{dt}\|e\|^2 \le C\left(\|e\|^2 + N^{-4}\right). \tag{2.57}$$

By Gronwall inequality, we have

$$\|e\| \le \exp^{Ct}\left(\|e_0\| + N^{-2}\right) \le \exp^{CT}\left(\|e_0\| + N^{-2}\right) \le C\left(\|e_0\| + N^{-2}\right), \tag{2.58}$$

where $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ is a constant. □

Summig up the properties above, we obtain the following.

Theorem 2.10. *Assume $m(s)$ is sufficiently smooth and satisfies* (1.6), *u is the solution of* (1.2)–(1.4), *and u_N is the solution of* (2.10) *and* (2.11). *Then there exists a positive constant $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ such that*

$$\|u - u_N\| \le C\left(N^{-2} + \|u_0 - u_{0N}\|\right), \quad \forall t \in (0, T). \tag{2.59}$$

3. Full-Discretization Spectral Scheme

In this section, we apply implicit midpoint Euler scheme to discretize time variable and get a full-discrete form. Furthermore, we investigate the boundedness of numerical solution and the convergence of the numerical solutions of the full-discrete system. We also obtain the error estimates between the numerical solution and the exact ones.

Firstly, we introduce a partition of $[0, T]$. Let $0 = t_0 < t_1 < \cdots < t_\Lambda$, where $t_j = jh$ and $h = T/\Lambda$ is time-step size. Then the full-discretization spectral method for (1.2)–(1.4) reads: $\forall v \in S_N$, find $U_N^j \in S_N$ $(j = 0, 1, 2, \ldots, \Lambda)$ such that

$$\left(\frac{U_N^{j+1} - U_N^j}{h}, v\right) + \left(D^2\overline{U}_N^{j+1/2} - P_N\widetilde{A}\left(U_N^{j+1}, U_N^j\right), D\left(m\left(\overline{U}_N^{j+1/2}\right)Dv\right)\right) = 0, \tag{3.1}$$

$$\left(U_N^0, v\right) - (u_0, v) = 0, \tag{3.2}$$

where $\overline{U}_N^{j+1/2} = (U_N^{j+1} + U_N^j)/2$ and

$$\widetilde{A}(\phi, \varphi) = \frac{\gamma_2}{4}\left(\phi^3 + \varphi^3 + \phi^2\varphi + \phi\varphi^2\right) + \frac{\gamma_1}{3}\left(\phi^2 + \phi\varphi + \varphi^2\right) - \frac{1}{2}(\phi + \varphi). \tag{3.3}$$

The solution U_N^j has the following property.

Lemma 3.1. *Assume that* $U_N^j \in S_N$ $(j = 1, 2, \ldots, \Lambda)$ *is the solution of* (3.1)-(3.2). *Then there exists a constant* $C = C(u_0, m, \gamma_1, \gamma_2) > 0$ *such that*

$$\left\| U_N^j \right\|_\infty \leq C, \qquad \left\| DU_N^j \right\| \leq C. \tag{3.4}$$

Proof. Define a discrete energy function at time t_j by

$$F(j) = \frac{1}{2}\left\| DU_N^j \right\|^2 + \left(H\left(U_N^j\right), 1 \right). \tag{3.5}$$

Notice that

$$\begin{aligned}
\frac{1}{h}(F(j+1) - F(j)) &= \left(D\overline{U}_N^{j+1/2}, D\frac{U_N^{j+1} - U_N^j}{h} \right) + \left(P_N \widetilde{A}\left(U_N^{j+1}, U_N^j\right), \frac{U_N^{j+1} - U_N^j}{h} \right) \\
&= -\left(D^2\overline{U}_N^{j+1/2} - P_N \widetilde{A}\left(U_N^{j+1}, U_N^j\right), \frac{U_N^{j+1} - U_N^j}{h} \right).
\end{aligned} \tag{3.6}$$

Setting $v = D^2\overline{U}_N^{j+1/2} - P_N \widetilde{A}(U_N^{j+1}, U_N^j) \in S_N$ in (3.1), we obtain

$$\begin{aligned}
&\frac{1}{h}(F(j+1) - F(j)) \\
&\quad = -\left(m\left(\overline{U}_N^{j+1/2}\right)\left(D^3\overline{U}_N^{j+1/2} - DP_N \widetilde{A}\left(U_N^{j+1}, U_N^j\right)\right), D^3\overline{U}_N^{j+1/2} - DP_N \widetilde{A}\left(U_N^{j+1}, U_N^j\right) \right) \\
&\quad \leq -m_0 \left\| D^3\overline{U}_N^{j+1/2} - DP_N \widetilde{A}\left(U_N^{j+1}, U_N^j\right) \right\|^2 \leq 0,
\end{aligned} \tag{3.7}$$

which implies

$$F(j) \leq F(0) = \frac{1}{2}\left\| DU_N^0 \right\|^2 + \left(H\left(U_N^0\right), 1 \right). \tag{3.8}$$

By (2.18), we have

$$\int_0^1 H\left(U_N^j\right) dx \geq \frac{\gamma_2}{8} \int_0^1 \left(U_N^j\right)^4 dx - K_1. \tag{3.9}$$

Then

$$\begin{aligned}\frac{1}{2}\left\|DU_N^j\right\|^2+\frac{\gamma_2}{8}\int_0^1\left(U_N^j\right)^4dx-K_1\leq\frac{1}{2}\left\|DU_N^j\right\|^2+\int_0^1H\left(U_N^j\right)dx=F(j)\\ \leq F(0)=\frac{1}{2}\left\|DU_N^0\right\|^2+\left(H\left(U_N^0\right),1\right).\end{aligned}\tag{3.10}$$

So we obtain

$$\begin{gathered}\left\|DU_N^j\right\|^2\leq 2K_1+\left\|DU_N^0\right\|^2+2\left(H\left(U_N^0\right),1\right)\equiv C_1,\\ \int_0^1\left(U_N^j\right)^4dx\leq\frac{8}{\gamma_2}\left(K_1+\frac{1}{2}\left\|DU_N^0\right\|^2+\left(H\left(U_N^0\right),1\right)\right)\equiv C_2,\end{gathered}\tag{3.11}$$

where $C_1 = C_1(u_0, m, \gamma_1, \gamma_2)$ and $C_2 = C_2(u_0, m, \gamma_1, \gamma_2)$ are positive constants. By Hölder inequality, we get

$$\left\|U_N^j\right\|^2=\int_0^1\left(U_N^j\right)^2dx\leq\left(\int_0^1\left(U_N^j\right)^4dx\right)^{1/2}\leq C_2^{1/2}.\tag{3.12}$$

Therefore,

$$\left\|U_N^j\right\|\leq C_2^{1/4}.\tag{3.13}$$

By the embedding theorem, we obtain

$$\left\|U_N^j\right\|_\infty\leq C,\tag{3.14}$$

where $C = C(u_0, m, \gamma_1, \gamma_2) > 0$ is a constant. □

Lemma 3.2. *Assume that $U_N^j \in S_N$ $(j = 1, 2, \ldots, \Lambda)$ is the solution of the full-discretization scheme* (3.1)-(3.2), *then there is a constant $C = C(u_0, m, T, \gamma_1, \gamma_2) > 0$ such that*

$$\left\|DU_N^j\right\|_\infty\leq C,\qquad\left\|D^2U_N^j\right\|\leq C.\tag{3.15}$$

Proof. Setting $v = 2D^4\overline{U}_N^{j+1/2} \in S_N$ in (3.1), we have

$$\begin{aligned}
&\left(\frac{U_N^{j+1} - U_N^j}{h}, 2D^4\overline{U}_N^{j+1/2}\right) \\
&\quad + \left(D\left(m\left(\overline{U}_N^{j+1/2}\right)D^3\overline{U}_N^{j+1/2} - DP_N\widetilde{A}\left(U_N^{j+1}, U_N^j\right)\right)2D^4\overline{U}_N^{j+1/2}\right) \\
&\quad = \left(\frac{D^2U_N^{j+1} - D^2U_N^j}{h}, D^2U_N^{j+1} + D^2U_N^j\right) + \left(m\left(\overline{U}_N^{j+1/2}\right)D^4\overline{U}_N^{j+1/2}, 2D^4\overline{U}_N^{j+1/2}\right) \\
&\quad + \left(m'\left(\overline{U}_N^{j+1/2}\right)D\overline{U}_N^{j+1/2}D^3\overline{U}_N^{j+1/2}, 2D^4\overline{U}_N^{j+1/2}\right) \\
&\quad - \left(m\left(\overline{U}_N^{j+1/2}\right)D^2P_N\widetilde{A}\left(U_N^{j+1} + U_N^j\right), 2D^4\overline{U}_N^{j+1/2}\right) \\
&\quad - \left(m'\left(\overline{U}_N^{j+1/2}\right)D\overline{U}_N^{j+1/2}DP_N\widetilde{A}\left(U_N^{j+1}, U_N^j\right), 2D^4\overline{U}_N^{j+1/2}\right) = 0.
\end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned}
&\frac{1}{h}\left(\left\|D^2U_N^{j+1}\right\|^2 - \left\|D^2U_N^j\right\|^2\right) + 2\left(m\left(\overline{U}_N^{j+1/2}\right)D^4\overline{U}_N^{j+1/2}, D^4\overline{U}_N^{j+1/2}\right) \\
&\quad = 2\left(m'\left(\overline{U}_N^{j+1/2}\right)D^3\overline{U}_N^{j+1/2}D\overline{U}_N^{j+1/2}, D^4\overline{U}_N^{j+1/2}\right) \\
&\quad\quad + 2\left(m\left(\overline{U}_N^{j+1/2}\right)D^2P_N\widetilde{A}\left(U_N^{j+1}, U_N^j\right), D^4\overline{U}_N^{j+1/2}\right) \\
&\quad\quad + 2\left(m'\left(\overline{U}_N^{j+1/2}\right)D\overline{U}_N^{j+1/2}DP_N\widetilde{A}\left(U_N^{j+1}, U_N^{j+1}\right), D^4\overline{U}_N^{j+1/2}\right) \\
&\quad = I_1^j + I_2^j + I_3^j.
\end{aligned} \tag{3.17}$$

By Nirenberg inequality (2.7), we have

$$\begin{aligned}
\left\|DU_N^j\right\|_\infty &\le C\left(\left\|D^4U_N^j\right\|^{3/8}\left\|U_N^j\right\|^{5/8} + \left\|U_N^j\right\|\right), \\
\left\|DU_N^j\right\|_\infty &\le C\left(\left\|D^2U_N^j\right\|^{3/4}\left\|U_N^j\right\|^{1/4} + \left\|U_N^j\right\|\right), \\
\left\|D^3U_N^j\right\|_\infty &\le C\left(\left\|D^4U_N^j\right\|^{7/8}\left\|U_N^j\right\|^{1/8} + \left\|U_N^j\right\|\right).
\end{aligned} \tag{3.18}$$

According to (3.14), we obtain

$$\left\|\widetilde{A}_1'\left(U_N^{j+1},U_N^j\right)\right\|_\infty \le C, \qquad \left\|\widetilde{A}_2'\left(U_N^{j+1},U_N^j\right)\right\|_\infty \le C,$$
$$\left\|\widetilde{A}_{11}''\left(U_N^{j+1},U_N^j\right)\right\|_\infty \le C, \qquad \left\|\widetilde{A}_{12}''\left(U_N^{j+1},U_N^j\right)\right\|_\infty \le C, \qquad \left\|\widetilde{A}_{22}''\left(U_N^{j+1},U_N^j\right)\right\|_\infty \le C, \tag{3.19}$$

where $C = C(u_0, m, \gamma_1, \gamma_2)$ is a positive constant. By Young inequality, for any positive constant $\varepsilon > 0$, it follows that

$$\begin{aligned}
\left|I_1^j\right| &\le 2M_1\left\|D^3\overline{U}_N^{j+1/2}\right\|_\infty\left\|D\overline{U}_N^{j+1/2}\right\|\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^4\overline{U}_N^{j+1/2}\right\|^{7/8}+1\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left\|D^4\overline{U}_N^{j+1/2}\right\|^{15/8}+C\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le \frac{\varepsilon}{2}\left\|D^4\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon+\frac{\varepsilon}{2}\left\|D^4\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon \\
&\le \varepsilon\left\|D^4\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon, \\
\left|I_2^j\right| &\le 2M_0\left\|D^2\widetilde{A}\left(U_N^{j+1},U_N^j\right)\right\|\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le 2M_0\Big\|A_{11}''\left(DU_N^{j+1}\right)^2+2A_{12}''DU_N^{j+1}DU_N^j+A_{22}''\left(DU_N^j\right)^2 \\
&\qquad +A'_1D^2U_N^{j+1}+A'_2D^2U_N^j\Big\|\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2U_N^{j+1}\right\|+\left\|D^2U_N^j\right\|+\left\|\left(DU_N^{j+1}\right)^2\right\|+\left\|\left(DU_N^j\right)^2\right\|\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2U_N^{j+1}\right\|+\left\|D^2U_N^j\right\|+\left\|DU_N^{j+1}\right\|\left\|DU_N^{j+1}\right\|_\infty+\left\|DU_N^j\right\|\left\|DU_N^j\right\|_\infty\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2U_N^{j+1}\right\|+\left\|D^2U_N^j\right\|+\left\|DU_N^{j+1}\right\|_\infty+\left\|DU_N^j\right\|_\infty\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2U_N^{j+1}\right\|+\left\|D^2U_N^j\right\|+\left\|D^2U_N^{j+1}\right\|^{3/4}+\left\|D^2U_N^j\right\|^{3/4}+1\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2U_N^{j+1}\right\|+\left\|D^2U_N^j\right\|+1\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le \varepsilon\left\|D^4\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon\left(\left\|D^2U_N^{j+1}\right\|^2+\left\|D^2U_N^j\right\|^2+1\right),
\end{aligned}$$

$$\begin{aligned}
\left|I_3^j\right| &\le M_1\left\|D\overline{U}_N^{j+1/2}\right\|\left\|DP_N\widetilde{A}\left(U_N^{j+1},U_N^j\right)\right\|_\infty\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2P_N\widetilde{A}\left(U_N^{j+1},U_N^j\right)\right\|^{3/4}+1\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2P_N\widetilde{A}\left(U_N^{j+1},U_N^j\right)\right\|+1\right)\left\|D^4\overline{U}_N^{j+1/2}\right\| \\
&\le C\left(\left\|D^2\widetilde{A}\left(U_N^{j+1},U_N^j\right)\right\|\left\|D^4\overline{U}_N^{j+1/2}\right\|+\left\|D^4\overline{U}_N^{j+1/2}\right\|\right) \\
&\le \varepsilon\left\|D^4\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon\left(\left\|D^2U_N^{j+1}\right\|^2+\left\|D^2U_N^j\right\|^2+1\right)+\varepsilon\left\|D^4\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon \\
&\le 2\varepsilon\left\|D^4\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon\left(\left\|D^2U_N^{j+1}\right\|^2+\left\|D^2U_N^j\right\|^2+1\right),
\end{aligned} \tag{3.20}$$

where $C_\varepsilon = C_\varepsilon(u_0, m, \gamma_1, \gamma_2) > 0$ is a constant. Therefore,

$$\begin{aligned}
&\frac{1}{h}\left(\left\|D^2U_N^{j+1}\right\|^2-\left\|D^2U_N^j\right\|^2\right)+2m_0\left\|D^4\overline{U}_N^{j+1/2}\right\|^2 \\
&\quad\le \frac{1}{h}\left(\left\|D^2U_N^{j+1}\right\|^2-\left\|D^2U_N^j\right\|^2\right)+2\left(m\left(\overline{U}_N^{j+1/2}\right)D^4\overline{U}_N^{j+1/2},D^4\overline{U}_N^{j+1/2}\right) \\
&\quad\le 4\varepsilon\left\|D^2\overline{U}_N^{j+1/2}\right\|^2+C_\varepsilon\left(\left\|D^2U_N^{j+1}\right\|^2+\left\|D^2U_N^j\right\|^2+1\right).
\end{aligned} \tag{3.21}$$

Setting $\varepsilon = m_0/2$, there is a positive constant $C = C(u_0, m, \gamma_1, \gamma_2)$ such that

$$\left\|D^2U_N^{j+1}\right\|^2 \le \left(1+\frac{2Ch}{1-Ch}\right)\left\|D^2U_N^j\right\|^2+\frac{C}{1-Ch}h. \tag{3.22}$$

Denoted by $\widehat{C} = C/(1-Ch)$, if h is sufficiently small such that $Ch \le 1/2$, we have

$$\begin{aligned}
\left\|D^2U_N^{j+1}\right\|^2 &\le \left(1+\widehat{C}h\right)\left\|D^2U_N^j\right\|^2+\frac{\widehat{C}}{2}h \\
&\le \left(1+\widehat{C}h\right)^{j+1}\left\|D^2U_N^0\right\|^2+\frac{\widehat{C}}{2}h\sum_{i=1}^{j}\left(1+\widehat{C}h\right)^i \\
&\le \exp^{(j+1)\widehat{C}h}\left(\left\|D^2U_N^0\right\|^2+\widehat{C}jh\right) \\
&\le \exp^{\widehat{C}T}\left(\left\|D^2U_N^0\right\|^2+\widehat{C}T\right) \equiv C',
\end{aligned} \tag{3.23}$$

where $C' = C'(u_0, m, T, \gamma_1, \gamma_2) > 0$ is a positive constant. By the embedding theorem, the estimate (3.15) holds. □

Next, we investigate the error estimates for the numerical solution U_N^j to the exact solution $u(t_j)$. Our analysis is based on the error decomposition denoted by

$$u(t_j) - U_N^j = \eta^j + e^j, \qquad \eta^j = u(t_j) - P_N u(t_j), \qquad e^j = P_N u(t_j) - U_N^j. \tag{3.24}$$

The boundedness estimate of η^j follows from the inequality (2.6), that is, for any $0 \le j \le \Lambda$, there is a positive constant $C = C(u_0, m, \gamma_1, \gamma_2)$ such that

$$\left\|\eta^j\right\| \le CN^{-4}. \tag{3.25}$$

Hence, it remains to obtain the approximate bounds of e^j. If no confusion occurs, we denote the average of the two instant errors e^j and e^{j+1} by $\overline{e}^{j+1/2}$:

$$\overline{e}^{j+1/2} = \frac{e^j + e^{j+1}}{2}, \qquad \overline{\eta}^{j+1/2} = \frac{\eta^j + \eta^{j+1}}{2}. \tag{3.26}$$

For later use, we give some estimates in the next lemmas.

Lemma 3.3. *Assume that the solution of* (1.2)–(1.4) *is such that* $u_{ttt} \in L^2(Q_T)$, *then*

$$\left\|e^{j+1}\right\|^2 \le \left\|e^j\right\|^2 + 2h\left(u_t(t_{j+1/2}) - \frac{U_N^{j+1} - U_N^j}{h}, \overline{e}^{j+1/2}\right) + \frac{1}{320}h^4 \int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 dt + h\left\|\overline{e}^{j+1/2}\right\|^2. \tag{3.27}$$

Proof. Applying Taylor expansion about $t_{j+1/2}$, we have

$$\begin{aligned} u^j &= u^{j+1/2} - \frac{h}{2}u_t^{j+1/2} + \frac{h^2}{8}u_{tt}^{j+1/2} - \frac{1}{2}\int_{t_j}^{t_{j+1/2}} (t - t_j)^2 u_{ttt} dt, \\ u^{j+1} &= u^{j+1/2} + \frac{h}{2}u_t^{j+1/2} + \frac{h^2}{8}u_{tt}^{j+1/2} + \frac{1}{2}\int_{t_j+1/2}^{t_{j+1}} (t_{j+1} - t)^2 u_{ttt} dt. \end{aligned} \tag{3.28}$$

Then

$$u_t^{j+1/2} - \frac{u^{j+1} - u^j}{h} = -\frac{1}{2h}\left(\int_{t_j+1/2}^{t_{j+1}} (t_{j+1} - t)^2 u_{ttt} dt + \int_{t_j}^{t_{j+1/2}} (t - t_j)^2 u_{ttt} dt\right). \tag{3.29}$$

From Hölder inequality it follows that

$$\begin{aligned}\left\| u_t^{j+1/2} - \frac{u^{j+1}-u^j}{h} \right\|^2 &\le \frac{1}{2h^2}\left(\left\| \int_{t_j+1/2}^{t_{j+1}} (t_{j+1}-t)^2 u_{ttt} dt \right\|^2 + \left\| \int_{t_j}^{t_{j+1/2}} (t-t_j)^2 u_{ttt} dt \right\|^2 \right) \\ &\le \frac{h^3}{320}\int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 dt.\end{aligned} \tag{3.30}$$

Noticing that for any $v \in S_N$, we have

$$\left(u_t^{j+1/2} - \frac{U_N^{j+1}-U_N^j}{h}, v \right) = \left(u_t^{j+1/2} - \frac{u^{j+1}-u^j}{h}, v \right) + \frac{1}{h}\left(e^{j+1} - e^j, v \right). \tag{3.31}$$

Taking $v = 2\overline{e}^{j+1/2}$ in (3.31), we obtain

$$\begin{aligned}\left\| e^{j+1} \right\|^2 &= \left\| e^j \right\|^2 + 2h\left(u_t^{j+1/2} - \frac{U_N^{j+1}-U_N^j}{h}, \overline{e}^{j+1/2} \right) - 2h\left(u_t^{j+1/2} - \frac{u^{j+1}-u^j}{\Delta t}, \overline{e}^{j+1/2} \right) \\ &\le \left\| e^j \right\|^2 + 2h\left(u_t^{j+1/2} - \frac{U_N^{j+1}-U_N^j}{h}, \overline{e}^{j+1/2} \right) \\ &\quad + \frac{1}{320}h^4\int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 dt + h\left\| \overline{e}^{j+1/2} \right\|^2.\end{aligned} \tag{3.32}$$

□

Taking $v = \overline{e}^{j+1/2}$ in (2.2) and (3.1), respectively, we have

$$\left(\frac{\partial u(t_{j+1/2})}{\partial t}, \overline{e}^{j+1/2} \right) + \left(D^2 u^{j+1/2} - A\left(u^{j+1/2}\right), D\left(m\left(u^{j+1/2}\right) D\overline{e}^{j+1/2} \right) \right) = 0, \tag{3.33}$$

$$\left(\frac{U_N^{j+1}-U_N^j}{h}, \overline{e}^{j+1/2} \right) + \left(D^2 \overline{U}_N^{j+1/2} - P_N \widetilde{A}\left(U_N^{j+1}, U_N^j\right), D\left(m\left(\overline{U}_N^{j+1/2}\right) D\overline{e}^{j+1/2} \right) \right) = 0. \tag{3.34}$$

Comparing (3.33) and (3.34), we have

$$\begin{aligned}
&\left(u_t^{j+1/2}-\frac{U_N^{j+1}-U_N^j}{h},\overline{e}^{j+1/2}\right)\\
&\quad=-\left(D^2u^{j+1/2}-\frac{D^2U_N^{j+1}+D^2U_N^j}{2},D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right)\\
&\qquad+\left(A\left(u^{j+1/2}\right)-P_N\widetilde{A}\left(U_N^{j+1},U_N^j\right),D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right)\\
&\qquad+\left(D^2u^{j+1/2}-A\left(u^{j+1/2}\right),D\left(m\left(\overline{U}_N^{j+1/2}\right)-m\left(u^{j+1/2}\right)\right)D\overline{e}^{j+1/2}\right).
\end{aligned}\tag{3.35}$$

Now we investigate the error estimates of the three items in the right-hand side of the previous equation.

Lemma 3.4. *Assume that u is the solution of (1.2)–(1.4) such that $Du_{tt}\in L^2(Q_T)$, then there exists a positive constant $C_1=C_1(m,u_0,T,\gamma_1,\gamma_2)>0$ such that*

$$\begin{aligned}
&-\left(D^2u^{j+1/2}-\frac{D^2U_N^{j+1}+D^2U_N^j}{2},D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right)\\
&\quad\le-\frac{m_0}{2}\left\|D^2\overline{e}^{j+1/2}\right\|^2+C_1\left(N^{-4}+\left\|e^j\right\|^2+\left\|e^{j+1}\right\|^2+h^3\int_{t_j}^{t_{j+1}}\left\|D^2u_{tt}\right\|^2dt\right).
\end{aligned}\tag{3.36}$$

Proof. By Taylor expansion and Hölder inequality, we obtain

$$\begin{aligned}
u^j&=u^{j+1/2}-\frac{h}{2}u_t^{j+1/2}+\int_{t_j}^{t_{j+1/2}}(t-t_j)u_{tt}dt,\\
u^{j+1}&=u^{j+1/2}+\frac{h}{2}u_t^{j+1/2}+\int_{t_j+1/2}^{t_{j+1}}(t_{j+1}-t)u_{tt}dt.
\end{aligned}\tag{3.37}$$

Therefore,

$$\frac{1}{2}\left(u^j+u^{j+1}\right)-u^{j+1/2}=\frac{1}{2}\left(\int_{t_j}^{t_{j+1/2}}(t-t_j)u_{tt}dt+\int_{t_{j+1/2}}^{t_{j+1}}(t_{j+1}-t)u_{tt}dt\right).\tag{3.38}$$

By Hölder inequality, we have

$$
\begin{aligned}
&\left\| D^2\left(u^{j+1/2} - \frac{1}{2}\left(u^j + u^{j+1}\right)\right)\right\|^2 \\
&\quad = \frac{1}{4}\left(\left\| D^2\left(\int_{t_j}^{t_{j+1/2}} (t-t_j)u_{tt}dt + \int_{t_{j+1/2}}^{t_{j+1}} (t_{j+1}-t)u_{tt}dt\right)\right\|^2\right) \\
&\quad \le \frac{1}{4}\left(\left\|\left(\int_{t_j}^{t_{j+1/2}} (t-t_j)^2 dt \int_{t_j}^{t_{j+1/2}} \left(D^2 u_{tt}\right)^2 dt + \int_{t_{j+1/2}}^{t_{j+1}} (t_{j+1}-t)^2 dt \int_{t_{j+1/2}}^{t_{j+1}} \left(D^2 u_{tt}\right)^2 dt\right)\right\|^2\right) \\
&\quad = \frac{1}{4}\left(\left\|\left(\frac{h^3}{24}\int_{t_j}^{t_{j+1/2}} \left(D^2 u_{tt}\right)^2 dt + \frac{h^3}{24}\int_{t_{j+1/2}}^{t_{j+1}} \left(D^2 u_{tt}\right)^2 dt\right)\right\|^2\right) \\
&\quad \le \frac{1}{4}\cdot\frac{h^3}{24}\int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt}\right\|^2 dt.
\end{aligned}
\tag{3.39}
$$

Direct computation gives

$$
\left\| D^2\left(u^{j+1/2} - \frac{1}{2}\left(u^j + u^{j+1}\right)\right)\right\|^2 \le \frac{h^3}{96}\int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt}\right\|^2 dt. \tag{3.40}
$$

Then

$$
\begin{aligned}
&-\left(D^2\left(u^{j+1/2} - \frac{U_N^{j+1} + U_N^j}{2}\right), D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right) \\
&\quad = -\left(D^2\left(u^{j+1/2} - \frac{u^{j+1} + u^j}{2}\right), D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right) \\
&\qquad -\left(D^2\left(\frac{u^{j+1} + u^j}{2} - \frac{U_N^{j+1} + U_N^j}{2}\right), D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right) \\
&\quad \le \left\| D^2\left(u^{j+1/2} - \frac{u^{j+1} + u^j}{2}\right)\right\| \cdot \left\| D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right\| \\
&\qquad -\left(D^2\left(\overline{\eta}^{j+1/2} + \overline{e}^{j+1/2}\right), D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \cdot \left(\left\| m\left(\overline{U}_N^{j+1/2}\right) D^2 \overline{e}^{j+1/2} \right\| + \left\| m'\left(\overline{U}_N^{j+1/2}\right) D\overline{U}_N^{j+1/2} D\overline{e}^{j+1/2} \right\| \right) \\
&\quad - \left(D^2 \overline{\eta}^{j+1/2}, m\left(\overline{U}_N^{j+1/2}\right) D^2 \overline{e}^{j+1/2} \right) - \left(D^2 \overline{\eta}^{j+1/2}, m'\left(\overline{U}_N^{j+1/2}\right) D\overline{U}_N^{j+1/2} D\overline{e}^{j+1/2} \right) \\
&\quad - \left(D^2 \overline{e}^{j+1/2}, m\left(\overline{U}_N^{j+1/2}\right) D^2 \overline{e}^{j+1/2} \right) - \left(D^2 \overline{e}^{j+1/2}, m'\left(\overline{U}_N^{j+1/2}\right) D\overline{U}_N^{j+1/2} D\overline{e}^{j+1/2} \right) \\
&\triangleq \sigma_1^j + \sigma_2^j + \sigma_3^j.
\end{aligned}
\tag{3.41}
$$

By Cauchy inequality, it follows that

$$
\begin{aligned}
\left|\sigma_1^j\right| &\leq \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \cdot \left\| m\left(\overline{U}_N^{j+1/2}\right) D^2 \overline{e}^{j+1/2} \right\| \\
&\quad + \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \left\| m'\left(\overline{U}_N^{j+1/2}\right) D\overline{U}_N^{j+1/2} D\overline{e}^{j+1/2} \right\| \\
&\leq M_0 \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \left\| D^2 \overline{e}^{j+1/2} \right\| \\
&\quad + M_1 \left\| D\overline{U}_N^{j+1/2} \right\|_\infty \left\{ \frac{h^3}{96} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt \right\}^{1/2} \left\| D\overline{e}^{j+1/2} \right\| \\
&\leq h^3 C_{1\varepsilon} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt + \varepsilon \left\| D^2 \overline{e}^{j+1/2} \right\|^2 + \varepsilon \left\| D\overline{e}^{j+1/2} \right\|^2 \\
&\leq h^3 C_{1\varepsilon} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt + \varepsilon \left\| D^2 \overline{e}^{j+1/2} \right\|^2 + \varepsilon \left\| D^2 \overline{e}^{j+1/2} \right\| \left\| \overline{e}^{j+1/2} \right\| \\
&\leq h^3 C_{1\varepsilon} \int_{t_j}^{t_{j+1}} \left\| D^2 u_{tt} \right\|^2 dt + 2\varepsilon \left\| D^2 \overline{e}^{j+1/2} \right\|^2 + \varepsilon \left\| \overline{e}^{j+1/2} \right\|^2, \\
\left|\sigma_2^j\right| &\leq M_0 \left\| D^2 \eta^{j+1/2} \right\| \left\| D^2 \overline{e}^{j+1/2} \right\| + M_1 \left\| DU_N^{j+1/2} \right\|_\infty \left\| D^2 \eta^{j+1/2} \right\| \left\| D\overline{e}^{j+1/2} \right\| \\
&\leq \frac{M_0^2}{2\varepsilon} \left\| D^2 \eta^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D^2 \overline{e}^{j+1/2} \right\|^2 + \frac{M_1^2 \left\| D\overline{U}_N^{j+1/2} \right\|_\infty^2}{2\varepsilon} \frac{\varepsilon}{2} \left\| D\overline{e}^{j+1/2} \right\|^2 \\
&\leq C_{2\varepsilon} \| D^2 \eta^{j+1/2} \|^2 + \frac{\varepsilon}{2} \left\| D^2 \overline{e}^{j+1/2} \right\|^2 + \frac{\varepsilon}{2} \left\| D^2 \overline{e}^{j+1/2} \right\| \left\| \overline{e}^{j+1/2} \right\| \\
&\leq C_{2\varepsilon} \| D^2 \eta^{j+1/2} \|^2 + \varepsilon \| D^2 \overline{e}^{j+1/2} \|^2 + \frac{\varepsilon}{2} \| \overline{e}^{j+1/2} \|^2,
\end{aligned}
$$

$$\begin{aligned}
\sigma_3^j &\le -m_0\left\|D^2\overline{e}^{j+1/2}\right\|^2 + M_1\left\|D\overline{U}_N^{j+1/2}\right\|_\infty \left\|D^2\overline{e}^{j+1/2}\right\|\cdot\left\|D\overline{e}^{j+1/2}\right\| \\
&\le -m_0\left\|D^2\overline{e}^{j+1/2}\right\|^2 + \frac{\varepsilon}{2}\left\|D^2\overline{e}^{j+1/2}\right\|^2 + \frac{M_1^2\left\|D\overline{U}_N^{j+1/2}\right\|_\infty^2}{2\varepsilon}\left\|D\overline{e}^{j+1/2}\right\|^2 \\
&\le -m_0\left\|D^2\overline{e}^{j+1/2}\right\|^2 + \varepsilon\left\|D^2\overline{e}^{j+1/2}\right\|^2 + C_{3\varepsilon}\left\|\overline{e}^{j+1/2}\right\|^2.
\end{aligned} \tag{3.42}$$

Then we obtain

$$\begin{aligned}
&-\left(D^2u^{j+1/2} - \frac{D^2U_N^{j+1}+D^2U_N^j}{2}, D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right) \\
&\quad\le \left|\sigma_1^j\right| + \left|\sigma_2^j\right| + \sigma_3^j \le (4\varepsilon - m_0)\left\|D^2\overline{e}^{j+1/2}\right\|^2 + h^3C_{1\varepsilon}\int_{t_j}^{t_{j+1}}\left\|D^2u_{tt}\right\|^2dt \\
&\quad\quad + C_{2\varepsilon}\left\|D^2\eta^{j+1/2}\right\|^2 + C_{3\varepsilon}\left\|\overline{e}^{j+1/2}\right\|^2,
\end{aligned} \tag{3.43}$$

where $C_{1\varepsilon}, C_{2\varepsilon}$, and $C_{3\varepsilon}$ are positive constants. Choosing $\varepsilon = m_0/8$, and terms of the properties of the projection operator P_N, we complete the proof of the estimate (3.36). □

Lemma 3.5. *Assume that u is the solution of* (1.2)–(1.4) *such that $u_{tt} \in L^2(Q_T)$ and $u_t \in L^\infty(Q_T)$. Then for any positive constant $\varepsilon > 0$, there exists a constant $C_2 = C_2(m, u_0, T, \gamma_1, \gamma_2) > 0$, such that*

$$\begin{aligned}
&\left(A\left(u^{j+1/2}\right) - P_N\widetilde{A}\left(U_N^{j+1}, U_N^j\right), D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right) \\
&\quad\le \frac{m_0}{4}\left\|D^2\overline{e}^{j+1/2}\right\|^2 + C_2\left\{\left\|e^{j+1}\right\|^2 + \left\|e^j\right\|^2 + N^{-8}\right. \\
&\quad\quad \left. + h^3\left(\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2dt + \left\|u_t^{j+1/2}\right\|_\infty^2\int_{t_j}^{t_{j+1}}\|u_t\|^2dt\right)\right\}.
\end{aligned} \tag{3.44}$$

Proof. Firstly, we consider

$$\begin{aligned}
A\left(u^{j+1/2}\right) - \widetilde{A}\left(u^{j+1}, u^j\right) &= \gamma_2\left(u^{j+1/2}\right)^3 - \frac{\gamma_2}{4}\left[\left(u^{j+1}\right)^3 + \left(u^{j+1}\right)^2u^j + u^{j+1}\left(u^j\right)^2 + \left(u^j\right)^3\right] \\
&\quad + \gamma_1\left(u^{j+1/2}\right)^2 - \frac{\gamma_1}{3}\left[\left(u^{j+1}\right)^2 + u^{j+1}u^j + \left(u^j\right)^2\right] \\
&\quad - \left(u^{j+1/2} - \frac{1}{2}\left(u^{j+1} + u^j\right)\right) \triangleq \gamma_2\rho_1^j + \gamma_1\rho_2^j - \rho_3^j.
\end{aligned} \tag{3.45}$$

Direct computation gives

$$
\begin{aligned}
\left\|\rho_3^j\right\| &= \frac{1}{2}\left\|\int_{t_j}^{t_{j+1/2}}(t-t_j)u_{tt}dt+\int_{t_{j+1/2}}^{t_{j+1}}(t_{j+1}-t)u_{tt}dt\right\| \le \left(\frac{\Delta t^3}{96}\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2dt\right)^{1/2},\\
\left\|\rho_2^j\right\| &= \left\|\frac{1}{6}\left[\left(2u^{j+1/2}\right)^2-\left(u^{j+1}+u^j\right)^2\right]+\frac{1}{6}\left[2\left(u^{j+1/2}\right)^2-\left(u^{j+1}\right)^2-\left(u^j\right)^2\right]\right\|\\
&\le \frac{1}{6}\left\|\left[2u^{j+1/2}-\left(u^{j+1}+u^j\right)\right]\left[2u^{j+1/2}+\left(u^{j+1}+u^j\right)\right]\right\|\\
&\quad+\frac{1}{6}\left\|\left(u^{j+1/2}-u^{j+1}\right)\left(u^{j+1/2}+u^{j+1}\right)+\left(u^{j+1/2}-u^j\right)\left(u^{j+1/2}+u^j\right)\right\|\\
&\le \frac{1}{6}\left\|2u^{j+1/2}-\left(u^{j+1}+u^j\right)\right\|\cdot\left\|2u^{j+1/2}+\left(u^{j+1}+u^j\right)\right\|_\infty\\
&\quad+\frac{1}{6}\left\|\left(\frac{-\Delta t}{2}u_t^{j+1/2}-\int_{t_{j+1/2}}^{t_{j+1}}(t_{j+1}-t)u_{tt}dt\right)\left(u^{j+1/2}+u^{j+1}\right)\right.\\
&\qquad\left.+\left(\frac{\Delta t}{2}u_t^{j+1/2}-\int_{t_j}^{t_{j+1/2}}(t-t_j)u_{tt}dt\right)\left(u^{j+1/2}+u^j\right)\right\|\\
&\le C\Delta t^{3/2}\left(\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2dt\right)^{1/2}+\frac{\Delta t^{3/2}}{12}\left\|u_t^{j+1/2}\right\|_\infty\cdot\left(\int_{t_j}^{t_{j+1}}\|u_t\|^2dt\right)^{1/2},\\
\left\|\rho_1^j\right\| &= \frac{1}{12}\left\|\left(\left(2u^{j+1/2}\right)^3-\left(u^{j+1}+u^j\right)^3\right)+\frac{1}{6}\left[2\left(u^{j+1/2}\right)^3-\left(u^{j+1}\right)^3-\left(u^j\right)^3\right]\right\|\\
&\le \frac{1}{12}\left\|\left[2u^{j+1/2}-\left(u^{j+1}+u^j\right)\right]\cdot\left[4\left(u^{j+1/2}\right)^2+2\left(u^{j+1}+u^j\right)u^{j+1/2}+\left(u^{j+1}+u^j\right)^2\right]\right\|\\
&\quad+\frac{1}{6}\left\|\left(u^{j+1/2}-u^{j+1}\right)\left(\left(u^{j+1/2}\right)^2+u^{j+1}u^{j+1/2}+\left(u^{j+1}\right)^2\right)\right.\\
&\qquad\left.+\left(u^{j+1/2}-u^j\right)\left(\left(u^{j+1/2}\right)^2+u^ju^{j+1/2}+\left(u^j\right)^2\right)\right\|\\
&\le C\Delta t^{3/2}\left\{\left(\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2dt\right)^{1/2}+\left\|u_t^{j+1/2}\right\|_\infty\cdot\left(\int_{t_j}^{t_{j+1}}\|u_t\|^2dt\right)^{1/2}\right\}.
\end{aligned}
\tag{3.46}
$$

Then

$$
\begin{aligned}
&\left(A\left(u^{j+1/2}\right)-P_N\tilde{A}\left(U_N^{j+1},U_N^j\right),D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right)\\
&\quad\le\frac{3}{4\varepsilon}\left\|A\left(u^{j+1/2}\right)-P_NA\left(u^{j+1/2}\right)\right\|^2+\frac{\varepsilon}{3}\left\|D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{4\varepsilon}\left\|A\left(u^{j+1/2}\right)-\widetilde{A}\left(u^{j+1},u^{j}\right)\right\|^2+\frac{\varepsilon}{3}\left\|D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right\|^2\\
&+\frac{3}{4\varepsilon}\left\|\widetilde{A}\left(u^{j+1},u^{j}\right)-\widetilde{A}\left(U_N^{j+1},U_N^{j}\right)\right\|^2+\frac{\varepsilon}{3}\left\|D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right\|^2\\
\le{}&\varepsilon\left\|D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right\|^2+\frac{3}{4\varepsilon}\left\|A\left(u^{j+1/2}\right)-P_NA\left(u^{j+1/2}\right)\right\|^2\\
&+\frac{3}{4\varepsilon}\left\|A\left(u^{j+1/2}\right)-\widetilde{A}\left(u^{j+1},u^{j}\right)\right\|^2+\frac{3}{4\varepsilon}\left\|\widetilde{A}\left(u^{j+1},u^{j}\right)-\widetilde{A}\left(U_N^{j+1},U_N^{j}\right)\right\|^2.
\end{aligned}
\tag{3.47}
$$

Direct computation yields

$$
\begin{aligned}
\left\|D\left(m\left(\overline{U}_N^{j+1/2}\right)D\overline{e}^{j+1/2}\right)\right\|^2&\le\left(M_0^2+M_1^2\right)\left\|D^2\overline{e}^{j+1/2}\right\|^2+C\left\|\overline{e}^{j+1/2}\right\|^2,\\
\left\|A\left(u^{j+1/2}\right)-P_NA\left(u^{j+1/2}\right)\right\|^2&\le CN^{-8},\\
\left\|A\left(u^{j+1/2}\right)-\widetilde{A}\left(u^{j+1},u^{j}\right)\right\|^2&\le C\left(\left\|\rho_1^j\right\|^2+\left\|\rho_2^j\right\|^2+\left\|\rho_3^j\right\|^2\right),\\
\left\|\widetilde{A}\left(u^{j+1},u^{j}\right)-\widetilde{A}\left(U_N^{j+1},U_N^{j}\right)\right\|^2&\le\left(\left\|G_1^j\right\|_\infty^2+\left\|G_2^j\right\|_\infty^2\right)\left(\left\|e^{j+1}\right\|^2+\left\|e^{j}\right\|^2+CN^{-8}\right),
\end{aligned}
\tag{3.48}
$$

where

$$
\begin{aligned}
G_1^j&=\frac{\gamma_2}{8}\left(\left(u^{j+1}+U_N^{j+1}\right)^2+\left(U_N^{j+1}+U_N^{j}\right)^2+\left(u^{j+1}+U_N^{j}\right)^2\right)+\frac{\gamma_1}{3}\left(U_N^{j+1}+u^{j+1}+U_N^{j}\right)-\frac{1}{2},\\
G_2^j&=\frac{\gamma_2}{8}\left(\left(u^{j}+U_N^{j}\right)^2+\left(u^{j}+u^{j+1}\right)^2+\left(u^{j+1}+U_N^{j}\right)^2\right)+\frac{\gamma_1}{3}\left(u^{j+1}+u^{j}+U_N^{j}\right)-\frac{1}{2}.
\end{aligned}
\tag{3.49}
$$

Applying Lemma 2.3 and Theorem 3.7, we obtain that $\|G_1\|_\infty\le C(u_0,m,T,\gamma_1,\gamma_2)$ and $\|G_2\|_\infty\le C(u_0,m,T,\gamma_1,\gamma_2)$. Taking $\varepsilon=m_0/4(M_0^2+M_1^2)$ in (3.47), we have

$$
\begin{aligned}
&\left(A\left(u^{j+1/2}\right)-P_N\widetilde{A}\left(U_N^{j+1},U_N^{j}\right),D\left(m^{j+1/2}D\overline{e}^{j+1/2}\right)\right)\\
&\quad\le\frac{m_0}{4}\left\|D^2\overline{e}^{j+1/2}\right\|^2+C_2\Bigg\{\left\|e^{j+1}\right\|^2+\left\|e^{j}\right\|^2+N^{-8}\\
&\qquad\qquad+\Delta t^3\left(\int_{t_j}^{t_{j+1}}\|u_{tt}\|^2dt+\left\|u_t^{j+1/2}\right\|_\infty^2\int_{t_j}^{t_{j+1}}\|u_t\|^2dt\right)\Bigg\},
\end{aligned}
\tag{3.50}
$$

where $C_2=C_2(u_0,m,T,\gamma_1,\gamma_2)>0$ is a constant. □

Lemma 3.6. *Assume that u is the solution of* (1.2)–(1.4). *Then there exists a positive constant* $C = C(u_0, m, \gamma_1, \gamma_2) > 0$ *such that*

$$\begin{aligned}
&\left(D^2 u^{j+1/2} - A\left(u^{j+1/2}\right), D\left(m\left(\overline{U}_N^{j+1/2}\right) - m\left(u^{j+1/2}\right)\right) D\overline{e}^{j+1/2}\right) \\
&\quad \leq \frac{m_0}{4}\left\|D^2\overline{e}^{j+1/2}\right\|^2 + C\left(\left\|\overline{e}^{j+1/2}\right\|^2 + N^{-8} + h^3\int_{t_j}^{t_{j+1}}\|u_{tt}\|dt\right).
\end{aligned} \tag{3.51}$$

Proof. By (2.9), we have

$$\begin{aligned}
\left\|D^3 u^{j+1/2} - DA\left(u^{j+1/2}\right)\right\|_\infty &\leq \left\|D^3 u^{j+1/2}\right\|_\infty + \left\|A'\left(u^{j+1/2}\right)Du^{j+1/2}\right\|_\infty \\
&\leq \left\|D^3 u^{j+1/2}\right\|_\infty + \left\|A'\left(u^{j+1/2}\right)\right\|_\infty\left\|Du^{j+1/2}\right\|_\infty \leq C.
\end{aligned} \tag{3.52}$$

In the other hand,

$$\begin{aligned}
\left\|m\left(\overline{U}_N^{j+1/2}\right) - m\left(u^{j+1/2}\right)\right\|^2 &\leq M_1^2\left\|\left(\overline{U}_N^{j+1/2} - u^{j+1/2}\right)\right\|^2 \\
&\leq C\left\|\overline{U}_N^{j+1/2} - \frac{u^j + u^{j+1}}{2}\right\|^2 + \left\|\frac{u^j + u^{j+1}}{2} - u^{j+1/2}\right\|^2 \\
&\leq C\left\|\overline{e}^{j+1/2}\right\|^2 + \left\|\overline{\eta}^{j+1/2}\right\|^2 + \frac{h^3}{96}\int_{t_j}^{t_{j+1}}\|u_{tt}\|dt.
\end{aligned} \tag{3.53}$$

By Young inequality, we obtain

$$\begin{aligned}
&\left(D^2 u^{j+1/2} - A\left(u^{j+1/2}\right), D\left(m\left(\overline{U}_N^{j+1/2}\right) - m\left(u^{j+1/2}\right)\right) D\overline{e}^{j+1/2}\right) \\
&\quad = -\left(D^3 u^{j+1/2} - DA\left(u^{j+1/2}\right), m\left(\overline{U}_N^{j+1/2}\right) - m\left(u^{j+1/2}\right) D\overline{e}^{j+1/2}\right) \\
&\quad \leq \left\|D^3 u^{j+1/2} - DA\left(u^{j+1/2}\right)\right\|_\infty \left\|m\left(\overline{U}_N^{j+1/2}\right) - m\left(u^{j+1/2}\right)\right\| \left\|D\overline{e}^{j+1/2}\right\| \\
&\quad \leq \varepsilon\left\|D\overline{e}^{j+1/2}\right\|^2 + C_\varepsilon\left\|m\left(\overline{U}_N^{j+1/2}\right) - m\left(u^{j+1/2}\right)\right\|^2 \\
&\quad \leq \varepsilon\left\|D^2\overline{e}^{j+1/2}\right\|\left\|\overline{e}^{j+1/2}\right\| + C_\varepsilon\left\|m\left(\overline{U}_N^{j+1/2}\right) - m\left(u^{j+1/2}\right)\right\|^2 \\
&\quad \leq \varepsilon\left\|D^2\overline{e}^{j+1/2}\right\|^2 + C_\varepsilon\left(\left\|\overline{e}^{j+1/2}\right\|^2 + N^{-8} + h^3\int_{t_j}^{t_{j+1}}\|u_{tt}\|dt\right).
\end{aligned} \tag{3.54}$$

Choosing $\varepsilon = m_0/4$ in the previous inequality leads to (3.51). □

Finally, we obtain the main theorem of this paper.

Theorem 3.7. *Assume that $u(x,t)$ is the solution of* (1.2)–(1.4) *and satisfies that*

$$\begin{aligned} &D^4 u \in L^\infty(Q_T), \quad u_t \in L^\infty(Q_T), \\ &D^2 u_{tt} \in L^2(Q_T), \quad u_{ttt} \in L^2(Q_T). \end{aligned} \tag{3.55}$$

$U_N^j \in S_N$ $(j = 1, 2, \ldots, \Lambda)$ *is the solution of the full-discretization* (3.1) *and* (3.2). *If h is sufficiently small, there exists a positive constant C such that, for any $j = 0, 1, 2, \ldots, \Lambda$,*

$$\left\| e^{j+1} \right\| = \left\| P_N u(t_{j+1}) - U_N^{j+1} \right\| \le C\left(N^{-2} + \left\| e^0 \right\| + Bh^2 \right), \tag{3.56}$$

where $B = \int_0^{t_{j+1}} (\|D^2 u_{tt}\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \max_{0 \le l \le j} \{\|u_t^{l+1/2}\|_\infty^2\} \cdot \|u_t\|^2) dt$.

Proof. By (3.27), (3.36), (3.44), and (3.51), we have

$$\begin{aligned} \left\| e^{j+1} \right\|^2 \le{}& \left\| e^j \right\|^2 + \frac{h^4}{320} \int_{t_j}^{t_{j+1}} \|u_{ttt}\|^2 dt + h \left\| \overline{e}^{j+1/2} \right\|^2 \\ &+ 2h \Bigg\{ -\left(D^2 u^{j+1/2} - \frac{D^2 U_N^{j+1} + D^2 U_N^j}{2}, D\left(m\left(\overline{U}_N^{j+1/2} \right) D\overline{e}^{j+1/2} \right) \right) \\ &\quad + \left(A\left(u^{j+1/2} \right) - \widetilde{A}\left(U_N^{j+1}, U_N^j \right), D\left(m\left(\overline{U}_N^{j+1/2} \right) D\overline{e}^{j+1/2} \right) \right) \\ &\quad + \left(D^2 u^{j+1/2} - A\left(u^{j+1/2} \right), D\left(m\left(\overline{U}_N^{j+1/2} \right) - m\left(u^{j+1/2} \right) \right) D\overline{e}^{j+1/2} \right) \Bigg\} \\ \le{}& \left\| e^j \right\|^2 + hC_1 \left(N^{-4} + \left\| e^{j+1} \right\|^2 + \left\| e^j \right\|^2 \right) \\ &+ C_2 h^4 \int_{t_j}^{t_{j+1}} \left(\left\| D^2 u_{tt} \right\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \left\| u_t^{j+1/2} \right\|_\infty^2 \|u_t\|^2 \right) dt, \end{aligned} \tag{3.57}$$

where $C_1 = C_1(m, u_0, T, \gamma_1, \gamma_2) > 0$ and $C_2 = C_2(m, u_0, T, \gamma_1, \gamma_2) > 0$ are constants. For sufficiently small h such that $C_1 h \le 1/2$, denoting $\widetilde{C} = 2(C_1 + C_2)$, we obtain

$$\left\| e^{j+1} \right\|^2 \le \left(1 + \widetilde{C} h \right) \left\| e^j \right\|^2 + \widetilde{C}\left(hN^{-4} + h^4 B^j \right), \tag{3.58}$$

where

$$B^j = \int_{t_j}^{t_{j+1}} \left(\left\| D^2 u_{tt} \right\|^2 + \|u_{tt}\|^2 + \|u_{ttt}\|^2 + \left\| u_t^{j+1/2} \right\|_\infty^2 \|u_t\|^2 \right) dt. \tag{3.59}$$

By the Gronwall inequality of the discrete form, we obtain

$$\begin{aligned}\left\|e^{j+1}\right\|^2 &\leq \left(1+\widetilde{C}h\right)^{j+1}\left\|e^0\right\|^2+\widetilde{C}\sum_{l=0}^{j}\left(1+\widetilde{C}h\right)^{l}\left(htN^{-4}+h^4B^l\right)\\ &\leq \exp\left\{\widetilde{C}(j+1)h\right\}\left\{\left\|e^0\right\|^2+\widetilde{C}\sum_{l=0}^{j}\left(hN^{-4}+h^4B^l\right)\right\}\\ &\leq \exp\left\{\widetilde{C}(j+1)h\right\}\left\{\left\|e^0\right\|^2+\widetilde{C}\left(jhN^{-4}+h^4\sum_{l=0}^{j}B^l\right)\right\}.\end{aligned} \tag{3.60}$$

Direct computation gives

$$\sum_{l=0}^{j}B^l \leq \int_0^{t_{j+1}}\left(\left\|D^2u_{tt}\right\|^2+\|u_{tt}\|^2+\|u_{ttt}\|^2+\max_{0\leq l\leq j}\left\{\left\|u_t^{l+1/2}\right\|_\infty^2\right\}\cdot\|u_t\|^2\right)dt. \tag{3.61}$$

Then we complete the conclusion (3.56). □

Furthermore, we get the following theorem.

Theorem 3.8. *Assume u is the solution of* (1.2)–(1.4) *and satisfies that*

$$\begin{aligned} &D^4u\in L^\infty(Q_T), \quad u_t\in L^\infty(Q_T),\\ &D^2u_{tt}\in L^2(Q_T), \quad u_{ttt}\in L^2(Q_T).\end{aligned} \tag{3.62}$$

$U_N^j \in S_N$ ($j=1,2,\ldots,\Lambda$) is the solution of the full-discretization scheme (3.1)-(3.2)*, and U^0 satisfies $\|e^0\|=\|P_Nu^0-U^0\|\leq CN^{-4}$. If h is sufficiently small, there exists a constant $C=C(u_0,m,T,\gamma_1,\gamma_2)>0$ such that*

$$\left\|u(x,t_j)-U_N^j\right\|\leq C\left(N^{-2}+h^2\right), \quad j=1,2,\ldots,\Lambda. \tag{3.63}$$

4. Numerical Experiments

In this section, we apply the spectral method described in (3.1) and (3.2) to carry out numerical computations to illustrate theoretical estimations in the previous section. Consider (2.2) with settings:

$$m(s)=m_0+s^2, \qquad A(s)=s^3-s, \tag{4.1}$$

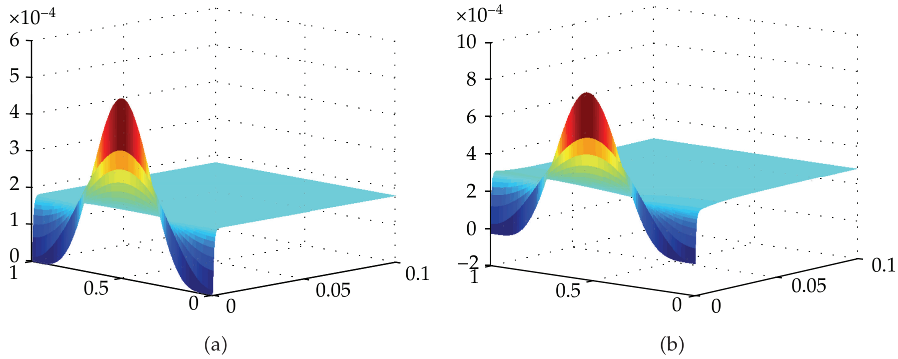


Figure 1: The development of the solution of the full-discrete scheme when initial value is $u_0(x) = x^5(1-x)^5$ (a) and $u_0(x) = x^5(1-x)^5 e^x$ (b).

where $m_0 > 0$ is a constant. The full-discretization spectral method of (2.2) and (2.3) reads: find $U_N^j = \sum_{l=0}^{N} \alpha_{jl} \cos l\pi x$ $(j = 1, 2, \ldots, \Lambda)$ such that

$$\left(\frac{U_N^{j+1} - U_N^j}{h}, v\right) + \left(\frac{D^2 U_N^{j+1} + D^2 U_N^j}{2} - P_N \tilde{A}\left(U_N^{j+1}, U_N^j\right),\right.$$
$$\left. D\left(m\left(\frac{D^2 U_N^{j+1} + D^2 U_N^j}{2}\right) Dv\right)\right) = 0, \tag{4.2}$$
$$\left(U_N^0, v\right) - (u_0, v) = 0.$$

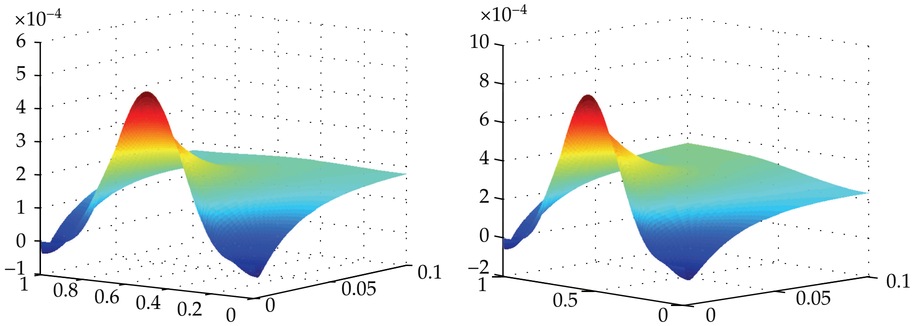
In our computations we fix $N = 32$ and choose five different time-step sizes h_k $(k = 1, 2, \ldots, 5)$. Let Λ_k be the integer with $h_k \Lambda_k = T$. Since we have no exact solution of (2.2) and (2.3), we take $N = 32$ and $h_0 = 0.15625 \times 10^{-4}$ to compute an approximating solution $U_N^{\Lambda_0}$ with $h_0 \Lambda_0 = T$ and regard this as an exact solution. we also choose five different time-step sizes h_k $(k = 1, 2, \ldots, 5)$ with $h_k \Lambda_k = T$ to obtain five approximating solutions $U_N^{\Lambda_k}$ $(k = 1, 2, \ldots, 5)$ and compute the error estimation. Define an error function:

$$\operatorname{err}(T, h_k) = \left(\int_0^1 \left(U_N^{\Lambda_k} - U_N^{\Lambda_0}\right)^2 dx\right)^{1/2}. \tag{4.3}$$

This function characterizes the estimations with respect to time-step size.

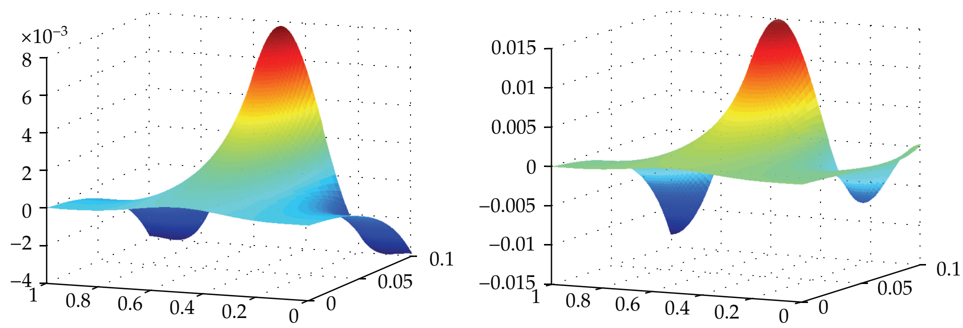
4.1. Example 1

Take $m_0 = 1$ and $T = 0.1$. We also take two different initial functions $u_0^1(x) = x^5(1-x)^5$ and $u_0^2 = x^5(1-x)^5 e^x$ to carry out numerical computations. Figure 1 shows the development of the solutions for time t from $t = 0$ to $t = 0.1$ with fixed step-size $h_0 = 0.15625 \times 10^{-4}$.



(a) (b)

Figure 2: The development of the solution of the full-discrete scheme with initial value $u_0(x) = x^5(1-x)^5$ (a) and $u_0(x) = x^5(1-x)^5 e^x$ (b).



(a) (b)

Figure 3: The development of the solution of the full-discrete scheme with initial value $u_0(x) = x^5(1-x)^5$ (a) and $u_0(x) = x^5(1-x)^5 e^x$ (b).

Table 1: The error and the convergence order.

h_k	$\text{err}_1(0.1, h_k)$	Order_1	$\text{err}_2(0.1, h_k)$	Order_2
0.1×10^{-2}	0.2441×10^{-6}	—	0.2114×10^{-6}	—
0.5×10^{-3}	0.6332×10^{-7}	1.9466	0.5414×10^{-7}	1.9651
0.25×10^{-3}	0.1186×10^{-7}	2.4159	0.1115×10^{-7}	2.2799
0.125×10^{-3}	0.1938×10^{-8}	2.6142	0.1901×10^{-8}	2.5515
0.625×10^{-4}	0.2504×10^{-9}	2.9518	0.2483×10^{-9}	2.9368

Table 2: The error and the convergence order.

h_k	$\text{err}_1(0.1, h_k)$	Order_1	$\text{err}_2(0.1, h_k)$	Order_2
0.1×10^{-2}	0.4748×10^{-8}	—	0.8864×10^{-8}	—
0.5×10^{-3}	0.8348×10^{-9}	2.5077	0.1577×10^{-8}	2.4907
0.25×10^{-3}	0.1370×10^{-9}	2.6069	0.2440×10^{-9}	2.6922
0.125×10^{-3}	0.2694×10^{-10}	2.3467	0.4457×10^{-10}	2.4525
0.625×10^{-4}	0.6398×10^{-11}	2.0741	0.1059×10^{-10}	2.0736

Table 3: The error and the convergence order.

h_k	$\text{err}_1(0.1, h_k)$	Order_1	$\text{err}_2(0.1, h_k)$	Order_2
0.1×10^{-2}	0.1150×10^{-5}	—	0.5224×10^{-5}	—
0.5×10^{-3}	0.2872×10^{-6}	2.0013	0.1304×10^{-5}	2.0019
0.25×10^{-3}	0.7158×10^{-7}	2.0043	0.3250×10^{-6}	2.0044
0.125×10^{-3}	0.1769×10^{-7}	2.0170	0.8031×10^{-7}	2.0171
0.625×10^{-4}	0.4211×10^{-8}	2.0704	0.1912×10^{-7}	2.0704

We also choose five different time-step sizes h_k to carry out numerical computations and apply the error function in (4.3) to illustrate the estimation and convergence order in time variable t, see Table 1.

4.2. Example 2

Take $m_0 = 0.05$ and $T = 0.1$. We also take two different initial functions $u_0^1(x) = x^5(1-x)^5$ and $u_0^2 = x^5(1-x)^5 e^x$ to carry out numerical computations. Figure 2 shows the development of the solutions for time t from $t = 0$ to $t = 0.1$ with fixed step-sizes $h_0 = 0.15625 \times 10^{-4}$.

We also choose five different time-step sizes h_k to carry out numerical computations and apply the error function in (4.3) to illustrate the estimation and convergence order in time variable t, see Table 2.

4.3. Example 3

Take $m_0 = 0.005$ and $T = 0.1$. We also take two different initial functions $u_0^1(x) = x^5(1-x)^5$ and $u_0^2 = x^5(1-x)^5 e^x$ to carry out numerical computations. Figure 3 shows the development of the solutions for time t from $t = 0$ to $t = 0.1$ with fixed step-sizes $h_0 = 0.15625 \times 10^{-4}$.

We also choose five different time-step sizes h_k to carry out numerical computations and apply the error function in (4.3) to illustrate the estimation and convergence order in time variable t, see Table 3.

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