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G-H-KKM SELECTIONS WITH APPLICATIONS TO MINIMAX THEOREMS

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Based on the G-H-KKM selections, some nonempty intersection theorems and their applications to minimax inequalities are presented.

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1. Introduction

Minimax inequalities have numerous applications to variational inequalities, while variational inequalities turn out to be a powerful tool to the solvability of problems in elasticity and plasticity theory, heat conduction, diffusion theory, optimization theory, mathematical economics, and others.

Chang and Zhang [1] introduced the notion of the generalized quasiconcavity and obtained some nonempty intersection theorems and their applications to minimax inequalities in a linear topological space setting. Recently, Tan [4] extended this notion to the case of a G-convex space with applications to minimax theorems and saddle points. Our aim here is to present some G-H-KKM selection theorems and related applications to minimax inequalities in a G-H-space setting.

Let X be a topological space, $P(X)$ denote the power set of X, and $\langle X\rangle$, a family of all nonempty finite subsets of X. Let Δ^n denote a standard $(n-1)$ simplex $\{e_1, e_2, \ldots, e_n\}$ of R^n.

Definition 1.1: A triple $(X, H, \{p\})$ is called a *G-H-space* [6] if X is a topological space and $H: \langle X\rangle \rightarrow P(X)\backslash\{\emptyset\}$ is a mapping such that:

(*i*) For each $F, G \in \langle X\rangle$, there exists $F_1 \subset F$ such that $F_1 \subset G$ implies $H(F_1) \subset H(G)$.

(*ii*) For $F = \{x_1, x_2, \ldots, x_n\} \in \langle X\rangle$, there is a continuous mapping $p: \Delta^n \rightarrow H(F)$ such that for $\{i1, i2, \ldots, ik\} \subset \{1, 2, \ldots, n\}$, we have $p(\{e_{i1}, e_{i2}, \ldots, e_{ik}\}) \subset H(\{x_{i1}, x_{i2}, \ldots, x_{ik}\})$, where $\{x_{i1}, x_{i2}, \ldots, x_{ik}] \subset F$.

A subset D of X is called *finitely G-H-closed in* X if for each $A \in \langle X \rangle$, there exists $\underset{\sim}{A} \subset A$ such that $D(^{-})H(\underset{\sim}{A})$ is closed in $H(\underset{\sim}{A})$.

A subset K of X is said to be *compactly closed in* X if $K(^{-})L$ is closed in L for all compact subsets L of X.

Definition 1.2: Let $(X, H, \{p\})$ be a G-H-space and $T: X \to P(X)$ a multivalued mapping. T is called a *G-H-KKM mapping* if for each $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$, there exists $\{x_{i1}, x_{i2}, \ldots, x_{ik}\} \subset \{x_1, x_2, \ldots, x_n\}$ such that

$$H(\{x_{i1}, x_{i2}, \ldots, x_{ik}\}) \subset (\underset{j=1}{\overset{k}{\quad}})T(x_{ij}) \text{ for } \{i1, i2, \ldots, ik\} \subset \{1, 2, \ldots, n\}.$$

Definition 1.3: Let $(X, H, \{p\})$ be a G-H-space and let $M_1, M_2, \ldots, M_n$ be subsets of X. A subset $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ is said to be a *G-H-KKM selection for* $M_1, M_2, \ldots, M_n$ if for any $\{x_{i1}, x_{i2}, \ldots, x_{ik}\} \subset \{x_1, x_2, \ldots, x_n\}$, we have

$$H(\{x_{i1}, x_{i2}, \ldots, x_{ik}\}) \subset (\underset{j=1}{\overset{k}{\quad}})M_{ij} \text{ for } \{i1, i2, \ldots, ik\} \subset \{1, 2, \ldots, n\}.$$

This generalizes the notion of a KKM selection in a pseudoconvex space by Joo' and Kassay [2].

Definition 1.4: Let X be nonempty set and $(Y, H, \{p\})$ a G-H-space. Let $f: X \times Y \to R$, $e: Y \to R$ and $h: X \to R$ be functions. The function f is said to be *0-generalized G-H-quasiconcave* (resp. *0-generalized G-H-quasiconvex*) in its first variable x if for each $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$, there exists $\{v_1, v_2, \ldots, v_n\} \in \langle Y \rangle$ such that for each $\{v_{i1}, v_{12}, \ldots, v_{ik}\} \subset \{v_1, v_2, \ldots, v_n\}$ and for any $y_0 \in H(\{v_{i1}, v_{i2}, \ldots, v_{ik}\})$, we have

$$\min_{1 \le j \le k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] \le 0$$

$$(\text{resp. } \max_{1 \le j \le k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] \ge 0),$$

where $\{i1, i2, \ldots, ik\} \subset \{1, 2, \ldots, n\}$.

This generalizes the notion of a 0-generalized quasiconcavity (0-generalized quasiconvexity) by Chang and Zhang [1].

2. G-H-KKM Theorems and Applications

In this section we first recall and obtain some auxiliary results and then establish some minimax theorems.

Lemma 2.1: [7] *Let* $(X, H, \{p\})$ *be a G-H-space and* $M_1, M_2, \ldots, M_n$ *be finitely G-H-closed subsets of* X. *Suppose that* $M_1, M_2, \ldots, M_n$ *have a G-KKM selection. Then* $(^{-})_{i=1}^{n} M_i \neq \emptyset$.

Proposition 2.1: *Let* X *be a nonempty set and* $(Y, H, \{p\})$ *a G-H-space. Let* $f: X \times Y \to R$, $e: Y \to R$ *and* $h: X \to R$ *be functions. Then the following statements are equivalent:*

(*a*) *A mapping* $T: X \to P(Y)$ *defined by*

$$T(x) = \{y \in Y: f(x, y) + e(y) - h(x) \le 0\}$$

$$(resp.\ T(x) = \{y \in Y: f(x,y) + e(y) - h(x) \geq 0\}),$$

is a G-H-KKM mapping.

(*b*) *f is* 0-*generalized G-H-quasiconcave* (*resp.* 0-*generalized G-H-quasiconvex*) *in its first variable x.*

Proof: $(a) \Rightarrow (b)$ Since T is G-H-KKM, it implies for each $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ and corresponding $\{v_1, v_2, \ldots, v_n\} \in \langle Y \rangle$ that there exist $\{x_{i1}, x_{i2}, \ldots, x_{ik}\} \subset \{x_1, x_2, \ldots, x_n\}$, $\{v_{i1}, v_{i2}, \ldots, v_{ik}\} \subset \{v_1, v_2, \ldots, v_k\}$ and any $y_0 \in H(\{v_{i1}, v_{i2}, \ldots, v_{ik}\})$ such that

$$H(\{v_{i1}, v_{i2}, \ldots, v_{ik}\}) \subset (\underset{j=1}{\overset{k}{\underline{\qquad}}})T(x_{ij}).$$

This implies $y_0 \in (\underset{j=1}{\overset{k}{\underline{\qquad}}})T(x_{ij})$, and as a result, there exists some index m $(i \leq m \leq k)$ such that $y_0 \in T(x_{im})$. Hence, $f(x_{im}, y_0) + e(y_0) - h(x_{im}) \leq 0$ (resp. $f(x_{im}, y_0) + e(y_0) - h(x_{im}) \geq 0$). It follows that

$$\min_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] \leq 0$$

$$(\text{resp. } \max_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] \geq 0).$$

$(b) \Rightarrow (a)$ Since f is 0-generalized G-H-quasiconcave (resp. 0-generalized G-H-quasiconvex) in x, it implies for any $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ and $\{v_1, v_2, \ldots, v_n\} \in \langle Y \rangle$, there exist $\{v_{i1}, v_{i2}, \ldots, v_{ik}\} \subset \{v_1, v_2, \ldots, v_n\}$, and any $y_0 \in H(\{v_{i1}, v_{i2}, \ldots, v_{ik}\})$ such that

$$\min_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] \leq 0$$

$$(\text{resp. } \max_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x) \geq 0).$$

It follows that there exists some index m $(1 \leq m \leq k)$ such that $y_0 \in T(x_m) \subset (\underset{j=1}{\overset{k}{\underline{\qquad}}})$ $T(x_{ij})$. This completes the proof.

Theorem 2.1: *Let X be a nonempty set and $(Y, H, \{p\})$ a G-H-space such that $H(F)$ is compact for all $F \in \langle Y \rangle$. Suppose that $f: X \times Y \rightarrow R$, $e: Y \rightarrow R$ and $h: X \rightarrow R$ are functions satisfying the following assumptions:*

(*i*) *f is lower semicontinuous in y on compact subsets of Y.*

(*ii*) *e is lower semicontinuous on compact subsets of Y.*

(*iii*) *f is* 0-*generalized G-H-quasiconcave in x.*

(*iv*) *There exists an element $x_0 \in X$ such that the set*

$$\{y \in Y: f(x_0, y) + e(y) - h(x_0) \leq 0\}$$

is a compact subset of Y.

Then there exists an element $\underset{\sim}{y} \in Y$ such that

$$f(x, \underset{\sim}{y}) + e(\underset{\sim}{y}) - h(x) \leq 0 \text{ for all } x \in X.$$

Proof: Let us define a mapping $T: X \rightarrow P(Y)$ by

$$T(x) = \{y \in Y: f(x,y) + e(y) - h(x) \leq 0\} \text{ for all } x \in X.$$

Since f is 0-generalized G-H-quasiconcave, it implies that $T(x)$ is nonempty. It follows from Proposition 2.1 that T is a G-H-KKM mapping. By (i) and (ii), each $T(x)$ is finitely G-H-closed, that is, for each $\underset{\sim}{A} \subset A \in \langle Y \rangle$, we have

$$T(x)(^{-})H(\underset{\sim}{A}) = \{y \in H(\underset{\sim}{A}) : f(x,y) + e(y) - h(x) \leq 0\}$$

$$= \{y \in H(\underset{\sim}{A}) : f(x,y) + e(y) \leq h(x)\},$$

is closed in $H(\underset{\sim}{A})$ by the lower semicontinuity of f and e, so the family $\{T(x) : x \in X\}$ has the finite intersection property by Lemma 2.1. Now applying (iv), we find that $\{T(x)(^{-})T(x_0) : x \in X\}$ is a family of compact subsets of Y. Hence, $(\,^{-}_{x \in X})T(x) \neq \emptyset$. That means, there exists an element $\underset{\sim}{y} \in Y$ such that

$$f(x, \underset{\sim}{y}) + e(\underset{\sim}{y}) - h(x) \leq 0 \text{ for all } x \in X.$$

This completes the proof.

For Y compact, Theorem 2.1 reduces to:

Theorem 2.2: *Let X be a nonempty set and $(Y, H, \{p\})$ a compact G-H-space with $H(F)$ compact for all $F \in \langle Y \rangle$. Suppose that $f: X \times Y \rightarrow R$, $e: Y \rightarrow R$ and $h: X \rightarrow R$ are functions such that:*

(i) f is lower semicontinuous in second variable y.

(ii) e is lower semicontinuous.

(iii) f is 0-generalized G-H-quasiconcave in first variable x.

Then there is an element $\underset{\sim}{y} \in Y$ such that

$$f(x, \underset{\sim}{y}) + e(\underset{\sim}{y}) - h(x) \leq 0 \textit{ for all } x \in X.$$

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