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Research Article

Electromagnetic Gyroscopic Motion

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A problem of the gyroscopic motions around a fixed point, under the action of a gyrostatic moment vector, in presence of electromagnetic field and Newtonian one, is considered. The small parameter technique is used to investigate the periodic solutions for the derived equations of such motion problem. A geometric interpretation of motion will be given in terms of Euler's angles (θ, ψ, ϕ). Computer programs are carried out to integrate the attained quasilinear autonomous system using a fourth-order Runge-Kutta method. A comparison between the obtained analytical solutions and the numerical ones is investigated to calculate the errors between them.

1. Introduction

The problem of motion of a nonsymmetric rigid body rotating around a fixed point, under the action of a central Newtonian field of force exerted by one center of attraction, is considered in [1]. The angular momentum principle is applied to deduce the equations of motion of the body [2]. These equations represent an autonomous system of six nonlinear ordinary differential equations describing the motion of the body [3]. The first integrals for such system are obtained in [4]. Euler, Lagrange and the kinetic symmetry cases are studied in [5–7]. Numerical solutions for this system are obtained using the fourth-order Runge-Kutta method [8]. The influence of the characteristic parameters of the body is obtained in [9] to describe the motion. Two cases of study are given: the first, when the attracting center lies on the vertical downward, and the second, when the attracting center lies on the vertical upward

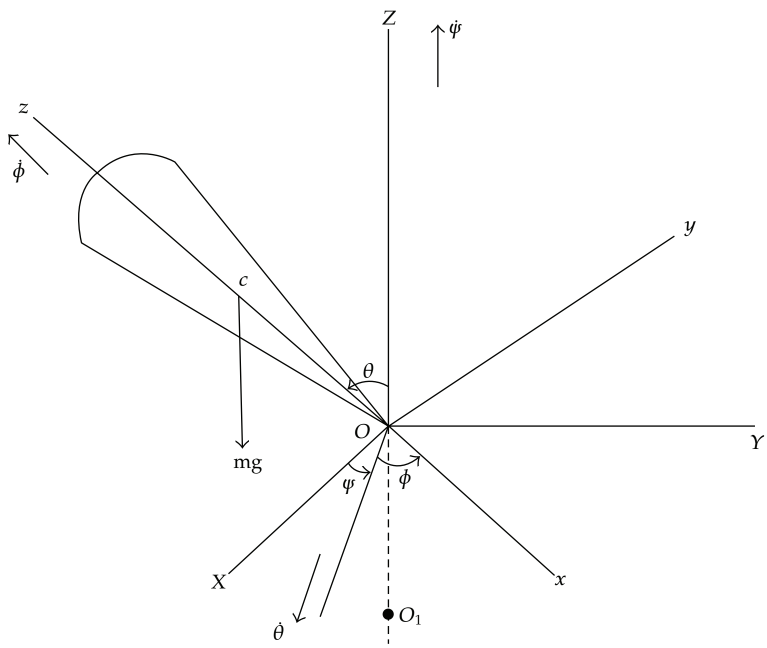

Figure 1: Description of the motion.

[10]. In [5, 6] the author showed that the fourth algebraic integral exists only in two special cases analogous to those of Euler and Lagrange besides the case of kinetic symmetry of the body. The other cases with single-valued integrals are really not new cases but can be reduced to the previous two cases. In [11] the authors study necessary and sufficient conditions for the existence of an additional algebraic integral named, the fourth first integral. In [12], Amer studied the motion of a gyrostat similar to Lagrange's gyroscope under the influence of a gyrostatic moment vector in the uniform gravity field (only the weight mg acted on the body).

In the present study, an electromagnetic gyroscopic motion is considered (in presence of uniform force field, Newtonian one, perturbed torques, and restoring ones) as one of the important problems in mechanics. The importance of this problem is due to its important applications in aeroplanes, space crafts, submarines, and compasses. The aim of this study is to give analytical solutions and numerical ones for such problem. The averaging technique [13] is used to investigate the first order approximate analytical solutions. On the other hand, fourth-order Runge-Kutta method [8] is used to investigate the numerical solutions for the derived system of equations of motion. Errors between both obtained solutions are considered.

2. Formulation of the Problem

Consider a dynamically symmetrical gyro of weight mg acted on its center of mass c directed along a fixed point O in space, (see Figure 1). Two systems of references are achieved: a fixed one $OXYZ$, in such a way that the point O_1 lies in the negative part of axis OZ at a constant distance $R = OO_1$ and another mobile one $Oxyz$ fixed in the body, and whose axes are directed along the principal axes of inertia of the body at O. The gyro was acted upon by the gravity mg; the Newtonian force N due to the center O_1 when the mutual potential V is approximated by $V^{(2)}$ where $V^{(2)}$ represents the second approximation term of V, that is, $V = V^{(0)} + \varepsilon V^{(1)} + \varepsilon^2 V^{(2)}$; the action of a variable restoring torques k_j, $j = 1, 2$ and perturbing torques τ_i, $(i = 1, 2, 3)$. Consider a restoring torque due to the gravity g in the form:

$$k_1 = \mathrm{mg}\ell; \quad \ell = Oc. \tag{2.1}$$

If the gyro rotates around the fixed point in an electromagnetic field of strength $\underline{B}$ ($\underline{B}$ is vertical) and a point charge Q on the axis of symmetry, the restoring torque k_2 comes from the Newtonian field and the Lorentz force $Q(\underline{V} \wedge \underline{B})$ [14], where $\underline{V}$ is the linear velocity vector of the gyro, that is, $\underline{V} = \underline{\omega} \wedge \underline{\ell'}$, $\underline{\ell'} = (0, 0, \ell')$, $\underline{\omega}$ is the angular velocity vector of such gyro, and ℓ' is the position of the point charge Q from the fixed point O. Let k_2 takes the form

$$k_2 = QB\,\ell'^2 \cos\theta \left| (\omega_y, -\omega_x, 0) \right| + N(I_{zz} - I_{xx})\cos\theta, \tag{2.2}$$

where k_2 depends on the components of the angular velocity vector (ω_x, ω_y); the principal moments of inertia I_{xx}, I_{zz}; and on the nutation angle θ. Equation (2.2) represents torques coming from Newtonian electromagnetic field of strength B and a point charge Q locating on the axis of symmetry. Thus the gyro rotates under the force of gravity, the central Newtonian force, and the restoring moments $k = k_1 + k_2$.

Suppose the center of mass of the body and the principal moments of inertia are

$$x_G = y_G = 0, \qquad z_G = \ell, \qquad I_{xx} = I_{yy} \neq I_{zz}. \tag{2.3}$$

The equations of motion take the form

$$\begin{gathered}
I_{xx}\dot{\omega}_x + (I_{zz} - I_{xx})\omega_y\omega_z = k\sin\theta\cos\phi + \tau_1 + \frac{1}{2}N(I_{zz} - I_{xx})\sin 2\theta\cos\phi, \\
I_{xx}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z = -k\sin\theta\sin\phi + \tau_2 + \frac{1}{2}N(I_{zz} - I_{xx})\sin 2\theta\sin\phi, \\
I_{zz}\dot{\omega}_z = \tau_3, \qquad \tau_i = \tau_i(\omega_x, \omega_y, \omega_z, \psi, \theta, \phi, t), \quad i = 1, 2, 3, \\
\dot{\theta} = \omega_x\cos\phi - \omega_y\sin\phi, \\
\dot{\phi} = \omega_z - (\omega_x\sin\phi + \omega_y\cos\phi)\cot\theta, \\
\dot{\psi} = \operatorname{cosec}\theta(\omega_x\sin\phi + \omega_y\cos\phi).
\end{gathered} \tag{2.4}$$

Here (2.4) are two vector equations represented in two groups; the first three equations represent the components of the first vector equation. The last three equations represent the components of the second vector equation. The symbols $(\omega_x, \omega_y, \omega_z)$ and τ_i, $i = 1, 2, 3$ are the projections of the vectors of angular velocity and perturbing torques onto the principal axes of inertia of the body passing through O; I_{xx} and I_{zz} are the equatorial and axial moments of inertia of the body relative to the fixed point; θ, ψ, and ϕ are the Euler's angles and $N = 3\lambda/R$, where λ is the gravitational constant.

The perturbing torques τ_i in (2.4) are assumed to be known functions of their arguments. For $\tau_i = 0$, $i = 1, 2, 3$, (2.4) correspond to the case analogous to that of Lagrange-Poisson [6], and, for $\tau = N = 0$, they give the Lagrange-Poisson case in the uniform gravity field.

Also (2.4), with $N = 0$, describe the motion of Lagrange's top acted upon by perturbations of various physical origin, as well as motion of a free rigid body relative to the center of mass when this body is acted upon by a restoring torque generated by aerodynamics forces and certain perturbing torques.

Consider the following initial assumptions:

$$\omega_x^2 + \omega_y^2 \ll \omega_z^2, \quad \omega_z^2 \gg k, \quad |\tau_i| \ll k, \quad i = 1, 2, \ \tau_3 \approx k. \tag{2.5}$$

The assumptions (2.5) mean that the direction of the angular velocity of the gyro is close to the axis of the dynamic symmetry, and the angular velocity is large, so that the kinetic energy of the gyro is much greater than the potential energy resulting from the restoring torque and two projections of the perturbing torque vector onto the principal axes of inertia of the gyro are small as compared to the restoring torque while the third is of the same order of magnitude as this torque. The assumptions (2.5) allow us to introduce the small parameter ε and to set

$$\begin{gathered} \omega_x = \varepsilon\Omega_x, \qquad \omega_y = \varepsilon\Omega_y, \qquad k = \varepsilon K, \\ \tau_i = \varepsilon^2 \tau_i^*(\omega_x, \omega_y, \omega_z, \psi, \theta, \phi, t), \quad i = 1, 2, \\ \tau_3 = \varepsilon \tau_3^*(\omega_x, \omega_y, \omega_z, \psi, \theta, \phi, t). \end{gathered} \tag{2.6}$$

The new variables Ω_x and Ω_y as well as the variables and constants ω_x, ψ, θ, ϕ, K, I_{xx}, I_{zz}, τ_i^*, $i = 1, 2, 3$ are assumed to be bounded quantities of order unity as ε tends to zero. The aim of this research is to investigate the asymptotic behavior of the solutions of system (2.4), for small ε, when conditions (2.5) and (2.6) are satisfied. This will be done by using the averaging method which is extensively applied in problems of dynamics of rigid bodies on a time interval of order ε^{-1}. This method was employed to investigate a variety of problems of dynamics, chiefly for bodies with dynamic symmetry.

The ensemble of simplifying assumptions (2.5) and (2.6) made in this work enables us to obtain a relatively simple averaging scheme in the general case and to exhaustively investigate the following cases.

2.1. The Case of Variable Restoring Torque

The resultant of restoring torque, K, taking into account (2.5) and (2.6), can be written in the form

$$K = \mathrm{mg}\ell + QB\,\ell'^2 \cos\theta \left|(\omega_y, -\omega_x, 0)\right| + N(I_{zz} - I_{xx})\cos\theta. \tag{2.7}$$

Equation (2.7) is the total restoring torque that comes out from the motion of the gyro under the action of uniform force field, Newtonian one, and the perturbed torques. Making use of (2.4), (2.6), and (2.7) and omitting ε on both sides of the first two equations, one obtains

$$\begin{gathered}
I_{xx}\dot{\Omega}_x + (I_{zz} - I_{xx})\Omega_y\omega_z = K\sin\theta\cos\phi + \varepsilon\tau_1^*, \\
I_{xx}\dot{\Omega}_y + (I_{xx} - I_{zz})\Omega_x\omega_z = -K\sin\theta\sin\phi + \varepsilon\tau_2^*, \\
I_{zz}\dot{\omega}_z = \varepsilon\tau_3^*, \\
\dot{\theta} = \varepsilon(\Omega_x\cos\phi - \Omega_y\sin\phi), \\
\dot{\phi} = \omega_z - \varepsilon(\Omega_x\sin\phi + \Omega_y\cos\phi)\cot\theta, \\
\dot{\psi} = \varepsilon(\Omega_x\sin\phi + \Omega_y\cos\phi)\operatorname{cosec}\theta.
\end{gathered} \tag{2.8}$$

The last four equations in (2.8) for the zero approximation give

$$\omega_z = (\omega_z)_o, \qquad \psi = \psi_o, \qquad \theta = \theta_o, \qquad \phi = (\omega_z)_o t + \phi_o, \tag{2.9}$$

where $(\omega_z)_0$, ψ_0, θ_0, and ϕ_0 are constants equal to the initial values of the corresponding variables for $t = 0$.

Substituting (2.9) into the first two equations of system (2.8) for $\varepsilon = 0$ yields

$$\begin{gathered}
\ddot{\Omega}_x + n_o^2\Omega_x = K_o[n_o - (\omega_z)_o](I_{xx})^{-1}\sin\theta_o\sin[(\omega_z)_o t + \phi_o], \\
\ddot{\Omega}_y + n_o^2\Omega_y = K_o[n_o - (\omega_z)_o](I_{xx})^{-1}\sin\theta_o\cos[(\omega_z)_o t + \phi_o].
\end{gathered} \tag{2.10}$$

Solving system (2.10), one obtains

$$\begin{gathered}
\Omega_x = a\cos\gamma_0 + b\sin\gamma_0 + K_0(I_{zz} - I_{xx})^{-1}(\omega_z)_0^{-1}\sin\theta_0\sin[(\omega_z)_0 t + \phi_0], \\
\Omega_y = a\sin\gamma_0 - b\cos\gamma_0 + K_0(I_{zz} - I_{xx})^{-1}(\omega_z)_0^{-1}\sin\theta_0\cos[(\omega_z)_0 t + \phi_0],
\end{gathered} \tag{2.11}$$

where

$$\begin{gathered}
a = (\Omega_x)_0 - K_0(I_{zz} - I_{xx})^{-1}(\omega_z)_0^{-1}\sin\theta_0\sin\phi_0, \\
b = -(\Omega_y)_0 + K_0(I_{zz} - I_{xx})^{-1}(\omega_z)_0^{-1}\sin\theta_0\cos\phi_0, \\
\gamma_0 = n_0 t, \quad n_0 = (I_{zz} - I_{xx})(I_{xx})^{-1}(\omega_z)_0 \neq 0, \quad \left|\frac{n_0}{(\omega_z)_0}\right| \leq 1.
\end{gathered} \tag{2.12}$$

Here $(\Omega_x)_0$ and $(\Omega_y)_0$ are the initial values of the new variables Ω_x and Ω_y introduced in accordance with (2.6), while γ_o is the oscillation phase of the generating system.

The last condition of (2.12) shows that the initial fast spin of the gyrostat is assumed to be given about the minor axis of the ellipsoid of inertia ($I_{xx} = I_{yy} < I_{zz}$).

System (2.8) is essentially nonlinear and therefore we introduce the additional variable γ defined by the relation

$$\frac{d\gamma}{dt} = n, \qquad \gamma(0) = 0. \tag{2.13}$$

For $\varepsilon = 0$, we have $\gamma = \gamma_o = n_o t$ in accordance with (2.12). Equations (2.9), (2.11) define the general solution of system (2.8) and (2.13) for $\varepsilon = 0$. Eliminating the constants with allowance of (2.9), it is possible to rewrite (2.11) in equivalent form

$$\begin{aligned}\Omega_x &= a\cos\gamma + b\sin\gamma + K_0(I_{zz}-I_{xx})^{-1}(\omega_z)^{-1}\sin\theta\sin\phi,\\ \Omega_y &= a\sin\gamma - b\cos\gamma + K_0(I_{zz}-I_{xx})^{-1}(\omega_z)^{-1}\sin\theta\cos\phi,\end{aligned} \tag{2.14}$$

where a and b are in the form

$$\begin{aligned}a &= \Omega_x\cos\gamma + \Omega_y\sin\gamma - K(I_{zz}-I_{xx})^{-1}(\omega_z)^{-1}\sin\theta\sin(\gamma+\phi),\\ b &= \Omega_y\sin\gamma - \Omega_y\cos\gamma + K_0(I_{zz}-I_{xx})^{-1}(\omega_z)^{-1}\sin\theta\cos(\gamma+\phi),\end{aligned} \tag{2.15}$$

which defines a change of variables Ω_x and Ω_y to variables a and b of Van der Pol type [15] and vice versa. Using (2.8) and (2.13), we convert from the variables $\Omega_x, \Omega_y, \omega_z, \psi, \theta, \phi, \gamma$ to the new variables $a, b, \omega_z, \psi, \theta, \phi, \alpha, \gamma$, where

$$\alpha = \gamma + \phi. \tag{2.16}$$

After some manipulations, we obtain a system of seven equations as follows:

$$\begin{aligned}\dot{a} &= \varepsilon(I_{xx})^{-1}\left[\tau_1^o\cos\gamma + \tau_2^o\sin\gamma\right] - \varepsilon K(I_{zz}\omega_z)^{-1}\cos\theta\left[b - K(I_{zz}\omega_z)^{-1}\sin\theta\cos\alpha\right]\\ &\quad + \varepsilon K(I_{zz}\omega_z)^{-2}\tau_3^o\sin\theta\sin\alpha + \varepsilon\sin\alpha(f_2 - f_1),\\ \dot{b} &= \varepsilon(I_{xx})^{-1}\left[\tau_1^o\sin\gamma - \tau_2^o\cos\gamma\right] + \varepsilon K(I_{zz}\omega_z)^{-1}\cos\theta\left[a - K(I_{zz}\omega_z)^{-1}\sin\theta\sin\alpha\right]\\ &\quad - \varepsilon K(I_{zz}\omega_z)^{-2}\tau_3^o\sin\theta\cos\alpha + \varepsilon\cos\alpha(f_2 - f_1),\\ \dot{\omega}_z &= \varepsilon(I_{zz})^{-1}\tau_3^0,\\ \dot{\psi} &= \varepsilon\operatorname{cosec}\theta(a\sin\alpha - b\cos\alpha) + \varepsilon K(I_{zz}-I_{xx})^{-1}(\omega_z)^{-1},\\ \dot{\theta} &= \varepsilon(a\cos\alpha + b\sin\alpha),\\ \dot{\alpha} &= I_{zz}(I_{xx})^{-1}\omega_z - \varepsilon\cot\theta(a\sin\alpha - b\cos\alpha) + \varepsilon K\cos\theta(I_{zz}-I_{xx})^{-1}(\omega_z)^{-1},\\ \dot{\gamma} &= (I_{zz}-I_{xx})(I_{xx})^{-1}\omega_z,\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}f_1 &= \frac{1}{2}(I_{zz}\omega_z)^{-1}BQ\,\ell'\sin\theta\left[\frac{1}{2}(I_{zz})^{-1}\tau_3^o(3-\cos 2\theta) - \left(\frac{I_{zz}}{I_{xx}} - 1\right)\omega_z\sin 2\theta(a\cos\alpha + b\sin\alpha)\right],\\ f_2 &= (I_{zz}\omega_z)^{-1}\sin^2\theta N(I_{zz}-I_{xx})(a\cos\alpha + b\sin\alpha).\end{aligned} \tag{2.18}$$

τ_i^o denote functions obtained from τ_i^* as a result of substitution of (2.14) into (2.17), that is,

$$\tau_i^o(a, b, \omega_z, \psi, \theta, \alpha, \gamma, t) = \tau_i^*(\Omega_x, \Omega_y, \omega_z, \psi, \theta, \phi, t), \quad i = 1, 2, 3. \tag{2.19}$$

We introduce a vector X whose components are the slow variables a, b, ω_z, ψ, and θ of system (2.17). Thus, this system can be written in the form

$$\begin{gathered} \dot{X} = \varepsilon X(x, \alpha, \gamma, t), \qquad \dot{\alpha} = I_{zz}(I_{xx})^{-1}\omega_z + \varepsilon Y(x, \alpha), \\ \dot{\gamma} = (I_{zz} - I_{xx})(I_{xx})^{-1}\omega_z, \qquad x(0) = x_o, \qquad \alpha(0) = \alpha_o, \qquad \gamma(0) = 0, \end{gathered} \tag{2.20}$$

where the vector-valued function X and the scalar function Y are defined by the right-hand sides of (2.17). The initial values of X and Y can be obtained in accordance with (2.9) to (2.13) and (2.16). Consider system (2.17) or (2.20) from the stand point of employing the averaging method. System (2.17) contains the slow variables a, b, ω_z, ψ, and θ and fast variables represented by the phases α, γ and time t. This system is essentially nonlinear and it is extremely difficult to employ the averaging method directly. Let us assume, for the sake of simplicity, that the perturbing torques τ_i^* are independent of t. Since τ_i, $i = 1, 2, 3$, are 2π-periodic in ϕ, it follows, in accordance with (2.14) to (2.17), that the functions τ_i^* from (2.19) will be 2π-periodic functions of α and γ. Then system (2.20) contains two rotating phases α and γ and two corresponding frequencies $I_{zz}(I_{xx})^{-1}\omega_z$ and $(I_{zz} - I_{xx})(I_{xx})^{-1}\omega_z$.

In averaging system (2.17) or (2.20), two cases should be distinguished.

(1) A nonresonant case, when frequencies $I_{zz}(I_{xx})^{-1}\omega_z$ and $(I_{zz} - I_{xx})(I_{xx})^{-1}\omega_z$ are noncommensurable.

(2) A resonant case, when these frequencies are commensurable.

A very important feature of system (2.20) is the fact that the ratio of the frequencies is constant $(I_{zz} - I_{xx})(I_{xx})^{-1}\omega_z / I_{zz}(I_{xx})^{-1}\omega_z = 1 - I_{xx}(I_{zz})^{-1}$ and the resonant case occurs for

$$\frac{I_{zz}}{I_{xx}} = \frac{i}{j}, \quad \frac{i}{j} \leq 2, \tag{2.21}$$

where i and j are relatively prime natural numbers, while in the non-resonant case I_{zz}/I_{xx} is an irrational number.

As a result of (2.21), averaging of nonlinear system (2.20), in which X is independent of t, is equivalent to averaging of a quasilinear system with constant frequencies; this can be achieved by introducing the independent variable γ. In the non-resonant case $I_{zz}/I_{xx} \neq i/j$, we obtain the first approximation averaged system by averaging the right sides of system (2.17) with respect to the fast variables α and γ. As a result, we obtain the following equations for the slow variables:

$$\begin{gathered} \dot{a} = \varepsilon(I_{xx})^{-1}\mu_1 - \varepsilon K(I_{zz}\omega_z)^{-1} b\cos\theta + \varepsilon K(I_{zz}\omega_z)^{-2}\sin\theta\mu_3^s + \frac{1}{2}\varepsilon\sin\theta\mu^k, \\ \dot{b} = \varepsilon(I_{xx})^{-1}\mu_2 + \varepsilon K(I_{zz}\omega_z)^{-1} a\cos\theta - \varepsilon K(I_{zz}\omega_z)^{-2}\sin\theta\mu_3^o - \frac{1}{2}\varepsilon\sin\theta\mu^{k1}, \\ \dot{\omega}_z = \varepsilon(I_{zz})^{-1}\ \mu_3, \qquad \dot{\psi} = \varepsilon K(I_{zz}\omega_z)^{-1}, \qquad \dot{\theta} = 0, \end{gathered} \tag{2.22}$$

where

$$
\begin{gathered}
\mu_1 = \frac{1}{4\pi^2}\int\int_0^{2\pi}\left[\tau_1^o \cos\gamma + \tau_2^o \sin\gamma\right] d\alpha d\gamma, \\
\mu_2 = \frac{1}{4\pi^2}\int\int_0^{2\pi}\left[\tau_1^o \sin\gamma - \tau_2^o \cos\gamma\right] d\alpha d\gamma, \\
\mu_3 = \frac{1}{4\pi^2}\int\int_0^{2\pi}\tau_3^o d\alpha d\gamma, \qquad \mu_3^s = \frac{1}{4\pi^2}\int\int_0^{2\pi}\tau_3^o \sin\alpha\, d\alpha d\gamma, \\
\mu_3^o = \frac{1}{4\pi^2}\int\int_0^{2\pi}\tau_3^o \cos\alpha\, d\alpha d\gamma, \\
\begin{aligned}
\mu^k &= (I_{zz}\omega_z)^{-1} N(I_{zz} - I_{xx}) b \sin\theta - \frac{1}{2}(I_{zz})^{-2}(\omega_z)^{-1} BQ\,\ell'(3 - \cos 2\theta)\mu_3^s \\
&\quad + \frac{1}{2}\left[(I_{xx})^{-1} - (I_{zz})^{-1}\right] BQ\,\ell' b \sin 2\theta, \\
\mu^{k1} &= (I_{zz}\omega_z)^{-1} N(I_{zz} - I_{xx}) a \sin\theta - \frac{1}{2}(I_{zz})^{-2}(\omega_z)^{-1} BQ\,\ell'(3 - \cos 2\theta)\mu_3^o \\
&\quad + \frac{1}{2}\left[(I_{xx})^{-1} - (I_{zz})^{-1}\right] BQ\,\ell' a \sin 2\theta.
\end{aligned}
\end{gathered}
\tag{2.23}
$$

Solving averaged system (2.22) for perturbing torques of specific form, we determine the motion of the gyrostat in the non-resonant case with an error of order ε on an interval of time variation of order ε^{-1}.

The integration of the last equation of system (2.22) yields $\theta = \theta_0 = \text{const}$. System (2.22) is equivalent to a two-frequency system with constant frequencies, since both frequencies are proportional to the axial component ω_z of the angular velocity vector; therefore, the applicability of the averaging method can be substantiated in the same way as for a quasilinear system; the principal assertion involves the following.

Assume that the function X is sufficiently smooth with respect to α and γ and that it satisfies a Lipschitz condition with respect to x , with a constant which is independent of α and γ. Then on the plane of permissible values of the parameters I_{zz} and I_{xx} there exists a set L of measure zero such that if $I_{zz}, I_{xx} \in L$, then for the solutions of system (2.20) and (2.22) we have the bound $|x(t,\varepsilon) - \zeta(t,\varepsilon)| \le R^*\varepsilon$, $t \in [0, O\varepsilon^{-1}]$ in which $\zeta(t,\varepsilon)$ is the solution of system (2.22) averaged with respect to the phases α and γ, where $\zeta = (a, b, \omega_z, \psi, \theta)$ and $R^* = \text{const}$. The proof can be carried out by using Gronwall's lemma, on the basis of the standard change of variable procedure of the averaging method, as well as the arithmetic lemma used to estimate the "small denominators" [16].

System (2.20) is a single frequency system in the resonant case (2.21). Indeed, instead of α we introduce a new slow variable, namely, a linear combination of the phases with coefficients

$$
\lambda = \alpha - i\gamma(i - j)^{-1}, \quad \left(\frac{i}{j}\right) \neq 1, \quad \left(\frac{i}{j}\right) \le 2; \quad i, j > 0. \tag{2.24}
$$

System (2.20) gives the following form of a standard system with a rotating phase

$$
\begin{aligned}
\dot{X} &= \varepsilon X\left(x, i\gamma(i-j)^{-1} + \lambda\ , \gamma\right), \\
\dot{\lambda} &= \varepsilon Y\left(x, i\gamma(i-j)^{-1} + \lambda\right), \\
\dot{\gamma} &= (I_{zz} - I_{xx})(I_{xx})^{-1}\omega_z,
\end{aligned}
\tag{2.25}
$$

and its right sides are $(2|i-j|\pi)$, periodic in γ. We set up first approximation system by averaging the right sides of system (2.25) with respect to the above period of variation of the argument γ. As a result, we obtain the following system of equations for the slow variables:

$$
\begin{aligned}
\dot{a} &= \varepsilon(I_{xx})^{-1}\,\mu_1^* - \varepsilon K(I_{zz}\omega_z)^{-1}\,b\cos\theta + \varepsilon K(I_{zz}\omega_z)^{-2}\ \sin\theta\mu_3^{*s} + \frac{1}{2}\varepsilon\sin\theta\mu^{*k}, \\
\dot{b} &= \varepsilon(I_{xx})^{-1}\,\mu_2^* + \varepsilon K(I_{zz}\omega_z)^{-1} a\cos\theta - \varepsilon K(I_{zz}\omega_z)^{-2}\sin\theta\mu_3^{*o} - \frac{1}{2}\varepsilon\sin\theta\mu^{*k1}, \\
\dot{\omega}_z &= \varepsilon(I_{zz})^{-1}\,\mu_3^*, \qquad \dot{\psi} = \varepsilon K(I_{zz}\omega_z)^{-1}\ , \qquad \dot{\theta} = 0, \qquad \dot{\lambda} = -\varepsilon K(I_{zz}\omega_z)^{-1}\ \cos\theta,
\end{aligned}
\tag{2.26}
$$

where

$$
\begin{aligned}
\mu_1^*(a,b,\omega_z,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}\left[\tau_1^o\cos\gamma + \tau_2^o\sin\gamma\right]d\gamma, \\
\mu_2^*(a,b,\omega_z,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}\left[\tau_1^o\sin\gamma + \tau_2^o\cos\gamma\right]d\gamma, \\
\mu_3^*(a,b,\omega_z,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}\tau_3^o d\gamma, \\
\mu_3^{*s}(a,b,\omega_z,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}\tau_3^o\sin\left[\lambda + i\gamma(i-j)^{-1}\right]d\gamma, \\
\mu_3^{*o}(a,b,\omega_z,\psi,\theta,\lambda) &= \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|}\tau_3^o\cos\left[\lambda + i\gamma(i-j)^{-1}\right]d\gamma, \\
\mu^{*k}(a,b,\omega_z,\psi,\theta,\lambda) &= (I_{zz}\omega_z)^{-1}N(I_{zz}-I_{xx})b\sin\theta - \frac{1}{2}(I_{zz})^{-2}(\omega_z)^{-1}BQ\,\ell'(3-\cos 2\theta)\mu_3^{*s} \\
&\quad + \frac{1}{2}\left[(I_{xx})^{-1} - (I_{zz})^{-1}\right]BQ\,\ell' b\sin 2\theta, \\
\mu^{*k1}(a,b,\omega_z,\psi,\theta,\lambda) &= (I_{zz}\omega_z)^{-1}N(I_{zz}-I_{xx})a\sin\theta - \frac{1}{2}(I_{zz})^{-2}(\omega_z)^{-1}BQ\,\ell'(3-\cos 2\theta)\mu_3^{*o} \\
&\quad + \frac{1}{2}\left[(I_{xx})^{-1} - (I_{zz})^{-1}\right]BQ\,\ell' b\sin 2\theta.
\end{aligned}
\tag{2.27}
$$

Therefore, the motion of the gyrostat in the resonant case can be substantiated.

3. The Case of the Sum of Constant and Linear Dissipative Perturbed Torques

Let us consider a perturbed motion analogous to that of Lagrange case with allowance for the torques acting on our gyrostat from the environment. We will assume that the perturbing torques τ_i, $i = 1, 2, 3$, are linear dissipative

$$\tau_1 = -\varepsilon I_1 \omega_x, \qquad \tau_2 = -\varepsilon I_1 \omega_y, \quad \tau_3 = -\varepsilon I_3 \omega_z + \varepsilon \tau_3^*; \quad I_1, I_3 > 0 \tag{3.1}$$

where I_1 and I_3 are constants depending on the properties of the medium and the shape of the gyro. Let us write the perturbing torques with allowance for expressions (2.6) for ω_x and ω_y

$$\tau_1 = -\varepsilon^2 I_1 \Omega_x, \qquad \tau_2 = -\varepsilon^2 I_1 \Omega_y, \quad \tau_3 = -\varepsilon I_3 \omega_z + \varepsilon \tau_3^*; \quad I_1, I_3 > 0, \tag{3.2}$$

For the fundamental oscillations (nonresonant case), we change over to new slow variables a, b, ω_z, ψ, and θ, the averaged system (2.22) takes the form

$$\begin{aligned}
\dot{a} &= \varepsilon a I_1 (I_{xx})^{-1} - \varepsilon b (I_{zz}\omega_z)^{-1} K \cos\theta \\
&\quad - \frac{1}{2} b\varepsilon \sin\theta \left\{ (I_{zz}\omega_z)^{-1} N \sin\theta (I_{xx} - I_{zz}) + \frac{1}{2} BQ\,\ell' \sin 2\theta \left[(I_{zz})^{-1} - (I_{xx})^{-1} \right] \right\}, \\
\dot{b} &= -\varepsilon b I_1 (I_{xx})^{-1} + \varepsilon a (I_{zz}\omega_z)^{-1} K \cos\theta \\
&\quad + \frac{1}{2} a\varepsilon \sin\theta \left\{ (I_{zz}\omega_z)^{-1} N \sin\theta (I_{xx} - I_{zz}) + \frac{1}{2} BQ\,\ell' \sin 2\theta \left[(I_{zz})^{-1} - (I_{xx})^{-1} \right] \right\}, \\
&\dot{\omega}_z = -\varepsilon (I_{zz})^{-1} \left(I_3 \omega_z - \tau_3^* \right), \qquad \dot{\psi} = \varepsilon K (I_{zz}\omega_z)^{-1}, \qquad \dot{\theta} = 0.
\end{aligned} \tag{3.3}$$

Integrating the third equation in (3.3), we obtain

$$\omega_z = \left[(\omega_z)_o - \tau_3^* I_3^{-1} \right] \exp\left[-\varepsilon (I_{zz})^{-1} I_3 t \right] + \tau_3^* I_3^{-1}. \tag{3.4}$$

Equation (3.3) for ψ can be integrated with allowance for (3.4) to yield

$$\psi = \psi_o + K\left(\tau_3^*\right)^{-1} \ln \left| \frac{1 + \left[(\omega_z)_o I_3 \tau_3^{*-1} - 1 \right] \exp\left[-\varepsilon (I_{zz}\ )^{-1} I_3 t \right]}{(\omega_z)_o I_3 \tau_3^{*-1}} \right| + \varepsilon K I_3 \left(I_{zz} \tau_3^* \right)^{-1} t. \tag{3.5}$$

From the last equation of (3.3), it is easy to see that the angle of nutation maintains constant value, that is,

$$\theta = \theta_o. \tag{3.6}$$

Making use of (3.3), (3.4), (3.5) and (3.6), one obtains

$$\begin{aligned}
a = & -bI_{xx}I_3K_1\left(I_{zz}I_1\tau_3^*\right)^{-1} - \frac{1}{2}BQ\,\ell' bI_1^{-1}\left[I_{xx}(I_{zz})^{-1} - 1\right]\sin^2\theta_o\cos\theta_o \\
& - \varepsilon^2 bI_3(I_{zz})^{-1}\tau_3^{*-1}K_1\left[I_1(I_{xx})^{-1} + (I_{zz})^{-1}\tau_3^*(\omega_z)_o^{-1}\right] + \chi_1\exp\left[-\varepsilon I_1(I_{xx})^{-1}t\right], \\
b = & -\frac{1}{2}I_1^{-1}BQ\,\ell'\left[1 - I_{xx}(I_{zz})^{-1}\right]\sin^2\theta_o\cos\theta_o + I_{xx}(\omega_z)_o^{-1}a(I_{zz})^{-1}K_1 \\
& - \varepsilon^2 a(\omega_z)_o^2(I_{zz})^{-2}\left[(\omega_z)_oI_3 - \tau_3^*\right]K_1 + \chi_2\exp\left[-\varepsilon I_1(I_{xx})^{-1}t\right],
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
K_1 &= \left[K + N(I_{xx} - I_{zz})\sin\theta_o\right]\cos\theta_o, \\
\chi_1 &= (\Omega_x)_o - K_o(I_{zz})^{-1}(\omega_z)_o^{-1}\sin\theta_o\sin\phi_o - I_1^{-1}(I_{zz})^{-1} \\
&\quad \times \left[\left(\Omega_y\right)_o - K_o(I_{zz})^{-1}(\omega_z)_o^{-1}\sin\theta_o\cos\phi_o\right]\left[AI_3K_1\tau_3^{*-1} + \frac{1}{2}BQ\,\ell'(I_{xx} - I_{zz})\sin^2\theta_o\cos\theta_o\right], \\
\chi_2 &= -\left(\Omega_y\right)_o + K_o(I_{zz})^{-1}(\omega_z)_o^{-1}\sin\theta_o\cos\phi_o - I_1^{-1}(I_{zz})^{-1} \\
&\quad \times \left[(\Omega_x)_o - K_o(I_{zz})^{-1}(\omega_z)_o^{-1}\sin\theta_o\sin\phi_o\right]\left[\frac{1}{2}BQ\,\ell'(I_{xx} - I_{zz})\sin^2\theta_o\cos\theta_o + I_{xx}(\omega_z)_o^{-1}K_1\right].
\end{aligned} \tag{3.8}$$

4. Discussion of the Solutions

In this section we give a qualitative analysis of the results obtained, several diagrams, and explanations.

The solutions of the first approximation system for the slow variables in the case of dissipative torque (3.1) are constructed. If resonance relation (2.21) is satisfied, then averaging should be performed in accordance with scheme (2.26). In this case, all the integrals μ_i^* from (2.26) coincide with the corresponding integrals μ_i of (2.22). Therefore resonance in effect does not accrue and the resultant solution is suitable for describing motion for any ratio $I_{zz}/I_{xx} \neq 1$. The motion considered in this paper is interpreted by obtaining Euler's angles of nutation θ, precession ψ, and pure rotation ϕ. We conclude from (3.6) and (3.5) that the nutation angle θ remains constant through the motion, while the precession angle ψ depends on time t. For zero-order approximation of ε, we note that

$$\dot{\theta} = 0, \qquad \dot{\psi} = 0, \qquad \dot{\phi} = (\omega_z)_o, \tag{4.1}$$

that is, the case of permanent rotation with fast spin r_o of the gyro about its axis of symmetry is obtained (see Figure 2).

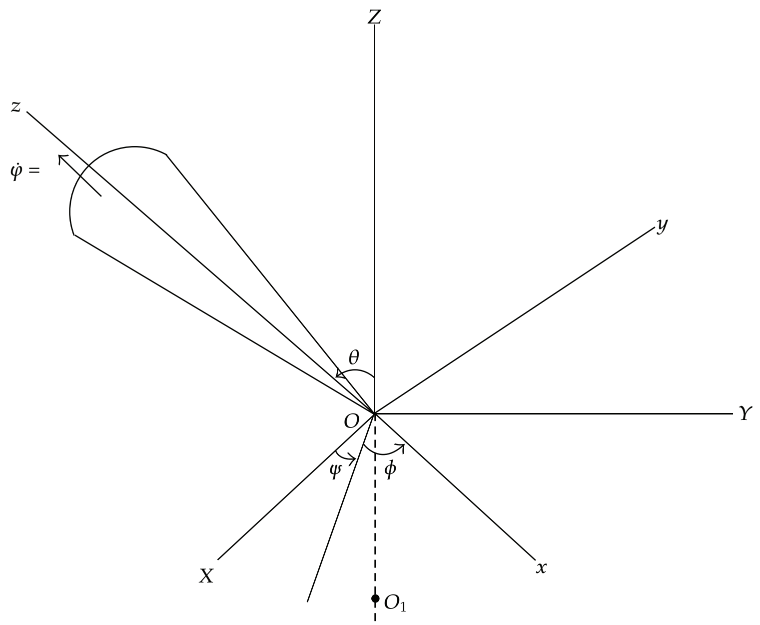

Figure 2: Euler's angles.

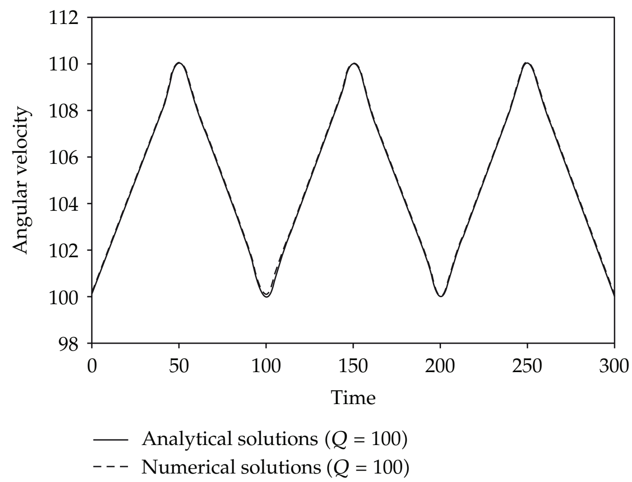

Figure 3: The angular velocity ω against time when $Q = 100$.

5. Numerical Results

The fourth-order Runge-Kutta method [8] is used through a computer program to investigate the numerical solutions for the derived system (2.4) of equations of motion. The angular velocity $\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ obtained from the numerical solutions is represented graphically, in the form of dashed curves, against the time t with different values of the point charge $Q = 100$ and 200 gauss and different initial values of the nutation angle. On the other hand the angular velocity ω obtained analytically from the averaging technique is graphed through continuous curves. Both dashes and continuous curves are given together in Figures 3 and 4.

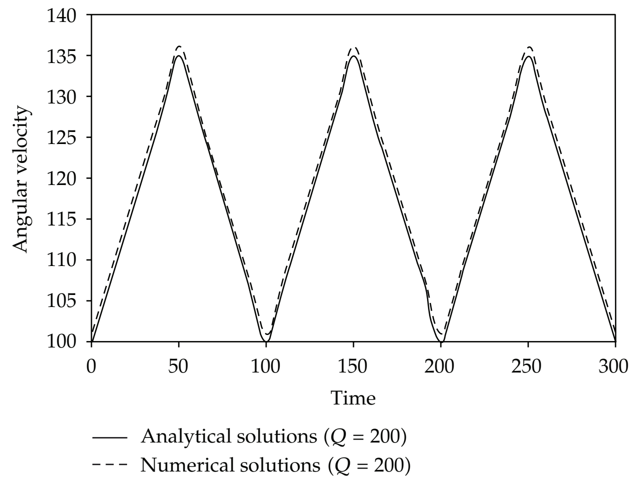

Figure 4: The angular velocity ω against time when $Q = 200$.

From these figures we conclude that the angular velocity ω increases when the point charge Q increases and vice versa and also that the analytical solutions are very close to the numerical ones especially when the charge Q is small (that is the errors between the analytical and the numerical solutions are negligible).

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References

[1] C. Rigby, *Geometry of Rotation: Understanding the Dynamics of Rigid Bodies*, DASET Publishing and Training, 2005.

[2] A. A. Burov, "The "restricted" formulation of the problem of the motion of a heavy rigid body," *Journal of Applied Mathematics and Mechanics*, vol. 68, no. 6, pp. 857–861, 2004.

[3] A. I. Ismail, "The motion of fast spinning rigid body about a fixed point with definite natural frequency," *Aerospace Science and Technology*, vol. 1, no. 3, pp. 183–190, 1997.

[4] A. A. Burov, "First integrals in the problem of the motion of a heavy rigid body about a fixed point under a unilateral constraint," *Journal of Applied Mathematics and Mechanics*, vol. 69, pp. 195–198, 2005.

[5] A. D. Bryuno, "Analysis of the Euler-Poisson equations by the methods of power geometry and the normal form," *Journal of Applied Mathematics and Mechanics*, vol. 71, pp. 168–199, 2007.

[6] T. S. Amer, "Motion of a rigid body analogous to the case of Euler and Poinsot," *Analysis*, vol. 24, no. 4, pp. 305–315, 2004.

[7] F. E. Udwadia and R. E. Kalaba, *Analytical Dynamics*, Cambridge University Press, Cambridge, UK, 1996.

[8] D. Faires and R. L. Burden, *Numerical Methods*, Thomson/Brooks-Cole, 2002.

[9] F. A. El-Barki and A. I. Ismail, "Limiting case for the motion of a rigid body about a fixed point in the Newtonian force field," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 75, no. 11, pp. 821–829, 1995.

[10] A. Rao, *Dynamics of Particles and Rigid Bodies: A Systematic Approach*, Cambridge University Press, 2005.

[11] A. I. Ismail and T. S. Amer, "A necessary and sufficient condition for solving a rigid body problem," *Technische Mechanik*, vol. 31, no. 1-2, pp. 50–57, 2011.

[12] T. S. Amer, "On the motion of a gyrostat similar to Lagrange's gyroscope under the influence of a gyrostatic moment vector," *Nonlinear Dynamics*, vol. 54, no. 3, pp. 249–262, 2008.

[13] J. A. Sanders, F. Verhulst, and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, vol. 59 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 2007.

[14] H. William and T. J. Hay, *Engineering Electromagnetics*, McGraw-Hill, 1989.

[15] Yu. A. Mitropolskii and Nguyen Van Dao, *Applied Asymptotic Methods in Nonlinear Oscillations*, vol. 55 of *Solid Mechanics and Its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 1997.

[16] V. I. Arnold, *Ordinary Differential Equations*, Universitext, Springer, Berlin, Germany, 2006.