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SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH COMPACT SEMIGROUPS

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In this paper we study the local and global existence of mild solutions to a class of integro-differential equations in an arbitrary Banach space associated with the operators generating compact semigroups on the Banach space.

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1. Introduction

In this paper we are concerned with the following integro-differential equation considered in a Banach space X:

$$\frac{du}{dt} + Au(t) = f(t, u(t)) + \int_{t_0}^{t} a(t-s)g(s, u(s))ds, \quad 0 \leq t_0 < T_0 \leq \infty, \tag{1.1}$$

$$u(t_0) = u_0,$$

where $-A$ is assumed to be an infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$, on X, the nonlinear maps $f, g: J \times U \rightarrow X$, $J = [t_0, T_0)$, $t_0 < T_0 \leq \infty$, are continuous where U is an open subset of X, $a \in L^1(J)$ and u_0 is in U.

The problem (1.1) for a particular case in which $g = 0$ has been considered by Pazy [4], Pavel [3] and others. The existence of a unique *mild solution* to (1.1) with $g = 0$ is assured under the conditions that $-A$ is the infinitesimal generator of a compact semigroup in $X, f(t, u)$ is continuous in both the variables and uniformly locally Lipschitz continuous in u. If the Lipschitz continuity of f in u is dropped, then the existence of a mild solution is no more guaranteed. Examples, in which $A = 0$ and f is continuous and the differential equations do not have solutions are given in Dieudonne [1] and Yorke [6].

Heard and Rankin [2] considered the following integro-differential equation in a Banach space X:

$$\frac{du}{dt} + A(t)u(t) = \int_{t_0}^{t} a(t,s)g(s,u(s))ds + f(t,u(t)), \quad t > t_0 \geq 0, \tag{1.2}$$
$$u(t_0) = u_0,$$

where for each $t \geq 0$, the linear operator $-A(t)$ is the infinitesimal generator of an analytic semigroup in X, the nonlinear operator f is defined from $[0,\infty) \times X$ into X and satisfies a Hölder condition of the form

$$\| f(t,y_1) - f(t,y_2) \| \leq C[\,|\,t-s\,|^{\eta} + \| y_1 - y_2 \|_{\mu}^{\gamma}],$$

$0 < \eta,\ \gamma,\ \mu < 1$, $\|\ \ \|$ is the norm on X and $\|\ \ \|_{\mu}$ is the graph norm on $X_{\mu} = D(A^{\mu}(0))$, the nonlinear map g is assumed to satisfy a local Lipschitz condition with respect to the norm of X (cf. (A6) in [2]). Also, the uniqueness of solutions is proved under the restriction that the space X is a Hilbert space and $\gamma = 1$.

We also consider the global existence of mild solutions to (1.1). Further assumptions are required for global existence of mild solutions as global existence fails quite frequently. We first prove a result related to maximal interval of existence $[t_0, T_{max})$ and show that, if $T_{max} < \infty$, then the solution blows up in a finite time. Then we establish the global existence under certain growth conditions of the maps f and g.

2. Preliminaries

In this section we mention some relevant notions and collect some results associated with the following initial value problem considered in a Banach space X:

$$\frac{du}{dt} + Au(t) = f(t,u(t)), \quad 0 < t_0 < t < T_0 \leq \infty, \tag{2.1}$$
$$u(t_0) = u_0,$$

where $-A$ is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$ and f is continuous from $J \times U$ into X, $J = [t_0, T_0)$, $t_0 < T_0 \leq \infty$, U is an open subset of X and u_0 is in U.

Let X be a Banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is called a *semigroup of bounded linear operators on* X if (*i*) $T(0) = I$, I is the identity operator on X and (*ii*) $T(t+s) = T(t)T(s)$ for every t, $s \geq 0$. The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ for } x \in D(A),$$

is called the *infinitesimal generator* of the semigroup $T(t)$. Here $D(A)$ denotes the domain of A. A semigroup $T(t)$ is called *uniformly continuous* if

$$\lim_{t \downarrow 0} \| T(t) - I \| = 0.$$

A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is called a *strongly continuous* semigroup of bounded linear operators if

$$\lim_{t \downarrow 0} T(t)x = x \text{ for every } x \in X.$$

A strongly continuous semigroup $T(t)$ is also called as a C_0 semigroup. A C_0 semigroup $T(t)$ is called *compact* for $t > t_0$ if for every $t > t_0$, $T(t)$ is a compact operator. $T(t)$ is called *compact* if it is compact for $t > 0$. We note that, if $T(t)$ is compact for $t \geq 0$, then in particular the identity operator is compact and therefore X in this case is finite dimensional. Also, if $T(t_0)$ is compact for some $t_0 > 0$, then $T(t)$ is compact for every $t \geq t_0$ since $T(t) = T(t-t_0)T(t_0)$ and $T(t-t_0)$ is bounded.

We shall use the following result on the compact semigroups.

Theorem 2.1: *Let $T(t)$ be a C_0 semigroup. If $T(t)$ is compact for $t > t_0$, then $T(t)$ is uniformly continuous for $t > t_0$.*

We have the following characterization of a compact semigroup in terms of the resolvent operators $R(\lambda : A)$ of its generator A.

Theorem 2.2: *Let $T(t)$ be a C_0 semigroup and let A be its infinitesimal generator. $T(t)$ is a compact semigroup if and only if $T(t)$ is uniformly continuous for $t > 0$ and $R(\lambda : A)$ is compact for $\lambda \in \rho(A)$.*

By a *mild solution* to (1.1) on J we mean a function $u \in C(J : X)$ satisfying the integral equation

$$u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s,u(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau,u(\tau))d\tau]ds. \quad (2.2)$$

For the problem (2.1) we have the following existence theorem due to Pazy [4, 5].

Theorem 2.3: *Let X be Banach space and U be an open subset of X. Let $-A$ be the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$. If $f: J \times U \to X$ is continuous then for every u_0 in U, there exists a $t_1, t_0 < t_1 < T_0$, such that (2.1) has a mild solution u on $J_0 = [t_0, t_1)$.*

The following result is due to Pavel [3] which extends the results of Theorem 2.3.

Theorem 2.4: *Suppose that D is a locally closed subset of $X, f: J \times D \to X$ is continuous where $J = [t_0, T_0)$, and the C_0 semigroup $T(t)$, $t \geq 0$ is compact for $t > 0$. A necessary and sufficient condition for the existence of a local mild solution $u: [t_0, T(t_0, u_0)) \to D$, $t_0 < T(t_0, u_0) < T_0$ to (2.1) for $u_0 \in D$ is*

$$\lim_{h \to 0} h^{-1} \text{dist}(S(h)z + hf(t,z) : D) = 0$$

for all $t \in [t_0, T_0)$ and $z \in D$.

3. Local Existence

Our aim is to extend the results of Theorem 2.3 to the initial value problem (1.1). Below we state and prove the following existence result for (1.1).

Theorem 3.1: *Let X be a Banach space, U be an open subset of X and $J = [t_0, T_0)$, $t_0 < T_0 \leq \infty$. Let $-A$ be the infinitesimal generator of a compact semi-*

group $T(t)$, $t \geq 0$. If the nonlinear maps $f, g: J \times U \rightarrow X$ are continuous and a is locally integrable in J, then for every $u_0 \in X$ there exists a $t_1, t_0 < t_1 < T_0$, such that (2.1) has a mild solution u on $[t_0, t_1)$.

Proof: Let T be such that $t_0 < T < T_0 \leq \infty$. Let M be a positive constant such that

$$\| T(t) \| \leq M \text{ for } 0 \leq t \leq T.$$

Let $\rho > 0$ be such that

$$B_\rho(u_0) = \{v \in X: \| v - u_0 \| \leq \rho\} \subset U.$$

Choose $t' > t_0$ such that

$$\| f(t,v) \| \leq N_1,$$

$$\| g(t,v) \| \leq N_2,$$

for $t_0 \leq t \leq t'$, $v \in B_\rho(u_0)$ with positive constants N_1 and N_2. Again choose $t'' > t_0$ such that

$$\| T(t - t_0)u_0 - u_0 \| < \frac{\rho}{2} \text{ for } t_0 \leq t \leq t''.$$

Let

$$t_1 = \min\left\{T, t', t'', t_0 + \frac{\rho}{2M(N_1 + a_T N_2)}\right\},$$

where $a_T = \int_{t_0}^{T} | a(s) | ds$. Now we set

$$Y = C([t_0, t_1]: X)$$

and

$$S = \{u \in Y: u(t_0) = u_0, u(t) \in B_\rho(u_0) \text{ for } t_0 \leq t \leq t_1\}.$$

We note that S is a bounded, closed and convex subset of Y. We define a map $F: S \rightarrow Y$ given by

$$(Fu)(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s, u(s)) + \int_{t_0}^{s} a(s - \tau)g(\tau, u(\tau))d\tau]ds. \tag{3.1}$$

For $u \in S$, we have

$$\| (Fu)(t) - u_0 \| \leq \| T(t - t_0)u_0 - u_0 \|$$

$$+ \| \int_{t_0}^{t} T(t-s)[f(s, u(s)) + \int_{t_0}^{s} a(s - \tau)g(\tau, u(\tau))d\tau]ds \|$$

$$\leq \frac{\rho}{2} + (t_1 - t_0)M(N_1 + a_T N_2)$$

$$\leq \rho.$$

Thus $F: S \to S$. Now we show that F is continuous from S into S. To show this, we first observe that since f and g are continuous in $[t_0, T] \times U$, it follows that any $\epsilon > 0$ and for a fixed $u \in B_\rho(u_0)$ there exist $\delta_1(u)$, $\delta_2(u) > 0$ such that for any $v \in B_\rho(u_0)$, we have

$$\| u - v \|_Y \leq \delta_1(u) \Rightarrow \| f(t, u(t)) - f(t, v(t)) \| \leq \frac{\epsilon}{2TM}$$

and

$$\| u - v \|_Y \leq \delta_2(u) \Rightarrow \| g(t, u(t)) - g(t, v(t)) \| \leq \frac{\epsilon}{2a_T TM}.$$

Let

$$\delta(u) = \min\{\delta_1(u), \delta_2(u)\}.$$

Then for any $v \in S$, $\| u - v \|_Y < \delta(u)$ implies that

$$\| (Fu)(t) - (Fv)(t) \| \leq \int_{t_0}^{t} \| T(t-s) \| \; \| f(s, u(s)) - f(s, v(s)) \| ds$$

$$+ \int_{t_0}^{t} \| T(t-s) \| \left(\int_{t_0}^{s} | a(s-\tau) | \; \| g(\tau, u(\tau)) - g(\tau, v(\tau)) \| d\tau \right) ds. \tag{3.2}$$

Thus, $F: S \to S$ is continuous. Let

$$\widetilde{S} = F(S),$$

and for fixed $t \in [t_0, t_1]$, let

$$S(t) = \{(Fu)(t): u \in S\}.$$

Since $S(t_0) = \{u_0\}, S(t_0)$ is precompact in X. For $t > t_0$ and $0 < \epsilon < t - t_0$, let

$$(F_\epsilon u)(t) = T(t-t_0)u_0 + \int_{t_0}^{t-\epsilon} T(t-s)[f(s, u(s)) + \int_{t_0}^{s} a(s-\tau) g(\tau, u(\tau)) d\tau] ds$$

$$= T(t-t_0)u_0 + T(\epsilon) \int_{t_0}^{t-\epsilon} T(t-s-\epsilon)[f(s, u(s))$$

$$+ \int_{t_0}^{s} a(s-\tau) g(\tau, u(\tau)) d\tau] ds. \tag{3.3}$$

The compactness of the semigroup $T(t)$ for every $t > 0$ and (3.3) imply that for every ϵ, $0 < \epsilon < t - t_0$, the set

$$S_\epsilon(t) = \{(F_\epsilon u)(t): u \in S\}$$

is precompact in X. Now, for any $u \in S$, we have

$$\| (Fu)(t) - (F_\epsilon u)(t) \| \leq \int_{t-\epsilon}^{t} \| T(t-s)[fs, u(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau, u(\tau))d\tau] \| ds$$

$$\leq \epsilon M(N_1 + a_T N_2). \tag{3.4}$$

From (3.4) it follows that the set $S(t)$ is precompact. Now we show that $\widetilde{S}$ is equicontinuous. For $r_1, r_2 \in [t_0, t_1]$ with $r_1 < r_2$, we have

$$\| (Fu)(r_2) - (Fu)(r_1) \| \leq \| (T(r_2 - t_0) - T(r_1 - t_0))u_0 \|$$

$$(N_1 + a_T N_2) \int_{t_0}^{r_1} \| T(r_2 - s) - T(r_1 - s) \| ds$$

$$+ (r_2 - r_1)M(N_1 + a_T N_2). \tag{3.5}$$

Since $T(t)$ is compact, Theorem 2.1 implies that $T(t)$ is continuous in the uniform operator topology for $t > 0$. Therefore, the right-hand side of (3.5) tends to zero as $r_2 - r_1$ tends to zero. Thus $\widetilde{S}$ is equicontinuous. Also, $\widetilde{S}$ is bounded. It follows from the Arzela-Ascoli theorem (cf. see Dieudonne [1]), that $\widetilde{S}$ is precompact. The existence of a fixed point of F in S is a consequence of Schauder's fixed point theorem and any fixed point of F in S is a mild solution to (1.1) on $[t_0, t_1)$.

4. Global Existence

In this section we consider the global existence of mild solution to (1.1). For (2.1) we have the following result.

Theorem 4.1: *Suppose $-A$ is the infinitesimal generator of a compact semigroup $T(t)$, $t > 0$ on X. If $f:[t_0, \infty) \times X \rightarrow X$ is continuous and maps bounded subsets of $[t_0, \infty) \times X$ into bounded subsets in X, then for every $u_0 \in X$ the equation (2.1) has a mild solution u on a maximal interval of existence $[t_0, T_{max})$ and, if $T_{max} < \infty$, then*

$$\lim_{T \uparrow T_{max}} \| u(t) \| = \infty.$$

In the following theorem we extend the results of Theorem 4.1 to the problem (1.1).

Theorem 4.2: *Suppose $-A$ is the infinitesimal generator of a compact semigroup $T(t)$, $t > 0$ on X. If $f, g:[t_0, \infty) \times X \rightarrow X$ are continuous and map bounded subsets of $[t_0, \infty) \times X$ into bounded subsets in X and a is locally integrable in $[t_0, \infty)$,* *then for every $u_0 \in X$ the equation (1.1) has a mild solution u on a maximal interval of existence $[t_0, T_{max})$ and, if $T_{max} < \infty$, then*

$$\lim_{t \uparrow T_{max}} \| u(t) \| = \infty.$$

Proof: From Theorem 3.1 we have the existence of a local mild solution $u \in C([t_0, t_1): X)$ for some $t_0 < t_1$ to (1.1) given by

$$u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} [f(s, u(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau, u(\tau))d\tau]ds.$$

Suppose that $u(t_1) < \infty$. Consider the problem

$$\frac{dv}{dt} + Av(t) = f(t, v(t)) + \int_{t_1}^{t} a(t-s)g(s, v(s0)ds, \tag{4.1}$$

$$v(t_1) = u(t_1).$$

From Theorem 3.1 we have that there exists a mild solution $v \in C([t_1, t_2): X)$ for some $t_2, t_1 < t_2 < \infty$ to (4.1) given by

$$v(t) = T(t-t_1)u(t_1) + \int_{t_1}^{t} T(t-s)[f(s, v(s)) + \int_{t_1}^{s} a(s-\tau)g(\tau, v(\tau))d\tau]ds$$

$$= T(t-t_1)\left[T(t_1-t_0)u_0 + \int_{t_0}^{t_1} T(t_1-s)[f(s, u(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau, u(\tau))d\tau]ds \right]$$

$$+ \int_{t_1}^{t} T(t-s)[f(s, v(s)) + \int_{t_1}^{s} a(s-\tau)g(\tau, v(\tau))d\tau]ds$$

$$= T(t-t_0)u_0 + \int_{t_0}^{t_1} T(t-s)[f(s, u(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau, u(\tau))d\tau]ds$$

$$+ \int_{t_1}^{t} T(t-s)[f(s, v(s)) + \int_{t_1}^{s} a(s-\tau)g(\tau, v(\tau))d\tau]ds.$$

We define $\tilde{u} : [t_0, t_2) \rightarrow X$ by

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [t_0, t_1), \\ v(t), & t \in [t_1, t_2). \end{cases}$$

Then $\widetilde{u} \in C([t_0, t_2): X)$ and for $t_1 < t < t_2$, we have

$$\widetilde{u}(t) = T(t-t_0)u_0 + \int_{t_0}^{t_1} T(t-s)[f(s,\widetilde{u}(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau,\widetilde{u}(\tau))d\tau]ds$$

$$+ \int_{t_1}^{t} T(t-s)[f(s,\widetilde{u}(s)) + \int_{t_1}^{s} a(s-\tau)g(\tau,\widetilde{u}(\tau))d\tau]ds$$

$$= T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,\widetilde{u}(s))ds$$

$$+ \int_{t_0}^{t_1} \int_{t_0}^{s} T(t-s)a(s-\tau)g(\tau,\widetilde{u}(\tau))d\tau ds$$

$$+ \int_{t_1}^{t} \int_{t_1}^{s} T(t-s)a(s-\tau)g(\tau,\widetilde{u}(\tau))d\tau ds. \tag{4.2}$$

Changing the order of integration in (4.2), we get

$$\widetilde{u}(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,\widetilde{u}(s))ds$$

$$+ \int_{t_0}^{t_1} \int_{\tau}^{t_1} T(t-s)a(s-\tau)g(\tau,\widetilde{u}(\tau))ds d\tau$$

$$+ \int_{t_1}^{t} \int_{\tau}^{t} T(t-s)a(s-\tau)g(\tau,\widetilde{u}(\tau))ds d\tau$$

$$= T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s,\widetilde{u}(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau,\widetilde{u}(\tau))d\tau]ds. \tag{4.3}$$

From (4.3) we have that $\widetilde{u}$ is a mild solution to (1.1) on $[t_0, t_2)$. Now, suppose that $[t_0, T_{max})$ is the maximal interval to which the solution u of (1.1) can be extended. If $T_{max} < \infty$, then we show that $\| u(t) \| \to \infty$ as $t \uparrow T_{max}$. It suffices to show that $\overline{\lim}_{t \uparrow T_{max}} \| u(t) \| - \infty$. If $\overline{\lim}_{t \uparrow T_{max}} \| u(t) \| < \infty$, then there exists a sequence $t_n \uparrow T_{max}$ such that $\| u(t_n) \| \leq K$ for some constant K and for all n. Suppose that $\| T(t) \| \leq M$ for $t \leq t \leq T_{max}$ and let

$$N_1 = \sup\{ \| f(t,v) \| : t_0 \leq t \leq T_{max}, \| v | \leq M(K+1)\}$$

and

$$N_2 = \sup\{ \| g(t,v) \| : t_0 \le t \le T_{max}, \| v \| \le M(K+1)\}.$$

Using the continuity of u and the assumption that $\overline{\lim}_{t\uparrow T_{max}} \| u(t) \| < \infty$, we can find a sequence $\{h_n\}$ such that $h_n \to 0$, $\| u(t) \| \le M(K+1)$ for $t_n \le t \le t_n + h_n$ and $\| u(t_n + h_n) \| = M(K+1)$. But then we have

$$\begin{aligned} M(K+1) &= \| T(t_n + h_n) \| \\ &\le \| T(h_n)u(t_n) \| \\ &+ \int_{t_n}^{t_n+h_n} \| T(t_n + h_n - s)[f(s,u(s)) + \int_{t_n}^{s} a(s-\tau)g(\tau,u(\tau))d\tau] \| ds \\ &\le MK + h_n(N_1 + a_{T_{max}} N_2)M \end{aligned}$$

which gives a contradiction as $h_n \to 0$. Hence,

$$\overline{\lim}_{t\uparrow T_{max}} \| u(t) \| = \infty.$$

This completes the proof.

Finally, we prove the following global existence result for (1.1).

Theorem 4.3: *Let $-A$ be the infinitesimal generator of a compact semigroup, $T(t)$, $t \ge 0$ on X. Let $f,g:[t_0,\infty) \times X \to X$ be continuous functions mapping bounded subsets $[t_0,\infty) \times X$ into bounded subsets of X and a be locally integrable in $[t_0,\infty)$, then any one of the following two conditions is sufficient for the global existence of a mild solution u to* (1.1):

(*i*) *There exist a continuous function $k_0:[t_0,\infty) \to [0,\infty)$ such that $\| u(t) \| \le k_0(t)$ for every t in the interval of existence of u.*

(*ii*) *There exist functions $k_i:[t_0,\infty) \to [0,\infty)$, $i = 1,2,3,4$; such that $k_1, k_2, a*k_3$, and $a*k_4$ are locally integrable on $[t_0,\infty)$, and for $t_0 \le t < \infty$, $v \in X$*

$$\| f(t,v) \| \le k_1(t) \| v \| + k_2(t), \tag{4.4}$$

$$\| g(t,v) \| \le k_3(t) \| v \| + k_4(t), \tag{4.5}$$

where

$$a*k_i(t) = \int_{t_0}^{t} a(t-s)k_i(s)ds$$

for $i = 3,4$.

Proof: (*i*) Since for any $t_1, t_0 < t_1 < \infty$ $\| u(t_1) \| \le k_0(t_1) < \infty$, from Theorem 4.2, it follows that the solution u can be extended beyond t_1, hence the solution u exists globally.

(*ii*) The mild solution u to (1.1) is given by

$$u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s,u(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau,u(\tau))d\tau]ds.$$

Let $\| T(t) \| \leq Me^{\omega t}$. Multiplying the above equation by $e^{-\omega(t-t_0)}$ and taking the norm, we have

$$e^{-\omega(t-t_0)} \| u(t) \| \leq M \| u_0 \| + M \int_{t_0}^{t} e^{-\omega(s-t_0)}[\, \| f(s,u(s)) \| + \int_{t_0}^{s} | a(s-\tau) | \, \| g(\tau,u(\tau)) \| \, d\tau]ds. \tag{4.6}$$

For $t \in [t_0, \infty)$, set

$$\xi(t) = M \| u_0 \| + M \int_{t_0}^{t} e^{-\omega(s-t_0)}[k_2(s) + \int_{t_0}^{s} | a(s-\tau) | k_4(\tau)d\tau]ds.$$

From (4.6) we have

$$e^{-\omega(t-t_0)} \| u(t) \| \leq \xi(t) + M \int_{t_0}^{t} e^{-\omega(s-t_0)}[k_1(s) \| u(s) \| + \int_{t_0}^{s} | a(s-\tau) | k_3(s) \| u(\tau \| \, d\tau]ds.$$

For $t_0 \leq r \leq t$, we have

$$e^{-\omega(t-t_0)} \| u(r) \| \leq \xi(r) + M \int_{t_0}^{r} e^{-\omega(s-t_0)}[k_1(s) \| u(s) \| + \int_{t_0}^{s} | a(s-\tau) | k_3(s) \| u(\tau \| \, d\tau]ds$$

$$\leq \xi(r) + M \int_{t_0}^{r} e^{-\omega(s-t_0)}[k_1(s) + \int_{t_0}^{s} | a(s-\tau) | k_3(s)] \sup_{t_0 \leq \tau \leq s} \| u(\tau) \| ds. \tag{4.7}$$

Taking the supremum over $[t_0, t]$ on both the sides of (4.7), we get

$$e^{-\omega(t-t_0)} \sup_{t_0 \leq r < t} \| u(r) \| \leq \sup_{t_0 \leq r \leq t} \xi(r) + M \int_{t_0}^{t} e^{-\omega(s-t_0)}[k_1(s) + \int_{t_0}^{s} | a(s-\tau) | k_3(\tau)d\tau] \sup_{t_0 \leq r \leq s} \| u(\tau) \| ds. \tag{4.8}$$

Gronwall's inequality implies that

$$e^{-\omega(t-t_0)} \sup_{t_0 \le r \le t} \| u(r) \|$$

$$\le \sup_{t_0 \le r \le t} \xi(r) + M \int_{t_0}^{t} \left[e^{-\omega(s-t_0)}[k_1(s) + \int_{t_0}^{s} | a(s-\tau) | k_3(\tau)d\tau]\exp\left\{ \int_{s}^{t} [k_1(u) + \int_{t_0}^{s} | a(u-\tau) | k_3(\tau)d\tau]du \right\} \right] \sup_{t_0 \le r \le s} \xi(\tau)ds. \qquad (4.9)$$

Inequality (4.9) implies that $\| u(t) \|$ is bounded by a continuous function and from (i) we get the global existence of the mild solution u to (1.1). This completes the proof.

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References

[1] Dieudonne, J., *Foundation of Modern Analysis*, Academic Press, New York 1960.

[2] Heard, M.L. and Rankin, S.M., A semilinear parabolic integro-differential equation, *J. Diff. Eqns.* **71** (1988), 201-233.

[3] Pavel, N.H., Invariant sets for a class of semilinear equations of evolution, *Nonlinear Anal.* **1** (1977), 187-196.

[4] Pazy, A., A class of semilinear equations of evolution, *Israel J. Math.* **20** (1975), 23-36.

[5] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York 1983.

[6] Yorke, J., A continuous differential equation in Hilbert space without existence, *Funk. Ekvacioj* **12** (1970), 19-21.