Research Article

Chover-Type Laws of the Iterated Logarithm for Continuous Time Random Walks

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A continuous time random walk is a random walk subordinated to a renewal process used in physics to model anomalous diffusion. In this paper, we establish Chover-type laws of the iterated logarithm for continuous time random walks with jumps and waiting times in the domains of attraction of stable laws.

1. Introduction

Let $\{Y_i, J_i\}$ be a sequence of independent and identically distributed random vectors, and write $S(n) = Y_1 + Y_2 + \cdots + Y_n$ and $T(n) = J_1 + J_2 + \cdots + J_n$. Let $N_t = \max\{n \geq 0 : T(n) \leq t\}$ the renewal process of J_i. A *continuous time random walk* (CTRW) is defined by

$$X(t) = S(N_t) = \sum_{i=1}^{N_t} Y_i. \tag{1.1}$$

In this setting, Y_i represents a particle jump, and $J_i > 0$ is the waiting time preceding that jump, so that $S(n)$ represents the particle location after n jumps and $T(n)$ is the time of the nth jump. Then N_t is the number of jumps by time $t > 0$, and the CTRW $X(t)$ represents the particle location at time $t > 0$, which is a random walk subordinated to a renewal process.

It should be mentioned that the subordination scheme of CTRW processes is going back to Fogedby [1] and that it was expanded by Baule and Friedrich [2] and Magdziarz et al. [3]. It should also be mentioned that the theory of subordination holds for nonhomogeneous CTRW processes, that were introduced in the following works: Metzler et al. [4, 5] and Barkai et al. [6].

The CTRW is useful in physics for modeling anomalous diffusion. Heavy-tailed particle jumps lead to superdiffusion, where a cloud of particles spreads faster than the classical Brownian motion, and heavy-tailed waiting times lead to subdiffusion. CTRW models and the associated fractional diffusion equations are important in applications to physics, hydrology, and finance; see, for example, Berkowitz et al. [7], Metzler and Klafter [8], Scalas [9], and Meerchaert and Scalas [10] for more information. In applications to hydrology, the heavy tailed particle jumps capture the velocity irregularities caused by a heterogeneous porous media, and the waiting times model particle sticking or trapping. In applications to finance, the particle jumps are price changes or log returns, separated by a random waiting time between trades.

If the jumps Y_i belong to the domain of attraction of a stable law with index α, $(0 < \alpha < 2)$, and the waiting times J_i belong to the domain of attraction of a stable law with index β, $(0 < \beta < 1)$, Becker-Kern et al. [11] and Meerschaert and Scheffler [12] showed that as $c \to \infty$,

$$c^{-\beta/\alpha} X([ct]) \Longrightarrow A(E(t)) \tag{1.2}$$

a non-Markovian limit with scaling $A(E(ct)) \overset{\mathrm{d}}{=} c^{\beta/\alpha} A(E(t))$, where $A(t)$ is a stable Lévy motion and $E(t)$ is the inverse or hitting time process of a stable subordinator. Densities of the CTRW scaling limit $A(E(t))$ solve a space-time fractional diffusion equation that also involves a fractional time derivative of order β; see Meerschaert and Scheffler [13], Becker-Kern et al. [11], and Meerschaert and Scheffler [12] for complete details. Becker-Kern et al. [14], Meerschaert and Scheffler [15], and Meerschaert et al. [16] discussed the related limit theorems for CTRWs based on two time scales, triangular arrays and dependent jumps, respectively. The aim of the present paper is to investigate the laws of the iterated logarithm for CTRWs. We establish Chover-type laws of the iterated logarithm for CTRWs with jumps and waiting times in the domains of attraction of stable laws.

Throughout this paper we will use C to denote an unspecified positive and finite constant which may be different in each occurrence and use "i.o." to stand for "infinitely often" and "a.s." to stand for "almost surely" and "$u(x) \sim v(x)$" to stand for "$\lim u(x)/v(x) = 1$". Our main results read as follows.

Theorem 1.1. *Let $\{Y_i\}$ be a sequence of i.i.d. nonnegative random variables with a common distribution F, and let $\{J_i\}$, independent of $\{Y_i\}$, be a sequence of i.i.d. nonnegative random variables with a common distribution G. Assume that $1 - F(x) \sim x^{-\alpha} L(x)$, $0 < \alpha < 2$, where L is a slowly varying function, and that G is absolutely continuous and $1 - G(x) \sim C x^{-\beta}$, $0 < \beta < 1$. Let $\{B(n)\}$ be a sequence such that $nL(B(n))/B(n)^{\alpha} \to C$ as $n \to \infty$. Then one has*

$$\limsup_{t \to \infty} \left(\left(B\left(t^{\beta}\right) \right)^{-1} X(t) \right)^{1/(\log \log t)} = e^{1/\alpha} \quad a.s. \tag{1.3}$$

The following is an immediate consequence of Theorem 1.1.

Corollary 1.2. *If the tail distribution of Y_i satisfies $P(Y_1 > x) \sim C x^{-\alpha}$ in Theorem 1.1, then one has*

$$\limsup_{t \to \infty} \left(t^{-\beta/\alpha} X(t) \right)^{1/(\log \log t)} = e^{1/\alpha} \quad a.s. \tag{1.4}$$

In the course of our arguments we often make statements that are valid only for sufficiently large values of some index. When there is no danger of confusion, we omit explicit mention of this proviso.

2. Chung Type LIL for Stable Summands

In this section we consider a Chung-type law of the iterated logarithm for sums of random variables in the domain of attraction of a stable law, which will take a key role to show Theorem 1.1. When J_i has a symmetric stable distribution function G characterized by

$$E \exp(it J_i) = \exp\left(-|t|^\beta\right) \quad \text{for } t \in \mathbb{R}, \tag{2.1}$$

$0 < \beta < 2$. Chover [17] established that

$$\limsup_{n \to \infty} \left| n^{-1/\beta} T(n) \right|^{1/(\log \log n)} = e^{1/\beta} \quad \text{a.s.} \tag{2.2}$$

We call (2.2) as Chover's law of the iterated logarithm. Since then, several papers have been devoted to develop Chover's LIL; see, for example, Hedye [18–20], Pakshirajan and Vasudeva [21], Vasudeva [22], Qi and Cheng [23], Scheffler [24], Chen [25], and Peng and Qi [26] for reference. For some reason the obvious corresponding statement for the "lim inf" result does not seem to have been recorded, and it is the purpose of this section to do so and may be of independent interest.

Theorem 2.1. *Let $\{J_i\}$ be a sequence of i.i.d. nonnegative random variables with a common distribution $G(x)$, and let $V(x) = \inf\{y > 0 : 1 - G(y) \le 1/x\}$. Assume that G is absolutely continuous and $1 - G(x) \sim x^{-\beta} l(x)$, $0 < \beta < 1$, where l is a slowly varying function. Then one has*

$$\liminf_{n \to \infty} \left(V(n)^{-1} T(n) \right)^{1/(\log \log n)} = 1 \quad \text{a.s.} \tag{2.3}$$

In order to prove Theorem 2.1, we need some lemmas.

Lemma 2.2. *Let $h(x)$ be a slowly varying function. Then, if $y_n \to \infty$, $z_n \to \infty$, one has for any given $\tau > 0$,*

$$\lim z_n^{-\tau} \frac{h(y_n z_n)}{h(y_n)} = 0, \qquad \lim z_n^{\tau} \frac{h(y_n z_n)}{h(y_n)} = \infty. \tag{2.4}$$

Proof. See Seneta [27]. □

Lemma 2.3. *Let $\{J_i\}$ be a sequence of i.i.d. nonnegative random variables with a common distribution G and let $M(n) = \max\{J_1, J_2, \ldots, J_n\}$. Assume that G is absolutely continuous and $1 - G(x) \sim x^{-\beta} l(x)$, $0 < \beta < 1$, where l is a slowly varying function. Then one has for some given small $t > 0$*

$$\lim_{n \to \infty} E e^{t T(n)/M(n)} = \frac{e^t}{1 - t \int_0^1 e^{tx} (x^{-\beta} - 1) dx}. \tag{2.5}$$

Proof. We will follow the argument of Lemma 2.1 in Darling [28]. Without loss of generality we can assume $J_1 = \max\{J_1, J_2, \ldots, J_n\} = M(n)$ since each J_i has a probability of $1/n$ of being the largest term, and $P(J_i = J_j) = 0$ for $i \neq j$ since $G(x)$ is presumed continuous.

For notational simplicity we will use the tail distribution $\overline{G}(x) = 1 - G(x) = P(J_1 > x)$ and denote by $g(x)$ the corresponding density, so that $\overline{G}(x) = \int_x^\infty g(z)dz$. Then, the joint density of $J_1, J_2, \ldots, J_n$, given $J_1 = M(n)$, is

$$
g(x_1, x_2, \ldots, x_n) = \begin{cases} ng(x_1)g(x_2)\cdots g(x_n) & \text{if } x_1 = \max_i \{x_i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}
$$

Thus

$$
\begin{aligned}
Ee^{tT(n)/M(n)} &= \int\int\cdots\int e^{t(x_1+x_2+\cdots+x_n)/x_1} g(x_1, x_2, \ldots, x_n)dx_1 dx_2 \cdots dx_n \\
&= ne^t \int_0^\infty \int_0^y \cdots \int_0^y e^{t(x_2+x_3+\cdots+x_n)/y} g(x_2)g(x_3)\cdots g(x_n)g(y)dx_2 dx_3 \cdots dx_n dy \\
&= ne^t \int_0^\infty \left\{ \int_0^y e^{tx/y} g(x)dx \right\}^{n-1} g(y)dy.
\end{aligned}
\tag{2.7}
$$

Let us put

$$
\phi(y, t) = y \int_0^1 e^{tx} g(xy)dx \tag{2.8}
$$

so that

$$
Ee^{tT(n)/M(n)} = ne^t \int_0^\infty (\phi(y, t))^{n-1} g(y)dy. \tag{2.9}
$$

It follows from Doeblin's theorem that if $\lambda > 0$,

$$
\overline{G}(\lambda y) = \lambda^{-\beta}\overline{G}(y)(1 + o(1)) \tag{2.10}
$$

for $y \geq y_0$ with some large $y_0 > 0$. Then, for $y \leq y_0$, we can choose $t > 0$ small enough such that $t < -\log G(y_0)$ since G has regularly varying tail distribution, so that

$$
\phi(y, t) \leq e^t G(y_0) < 1. \tag{2.11}
$$

It follows that

$$
ne^t \int_0^{y_0} (\phi(y, t))^{n-1} g(y)dy \longrightarrow 0. \tag{2.12}
$$

Consider the case $y \geq y_0$. By a slight transformation we find that

$$\phi(y, t) = 1 - \overline{G}(y) + t \int_0^1 e^{tx} \left(\overline{G}(xy) - \overline{G}(y) \right) dx$$

$$= 1 - \overline{G}(y) + t\overline{G}(y)(1 + o(1)) \int_0^1 e^{tx} \left(x^{-\beta} - 1 \right) dx. \tag{2.13}$$

Putting

$$\eta = \eta(t) = t \int_0^1 e^{tx} \left(x^{-\beta} - 1 \right) dx, \tag{2.14}$$

we have $\eta < 1$ since $0 < \beta < 1$ and t is small. Thus

$$\phi(y, t) = 1 - \overline{G}(y)(1 - \eta) + o\left(\overline{G}(y) \right). \tag{2.15}$$

By (2.9) and making the change of variable $n\overline{G}(y) = v$ to give

$$E e^{tT(n)/M(n)} = e^t \int_0^n \left(1 - \frac{v}{n}(1 - \eta) + vo\left(\frac{1}{n} \right) \right)^{n-1} dv \longrightarrow e^t \int_0^\infty e^{-v(1-\eta)} dv$$

$$= \frac{e^t}{1 - \eta}, \tag{2.16}$$

which yields the desired result. □

The following large deviation result for stable summands is due to Heyde [19].

Lemma 2.4. *Let $\{\xi_i\}$ be a sequence of i.i.d. nonnegative random variables with a common tail distribution satisfying $P(\xi_1 > x) \sim x^{-r}h(x)$, $0 < r < 2$, where h is a slowly varying function. Let $\{\lambda_n\}$ be a sequence such that $nh(\lambda_n)/\lambda_n^r \to C$ as $n \to \infty$, and let $\{x_n\}$ be a sequence with $x_n \to \infty$ as $n \to \infty$. Then*

$$0 < \liminf_{n \to \infty} \frac{x_n^r h(\lambda_n)}{h(x_n\lambda_n)} P\left(\sum_{i=1}^n \xi_i > x_n\lambda_n \right) \leq \limsup_{n \to \infty} \frac{x_n^r h(\lambda_n)}{h(x_n\lambda_n)} P\left(\sum_{i=1}^n \xi_i > x_n\lambda_n \right) < \infty. \tag{2.17}$$

Now we can show Theorem 2.1.

Proof of Theorem 2.1. In order to show (2.3), it is enough to show that for all $\varepsilon > 0$

$$\liminf_{n \to \infty} (\log n)^\varepsilon V(n)^{-1} T(n) \geq 1 \quad \text{a.s.,} \tag{2.18}$$

$$\liminf_{n \to \infty} (\log n)^{-\varepsilon} V(n)^{-1} T(n) \leq 1 \quad \text{a.s.} \tag{2.19}$$

We first show (2.18). Let $n_k = [\theta^k]$, $1 < \theta < 2$. Put again $\overline{G}(x) = 1 - G(x) = P(J_1 > x)$. Let $\overline{G}^*$ be the inverse of $\overline{G}$. Obverse that $\overline{G}^*(y) \sim y^{-1/\beta} H(1/y)$, $0 < y \leq 1$, where H is a slowly varying function and $V(n) = \overline{G}^*(1/n) \sim n^{1/\beta} H(n)$, so that

$$\frac{V(n_k)}{V(n_{k+1})} \longrightarrow \theta^{-1/\beta} \tag{2.20}$$

$$\frac{(\log n_k)^{-\varepsilon} V(n_k)}{\overline{G}^* \left((\log n_k)^{\beta\varepsilon/2} n_k^{-1} \right)} \sim (\log n_k)^{-\varepsilon} \frac{n_k^{1/\beta} H(n_k)}{n_k^{1/\beta} (\log n)^{-\varepsilon/2} H\left(n_k (\log n_k)^{-\beta\varepsilon/2} \right)}$$

$$= (\log n_k)^{-\varepsilon/2} \frac{H(n_k)}{H\left(n_k (\log n_k)^{-\beta\varepsilon/2} \right)} \longrightarrow 0, \tag{2.21}$$

by Lemma 2.2. Let $U, U_1, U_2, \ldots, U_n$ be i.i.d. random variables with the distribution of U Uniform over $(0, 1)$, and let $M^*(n) = \max\{U_1, U_2, \ldots, U_n\}$. Then, from the fact that $G(J_n)$ is a Uniform $(0, 1)$ random variable, we note that $M^*(n) \stackrel{d}{=} G(M(n))$, $n \geq 1$. From (2.21), J_i nonnegative, and $\overline{G}$ and $\overline{G}^*$ nonincreasing, it follows that

$$P\left(T(n_k) \leq (\log n_k)^{-\varepsilon} V(n_k) \right)$$

$$\leq P\left(M(n_k) \leq (\log n_k)^{-\varepsilon} V(n_k) \right)$$

$$\leq P\left(\overline{G}^* \left(\overline{G}(M(n_k)) \right) \leq \overline{G}^* \left((\log n_k)^{\beta\varepsilon/2} n_k^{-1} \right) \right)$$

$$= P\left(\overline{G}\left(M(n_k) \right) \geq (\log n_k)^{\beta\varepsilon/2} n_k^{-1} \right)$$

$$= P\left(1 - M^*(n_k) \geq (\log n_k)^{\beta\varepsilon/2} n_k^{-1} \right) \tag{2.22}$$

$$= P\left(M^*(n_k) \leq 1 - (\log n_k)^{\beta\varepsilon/2} n_k^{-1} \right)$$

$$= \left(P\left(U \leq 1 - (\log n_k)^{\beta\varepsilon/2} n_k^{-1} \right) \right)^{n_k}$$

$$\leq \exp\left(-(\log n_k)^{\beta\varepsilon/2} \right).$$

Hence, the sum of the left hand side of the previously mentioned probability is finite; by the Borel-Cantelli lemma, we get

$$\liminf_{k \to \infty} (\log n_k)^{\varepsilon} V(n_k)^{-1} T(n_k) \geq 1 \quad \text{a.s.} \tag{2.23}$$

Thus, by (2.20) we have

$$
\begin{aligned}
\liminf_{n \to \infty} (\log n)^{\varepsilon} V(n)^{-1} T(n) \\
\geq \liminf_{k \to \infty} \min_{n_k \leq n \leq n_{k+1}} (\log n)^{\varepsilon} V(n)^{-1} T(n) \\
\geq \liminf_{k \to \infty} \left(\frac{V(n_k)}{V(n_{k+1})} \right) (\log n_k)^{\varepsilon} V(n_k)^{-1} T(n_k) \\
\geq \theta^{-1/\beta} \quad \text{a.s.}
\end{aligned}
\tag{2.24}
$$

Therefore, by the arbitrariness of $\theta > 1$, (2.18) holds.

We now show (2.19). Let $n_k = [e^{k^{1+\delta}}]$, $\delta > 0$. For notational simplicity, we introduce the following notations:

$$
\zeta_k = \frac{T(n_k - n_{k-1})}{M(n_k - n_{k-1})},
$$

$$
E_k = \{ T(n_k) - T(n_{k-1}) \leq (\log n_k)^{\varepsilon} V(n_k) \},
$$

$$
\tilde{E}_k = \{ T(n_{k-1}) \geq \varepsilon (\log n_k)^{\varepsilon} V(n_k) \},
\tag{2.25}
$$

$$
F_k = \left\{ M(n_k - n_{k-1}) \leq (\log \log n_k)^{(1-\varepsilon)/\beta} V(n_k) \right\},
$$

$$
O_k = \left\{ \zeta_k \geq (\log n_k)^{\varepsilon} (\log \log n_k)^{-(1-\varepsilon)/\beta} \right\}.
$$

By Lemma 2.3, we have

$$
P(O_k) \leq \exp\left(-t(\log n_k)^{\varepsilon} (\log \log n_k)^{-1-\varepsilon/\beta} \right) E e^{t \zeta_k} \leq C \exp\left(-t(\log n_k)^{\varepsilon} (\log \log n_k)^{-(1-\varepsilon)/\beta} \right).
\tag{2.26}
$$

Thus, we get $\sum P(O_k) < \infty$.

Observe again that $\overline{G}^*(y) \sim y^{-1/\beta} H(1/y)$ and $V(n) \sim n^{1/\beta} H(n)$, so that

$$
\frac{V(n_k)}{V(n_{k-1})} \geq e^{(1/\beta)k^{\delta}},
\tag{2.27}
$$

$$
\frac{(\log \log n_k)^{(1-\varepsilon)/\beta} V(n_k)}{\overline{G}^*\left((\log \log n_k)^{(1-\varepsilon)} n_k^{-1} \right)} \sim \frac{(\log \log n_k)^{2(1-\varepsilon)/\beta} H(n_k)}{H\left((\log \log n_k)^{-(1-\varepsilon)} n_k \right)} \longrightarrow \infty,
\tag{2.28}
$$

by Lemma 2.2. Thus, we note

$$
\begin{aligned}
P(F_k) &\geq P(\overline{G}^*\left(\overline{G}(M(n_k - n_{k-1})) \leq \overline{G}^*\left((\log\log n_k)^{(1-\varepsilon)} n_k^{-1}\right)\right) \\
&= P(\overline{G}\left(M(n_k - n_{k-1}) \geq (\log\log n_k)^{(1-\varepsilon)} n_k^{-1}\right) \\
&= P\left(1 - M^*(n_k - n_{k-1}) \geq (\log\log n_k)^{(1-\varepsilon)} n_k^{-1}\right) \\
&= P\left(M^*(n_k - n_{k-1}) \leq 1 - (\log\log n_k)^{(1-\varepsilon)} n_k^{-1}\right) \\
&= \left(P\left(U \leq 1 - (\log\log n_k)^{(1-\varepsilon)} n_k^{-1}\right)\right)^{n_k - n_{k-1}} \\
&= \left(1 - (\log\log n_k)^{(1-\varepsilon)} n_k^{-1}\right)^{n_k - n_{k-1}} \\
&\geq \exp\left(-C(\log\log n_k)^{(1-\varepsilon/2)}\right),
\end{aligned}
\tag{2.29}
$$

which yields easily $\sum P(F_k) = \infty$. Hence, since $P(E_k) \geq P(F_k) - P(O_k)$, we get $\sum P(E_k) = \infty$. Since E_k are independent, by the Borel-Cantelli lemma, we get

$$
\liminf_{k \to \infty} (\log n_k)^{-\varepsilon} V(n_k)^{-1}(T(n_k) - T(n_{k-1})) \leq 1 \quad \text{a.s.}
\tag{2.30}
$$

By applying Lemma 2.4 and (2.27) and some simple calculation, we have easily that $\sum P(\widetilde{E}_k) < \infty$, so that

$$
\limsup_{k \to \infty} (\log n_k)^{-\varepsilon} V(n_k)^{-1} T(n_{k-1}) = 0 \quad \text{a.s.},
\tag{2.31}
$$

which, together with (2.30), implies

$$
\liminf_{k \to \infty} (\log n_k)^{-\varepsilon} V(n_k)^{-1} T(n_k) \leq 1 \quad \text{a.s.}
\tag{2.32}
$$

This yields (2.19). The proof of Theorem 2.1 is now completed. $\qquad\square$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We have to show that for all $\varepsilon > 0$

$$
\limsup_{t \to \infty} (\log t)^{-(1+\varepsilon)/\alpha} \left(B\left(t^\beta\right)\right)^{-1} X(t) \leq 1 \quad \text{a.s.},
\tag{3.1}
$$

$$
\limsup_{t \to \infty} (\log t)^{-(1-\varepsilon)/\alpha} \left(B\left(t^\beta\right)\right)^{-1} X(t) \geq 1 \quad \text{a.s.}
\tag{3.2}
$$

We first show (3.1). Let $t_k = \theta^k$, $1 < \theta < 2$. For notational simplicity, we introduce the following notations:

$$Q_k = \left\{ (\log t_k)^{-(1+\varepsilon)/\alpha} \left(B\left(t_k^\beta \right) \right)^{-1} S(N_{t_k}) \geq 1 \right\},$$

$$U(x) = (\log x)^{-\rho} x^{1/\beta}, \quad \gamma_1(x) = \sup\{y : U(y) \leq x\}, \quad \rho = \frac{\varepsilon}{5\beta},$$

$$\tilde{Q}_k = \left\{ (\log t_k)^{-(1+\varepsilon)/\alpha} \left(B\left(t_k^\beta \right) \right)^{-1} S(\gamma_1(t_k)) \geq 1 \right\},$$

$$R_k = \{ N_{t_k} \geq \gamma_1(t_k) \}.$$

(3.3)

By (2.18), we have

$$P(R_k \text{ i.o.}) = P(\{T(\gamma_1(t_k)) \leq t_k\} \text{ i.o.}) = P\left(\left\{ T(t_k) \leq (\log t_k)^{-\rho} V(t_k) \right\} \text{ i.o.} \right) = 0. \quad (3.4)$$

Put $\overline{F}(x) = 1 - F(x) = P(Y_1 > x)$. Let $\overline{F}^*$ be the inverse of $\overline{F}$. Recall that $\overline{F}^*(y) \sim y^{-1/\alpha} \widetilde{H}(1/y)$, $0 < y \leq 1$, where $\widetilde{H}$ is a slowly varying function, so that $B(n) = \overline{F}^*(C/n) \sim C n^{1/\alpha} \widetilde{H}(n)$ and

$$\frac{B\left(t_k^\beta \right)}{B\left(t_{k-1}^\beta \right)} \longrightarrow \theta^{\beta/\alpha}. \quad (3.5)$$

Note that

$$U\left((\log t_k)^{\varepsilon/4} t_k^\beta \right) \sim (\log t_k)^{\varepsilon/(4\beta)} t_k \left(\log\left((\log t_k)^{\varepsilon/4} t_k^\beta \right) \right)^{-\rho} \geq U(\gamma_1(t_k)) = t_k. \quad (3.6)$$

Thus, by noting U increasing,

$$(\log t_k)^{\varepsilon/(4\alpha)} t_k^{\beta/\alpha} \geq \gamma_1(t_k)^{1/\alpha}. \quad (3.7)$$

Hence, by Lemma 2.2,

$$(\log t_k)^{\varepsilon/(2\alpha)} \frac{B\left(t_k^\beta \right)}{B(\gamma_1(t_k))} \geq C(\log t_k)^{\varepsilon/(2\alpha)} \frac{t_k^{\beta/\alpha} \left(\widetilde{H}\left(t_k^\beta \right) \right)^{1/\alpha}}{(\gamma_1(t_k))^{1/\alpha} \left(\widetilde{H}(\gamma_1(t_k)) \right)^{1/\alpha}} \geq 1. \quad (3.8)$$

Thus, by (3.8) and Lemma 2.4, we have

$$
\begin{aligned}
P\big(\tilde{Q}_k\big) &\le P\left(S(\gamma_1(t_k)) \ge \left((\log t_k)^{(1+\varepsilon)/\alpha}\frac{B\big(t_k^\beta\big)}{B(\gamma_1(t_k))}\right)B(\gamma_1(t_k))\right) \\
&\le P\Big(S(\gamma_1(t_k)) \ge (\log t_k)^{(1+\varepsilon/2)/\alpha}B(\gamma_1(t_k))\Big) \\
&\le C(\log t_k)^{-(1+\varepsilon/4)}.
\end{aligned}
\tag{3.9}
$$

Therefore, $\sum P(\tilde{Q}_k) < \infty$. By the Borel-Cantelli lemma, we get $P(\tilde{Q}_k \text{ i.o.}) = 0$.

Observe that

$$
\begin{aligned}
P\left(\bigcup_{k=n}^{\infty}Q_k\right) &= P\left(\bigcup_{k=n}^{\infty}Q_k \cap \bigcap_{k=n}^{\infty}R_k^c\right) + P\left(\bigcup_{k=n}^{\infty}Q_k \cap \left(\bigcap_{k=n}^{\infty}R_k^c\right)^c\right) \\
&\le P\left(\bigcup_{k=n}^{\infty}\tilde{Q}_k\right) + P\left(\bigcup_{k=n}^{\infty}R_k\right),
\end{aligned}
\tag{3.10}
$$

where E^c stands for the complement of E. Thus, letting $n \to \infty$, we have

$$
P(Q_k \text{ i.o.}) \le P\big(\tilde{Q}_k \text{ i.o.}\big) + P(R_k \text{ i.o.}) = 0,
\tag{3.11}
$$

which implies that

$$
\limsup_{k \to \infty}(\log t_k)^{-(1+\varepsilon)/\alpha}\big(B\big(t_k^\beta\big)\big)^{-1}X(t_k) \le 1 \quad \text{a.s.}
\tag{3.12}
$$

Thus, by (3.5), we have

$$
\begin{aligned}
\limsup_{t \to \infty}&(\log t)^{-(1+\varepsilon)/\alpha}\big(B\big(t^\beta\big)\big)^{-1}X(t) \\
&\le \limsup_{k \to \infty}\max_{t_{k-1}<t\le t_k}(\log t)^{-(1+\varepsilon)/\alpha}\big(B\big(t^\beta\big)\big)^{-1}X(t) \\
&\le \theta^{\beta/\alpha}\limsup_{k \to \infty}(\log t_k)^{-(1+\varepsilon)/\alpha}\big(B\big(t_k^\beta\big)\big)^{-1}X(t_k) \\
&\le \theta^{\beta/\alpha} \quad \text{a.s.}
\end{aligned}
\tag{3.13}
$$

This yields (3.1) immediately by letting $\theta \downarrow 1$.

We now show (3.2). Let $t_k = e^{k^{1+\delta}}$, $\delta > 0$. To show (3.2), it is enough to prove

$$
\limsup_{k \to \infty}(\log t_k)^{-(1-\varepsilon)/\alpha}\big(B\big(t_k^\beta\big)\big)^{-1}X(t_k) \ge 1 \quad \text{a.s.}
\tag{3.14}
$$

Put

$$\Lambda_k = \left\{ (\log t_k)^{-(1-\varepsilon)/\alpha} \left(B\left(t_k^\beta \right) \right)^{-1} (S(N_{t_k})) \ge 1 \right\},$$

$$U_1(x) = (\log x)^\rho x^{1/\beta}, \quad \gamma_2(x) = \sup\{ y : U_1(y) \le x \}, \quad \rho = \frac{\varepsilon}{5\beta},$$

$$W_k = \left\{ (\log t_k)^{-(1-\varepsilon)/\alpha} \left(B\left(t_k^\beta \right) \right)^{-1} (S(\gamma_2(t_k)) - S(\gamma_2(t_{k-1}))) \ge 1 \right\},$$

$$\widetilde{R}_k = \{ N_{t_k} \ge \gamma_2(t_k) \}.$$

(3.15)

By (2.19), we have

$$P\left(\widetilde{R}_k \text{ i.o.} \right) = P(\{ T(\gamma_2(t_k)) \le t_k \} \text{ i.o.}) = P\left(\left\{ T(t_k) \le (\log t_k)^\varepsilon t_k^{1/\beta} \right\} \text{ i.o.} \right) = 1. \qquad (3.16)$$

Note that

$$U_1\left((\log t_k)^{-\varepsilon/4} t_k^\beta \right) \sim (\log t_k)^{-\varepsilon/(4\beta)} t_k \left(\log\left((\log t_k)^{-\varepsilon/4} t_k^\beta \right) \right)^\rho \le U_1(\gamma_2(t_k)) = t_k. \qquad (3.17)$$

Thus, by noting U_1 increasing,

$$(\log t_k)^{-\varepsilon/(4\alpha)} t_k^{\beta/\alpha} \le \gamma_2(t_k)^{1/\alpha}. \qquad (3.18)$$

Hence, by Lemma 2.2,

$$(\log t_k)^{-\varepsilon/(2\alpha)} \frac{B\left(t_k^\beta \right)}{B(\gamma_2(t_k))} \le C (\log t_k)^{-\varepsilon/(2\alpha)} \frac{t_k^{\beta/\alpha} \left(\widetilde{H}\left(t_k^\beta \right) \right)^{1/\alpha}}{(\gamma_2(t_k))^{1/\alpha} \left(\widetilde{H}(\gamma_2(t_k)) \right)^{1/\alpha}} \longrightarrow 0. \qquad (3.19)$$

Similarly, by noting $t_k/t_{k-1} \to \infty$, one can have

$$\frac{B(\gamma_2(t_k))}{B(\gamma_2(t_k) - \gamma_2(t_{k-1}))} \longrightarrow 1. \qquad (3.20)$$

Thus, by Lemma 2.4, we have

$$\begin{aligned}
P(W_k) &\ge P\left(S(\gamma_2(t_k) - \gamma_2(t_{k-1})) \ge \left((\log t_k)^{(1-\varepsilon)/\alpha} \frac{B\left(t_k^\beta \right)}{B(\gamma_2(t_k))} \right) B(\gamma_2(t_k)) \right) \\
&\ge P\left(S(\gamma_2(t_k) - \gamma_2(t_{k-1})) \ge (\log t_k)^{(1-\varepsilon/2)/\alpha} B(\gamma_2(t_k)) \right) \\
&\ge C (\log t_k)^{-(1-\varepsilon/4)}.
\end{aligned} \qquad (3.21)$$

Therefore, $\sum P(W_k) = \infty$. Since the events $\{W_k\}$ are independent, by the Borel-Cantelli lemma, we get $P(W_k \text{ i.o.}) = 1$.

Now, observe that

$$
\begin{aligned}
P\left(\bigcup_{n=m}^{\infty} \Lambda_k \right) &\geq P\left(\bigcup_{n=m}^{\infty} \left(\Lambda_k \cap \widetilde{R}_k \right) \right) \\
&\geq P\left(\bigcup_{n=m}^{\infty} \left\{ (\log t_k)^{-(1-\varepsilon)/\alpha} \left(B\left(t_k^\beta\right) \right)^{-1} S(\gamma_2(t_k)) \geq 1 \right\} \right) \times P\left(\bigcap_{n=m}^{\infty} \widetilde{R}_k \right) \\
&\geq P\left(\bigcup_{n=m}^{\infty} W_k \right) \times P\left(\bigcap_{n=m}^{\infty} \widetilde{R}_k \right).
\end{aligned}
\tag{3.22}
$$

Therefore, by letting $m \to \infty$, we get

$$
P(\Lambda_k \text{ i.o.}) \geq \left(P(W_k \text{ i.o.}) - P\left(\widetilde{W}_k \text{ i.o.} \right) \right) P\left(\widetilde{R}_k \text{ i.o.} \right) = 1,
\tag{3.23}
$$

which implies (3.14). The proof of **Theorem 1.1** is now completed. □

Remark 3.1. By the proof **Theorem 1.1**, (1.3) can be modified as follows:

$$
\limsup_{t \to \infty} (\log t)^{-1/\alpha} \left(B\left(t^\beta\right) \right)^{-1} X(t) = 1 \quad \text{a.s.}
\tag{3.24}
$$

That is to say that the form of (1.3) is no rare and the variables $(B(t^\beta))^{-1} X(t)$ must be cut down additionally by the factors $(\log t)^{-1/\alpha}$ to achieve a finite lim sup.

Acknowledgments

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