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NEGATIVELY DEPENDENT BOUNDED RANDOM VARIABLE PROBABILITY INEQUALITIES AND THE STRONG LAW OF LARGE NUMBERS

M. AMINI and A. BOZORGNIA
Ferdowsi University
Faculty of Mathematical Sciences
Mashhad, Iran
E-mail: Bozorg@science2.um.ac.ir

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Let $X_1,\ldots,X_n$ be negatively dependent uniformly bounded random variables with d.f. $F(x)$. In this paper we obtain bounds for the probabilities $P(\mid \sum_{i=1}^{n} X_i \mid \geq nt)$ and $P(\mid \widehat{\xi}_{pn} - \xi_p \mid > \varepsilon)$ where $\widehat{\xi}_{pn}$ is the sample pth quantile and ξ_p is the pth quantile of $F(x)$. Moreover, we show that $\widehat{\xi}_{pn}$ is a strongly consistent estimator of ξ_p under mild restrictions on $F(x)$ in the neighborhood of ξ_p. We also show that $\widehat{\xi}_{pn}$ converges completely to ξ_p.

Key words: Probability Inequalities, Strong Law of Large Numbers, Complete Convergence.

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1. Introduction

In many stochastic models, the assumption that random variables are independent is not plausible. Increases in some random variables are often related to decreases in other random variables so an assumption of negative dependence is more appropriate than an assumption of independence. Lehmann [12] investigated various conceptions of positive and negative dependence in the bivariate case. Strong definitions of bivariate positive and negative dependence were introduced by Esary and Proschen [7]. Also Esary, Proschen and Walkup [8] introduced a concept of association which implied a strong form of positive dependence. Their concept has been very useful in reliability theory and applications. Multivariate generalizations of conceptions of dependence were initiated by Harris [9] and Brindley and Thompson [4]. These were later developed by Ebrahimi and Ghosh [6], Karlin [11], Block and Ting [2], and Block, Savits and Shaked [1]. Furthermore, Matula [13] studied the almost sure convergence of sums of negatively dependent (ND) random variables and Bozorgnia, Patterson and Taylor [3] studied limit theorems for dependent random variables. In this paper we study the asymptotic behavior of quantiles for negatively dependent random variables.

Definition 1: The random variables $X_1, \ldots, X_n$ are pairwise negatively dependent if

$$P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j) \tag{1}$$

for all $x_i, x_j \in R$, $i \neq j$. It can be shown that (1) is equivalent to

$$P(X_i > x_i, X_j > x_j) \leq p(X_i > x_i)P(X_j > x_j) \tag{2}$$

for all $x_j, x_i \in \mathbb{R}$, $i \neq j$.

Definition 2: The random variables $X_1, \ldots, X_n$ are said to be ND if we have

$$P(\bigcap_{j=1}^n (X_j \leq x_j)) \leq \prod_{j=1}^n P(X_j \leq x_j), \tag{3}$$

and

$$P(\bigcap_{j=1}^n (X_j > x_j)) \leq \prod_{j=1}^n P(X_j > x_j), \tag{4}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$.

Conditions (3) and (4) are equivalent for $n = 2$. However, Ebrahimi and Ghosh [6] show that these definitions do not agree for $n \geq 3$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be ND if every finite subset $\{X_1, \ldots, X_n\}$ is ND.

Definition 3: For parametric function $g(\theta)$, a sequence of estimators $\{T_n, n \geq 1\}$ is strongly consistent if

$$T_n \rightarrow g(\theta) \text{ a.e.}$$

Definition 4: The sequence $\{X_n, n \geq 1\}$ of random variables converges to zero completely (denoted $\lim_{n \rightarrow \infty} X_n = 0$ completely) if for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P[\mid X_n \mid > \varepsilon \mid < \infty. \tag{5}$$

In the following example, we will show that the ND properties are not preserved for absolute values and squares of random variables.

Example: Let (X, Y) have the following p.d.f:

$$f(-1,-1) = f(1,0) = 0,\ f(-1,0) = f(0,0) = f(0,-1) = f(0,1) = f(1,1) = 1/9,$$

$$f(-1,1) = f(1,-1) = 2/9.$$

Then

(i) X and Y are ND random variables since for each $x, y \in R$ we have

$$F(x,y) \leq F_X(x)F_Y(y).$$

(ii) X and $V = Y^2$ are not ND random variables because for $-1 \leq x < 0$ and $0 \leq v < 1$ we have

$$F(x, v) = 1/9 > F_X(x)F_V(v) = (3/9)(2/9).$$

(*iii*) $U = X^2$ and $V = Y^2$ are not ND random variables nor are $|X|$ and $|Y|$ since $0 \leq u < 1$, $0 \leq v < 1$ we have

$$F(u, v) = 1/9 > F_U(u)V_V(v) = (2/9)(3/9). \qquad \square$$

The following lemmas are used to obtain the main result in the next section. Detailed proofs of these lemmas can be founded in the Bozorgnia, Patterson and Taylor [3].

Lemma 1: *Let $\{X_n, n \geq 1\}$ be a sequence of ND random variables let $\{f_n, n \geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n), n \geq 1\}$ is a sequence of ND random variables.*

Lemma 2: *Let $X_1, \ldots, X_n$ be ND random variables and let $t_1, \ldots, t_n$ be all nonnegative (or all nonpositive). Then*

$$E[e^{\sum_{i=1}^n t_i X_i}] \leq \prod_{i=1}^n Ee^{t_i X_i}.$$

Lemma 3: *Let X be a r.v. such that $E(X) = 0$ and $|X| \leq c < \infty$ a.e. Then for every real number h we have*

$$Ee^{hX} \leq e^{h^2c^2}.$$

Proof: For $c = 1$, see Chow [5]. For general c, apply the $c = 1$ result with X replaced by X/c.

2. An Extension of the Theorem of Hoeffding for ND Random Variables

In this section we extended the theorem of Hoeffding (Theorem 1 below) and then obtain the strong law of large numbers for ND uniformly bounded random variables.

Theorem 1: (Hoeffding [10]) *Let $X_1, \ldots, X_n$ be independent random variables satisfying $P[a \leq X_i \leq b] = 1$ for each i where $a < b$, and let $S_n = \sum_{k=1}^n (X_k - EX_k)$. Then for any $t > 0$ and $c = b - a$*

$$P[S_n \geq nt] \leq \exp\left[\frac{-2nt^2}{c^2}\right].$$

Theorem 2: *Let $X_1, \ldots, X_n$ be ND random variables satisfying $P[a \leq X_i \leq b] = 1$ for each i where $a < b$, and let $S_n = \sum_{k=1}^n (X_k - EX_k)$. Then for any $t > 0$ and $c = b - a$,*

$$P[S_n \geq nt] \leq \exp\left[\frac{-nt^2}{4c^2}\right].$$

Proof: We define $Y_k = X_k - EX_k$ for $k = 1, \ldots, n$. Then we have $EY_k = 0$ and $|Y_k| \leq c$ a.e. Hence, by Lemmas 2 and 3, we have

$$p(S_n \geq nt) \leq \exp(-nth)Ee^{hS_n}$$
$$\leq \exp[-nth]\prod_{k=1}^{n} Ee^{hY_k} \leq \exp[-nth + nh^2c^2].$$

The right side of this inequality attains its minimum value $\exp[\frac{-nt^2}{4c^2}]$ for $h = \frac{t}{2c^2}$. Thus, for each $t > 0$,

$$P[S_n \geq nt] \leq \exp\left[\frac{-nt^2}{4c^2}\right]. \qquad \square$$

Corollary 1: *Under the assumptions of Theorem 1, for every $t > 0$*

$$P[\,|S_n| \geq nt] \leq 2\exp\left[\frac{-nt^2}{4c^2}\right].$$

Proof:

$$P(\,|S_n| > nt) = P[S_n \geq nt] + P[-S_n \geq nt]$$

$$= P[S_n \geq nt] + P[S'_n \geq nt] \leq 2\exp\left[\frac{-nt^2}{4c^2}\right]$$

where $S_n^2 = \sum_{k=1}^{n} Z_k$ and $Z_k - -Y_k, k = 1,\ldots,n$. $\square$

Theorem 3: *Under the assumptions of Theorem 2, for every $\alpha > 1/2$ we have*

$$\frac{1}{n^{\alpha}}\sum_{k=1}^{n}(X_k - EX_k) \to 0 \quad \text{completely.}$$

Proof: By Theorem 2 and Corollary 1, for each $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} P(\,|S_n| > n^{\alpha}\varepsilon) < 2\sum_{n=1}^{\infty}\exp\left(-\frac{\varepsilon^2 n^{2\alpha-1}}{4c^2}\right) < \infty. \qquad \square$$

Hence, for $\alpha = 1$ we obtain the strong law of large numbers for negatively dependent uniformly bounded random variables.

3. Asymptotic Behavior of Quantiles for ND Random Variables

The following two theorems and one corollary given conditions under which $\widehat{\xi}_{pn}$ is contained in a suitably small neighborhood of ξ_p with probability one for all sufficiently large n.

Theorem 4: *Let $X_1,\ldots,X_n$ be ND random variables with d.f. $F(x)$. Let $0 < p < 1$. Suppose that ξ_p is the unique solution x of $F(x^-) \leq p \leq F(x)$. Then for every $\varepsilon > 0$ and n we have*

$$P(\,|\widehat{\xi}_{pn} - \xi_p| > \varepsilon) \leq 2\exp(-n\delta_\varepsilon^2/4) \tag{6}$$

where $\delta_\varepsilon = min\{F(\xi_p + \varepsilon) - p, p - F(\xi_p - \varepsilon)\}$ *and* $\widehat{\xi}_{pn}$ *is the sample pth quantile.*

Proof: For every $\varepsilon > 0$ we have

$$P[\,|\,\widehat{\xi}_{pn} - \xi_p\,|\, > \varepsilon] = P[\widehat{\xi}_{pn} > \varepsilon + \xi_p] + P[\widehat{\xi}_{pn} < \xi_p - \varepsilon]$$

$$= P[p > F_n(\xi_p + \varepsilon)] + P[p < F_n(\xi_p - \varepsilon].$$

We define

$$V_i = I_{[X_i > \xi_p + \varepsilon]} \text{ and } U_i = I_{[X_i \leq \xi_p - \varepsilon]} \text{ for } i = 1, 2, \ldots, n.$$

Since $X_1, \ldots, X_n$ are ND random variables, by Lemma 1 $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ are ND random variables. Hence, by Theorem 2 23 have

$$P(p > F_n(\xi_p + \varepsilon)) = P\left(\sum_{i=1}^{n}(V_i - E(V_i)\right) > n\delta_1) \leq \exp\left(-\frac{n\delta_1^2}{4}\right)$$

where $\delta_1 = F(\xi_p + \varepsilon) - p$. Similarly we have

$$P[p < F_n(\xi_p - \varepsilon)] = P\left[\sum_{i=1}^{n}(U_i - EU_i) > n\delta_2\right] \leq \exp\left(\frac{-n\delta_2^2}{4}\right),$$

where $\delta = p - F(\xi_p - \varepsilon)$. Define $\delta_\varepsilon = \min\{\delta_1, \delta_2\}$. Thus we have (6). □

Corollary 2: *Let* $X_1, \ldots, X_n$ *be ND random variables with d.f.* $F(x)$ *and let* $\widehat{\xi}_{pn}$ *be the sample pth quantile. Then*

$$\widehat{\xi}_{pn} \rightarrow \xi_p \quad \textit{completely as } n \rightarrow \infty.$$

Proof: By Theorem 4 we have

$$\sum_{n=1}^{\infty} P[\,|\,\widehat{\xi}_{pn} - \xi_p\,|\, > \varepsilon] \leq 2\sum_{n=1}^{\infty}\exp\left(\frac{-n\delta^2\varepsilon}{4}\right) < \infty.$$

Hence $\widehat{\xi}_{pn} \rightarrow \xi_p$ completely. □

By the Borel-Cantelli lemma, we have $\widehat{\xi}_{pn} \rightarrow \xi_p$ a.e. Thus $\widehat{\xi}_{pn}$ is a strongly consistent estimator of ξ_p.

Theorem 5: *Let* $X_1, \ldots, X_n$ *be ND random variables with d.f.* $F(X)$. *Let* $0 < p < 1$. *Suppose that* F *is differentiable at* ξ_p *with* $F'(\xi_p) = f(\xi_p) > 0$. *Then for some* $\beta > 0$ *and* $0 < \alpha \leq \frac{1}{2}$

$$|\,\widehat{\xi}_{pn} - \xi_p\,| \leq \frac{(2+\beta)ln^{\alpha}(n)}{f(\xi_p)n^{\alpha}} \quad a.e.$$

Proof: Since F is continuous at ξ_p with $F'(\xi_p) > 0$, ξ_p is a unique solution of $F(x^-) < p < F(x)$ and $F(\xi_p) = p$. Thus, we may apply Theorem 4. Writing

$$\varepsilon_n = \frac{(2+\beta)\ln^{\alpha}(n)}{f(\xi_p)n^{\alpha}}$$

we have

$$F(\xi_p + \varepsilon_n) - p = \varepsilon_n f(\xi_p) + o(\varepsilon_n) \geq \frac{(2+\beta)\ln^{\alpha}(n)}{n^{\alpha}}$$

where n is sufficiently large. Similarly

$$p - F(\xi_p - \varepsilon_n) = \varepsilon_n f(\xi_p) + o(\varepsilon_n) \geq \frac{(2+\beta)^2 \ln(n)}{n^\alpha}.$$

Thus for all sufficiently large n,

$$n\delta^2_{\varepsilon_n}/4 \geq \frac{(2+\beta)^2 \ln^{2\alpha}(n)}{4n^{2\alpha-1}} \doteq \frac{(2+\beta)^2 \ln(n)}{4}$$

where

$$\delta_{\varepsilon_n} = \min\{F(\xi_p + \varepsilon_n) - p, p - F(\xi_p - \varepsilon_n)\}.$$

Hence, for a constant c we have

$$\sum_{n=1}^{\infty} p[\mid \widehat{\xi}_{pn} - \xi_p \mid > \varepsilon_n] \leq c + \sum_{n=1}^{\infty} \frac{2}{n^{(1+\beta/2)^2}} < \infty$$

which completes the proof. □

Let $X_1, \ldots, X_n$ be independent random variables with d.f. $F(x)$. Then in this case, all the above theorems and corollaries are true. In particular, Theorems 4 and 5 are extensions of Theorem 2.3.1 and Lemma B, respectively, pages 96 of Serfling [14].

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