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A MONOTONE ITERATIVE METHOD FOR BOUNDARY VALUE PROBLEMS OF PARAMETRIC DIFFERENTIAL EQUATIONS

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This paper studies boundary value problems for parametric differential equations. By using the method of upper and lower solutions, monotone sequences are constructed and proved to converge to the extremal solutions of the boundary value problem.

Key words: Upper and Lower Solutions, Monotone Iteration, Boundary Value Problems, Parametric Differential Equations.

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1. Introduction

Many problems in physical chemistry and physics, describing the exothermic and isothermal chemical reactions, the steady-state temperature distributions, the oscillation of a mass attached by two springs, lead to differential equations with a parameter [5]. Existence, uniqueness and approximate solutions of problems with a parameter have been discussed in [1] where several other references may be found.

Recently, the method of upper and lower solutions and monotone iterative technique was initiated for boundary value problems with a parameter [2]. In this paper, we shall extend the work of [2], under weaker assumptions in the general set up of the standard work [3] so that further advances can be made for such problems.

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2. Main Results

Consider the parametric equation

$$\begin{cases} x' = f(t,x,\lambda) \\ 0 = g(x,\lambda) \\ x(0) = A \end{cases} \tag{2.1}$$

where $f \in C[J \times R^2, R]$, $g \in C[\Omega \times R, R]$, where $J = [0,T]$, $\Omega = \{u: u \in C[J,R]\}$.

Definition 2.1: A pair (v,α), $v \in C^1[J,R]$ and $\alpha \in R$, is said to be a lower solution of (2.1) if

$$\begin{cases} v' \le f(t,v,\alpha), \\ 0 \le g(v,\alpha), \\ v(0) \le A. \end{cases} \tag{2.2}$$

An upper solution of (ω,β) of (2.1) can be defined similarly by reversing the above inequalities.

Theorem 2.1: *Assume that*

(*i*) $v, w \in C^1[J,R], \alpha, \beta \in R, (v,\alpha)$ *and* (w,β) *are lower solutions of* (2.1), *respectively, such that* $\alpha \le \beta$ *and* $v(t) \le w(t), t \in J$;

(*ii*) $f \in C[J \times R^2, R], f(t,x,\lambda)$ *is nondecreasing in* λ *and for* $M \ge 0$ *such that*

$$f(t,x,\lambda) - f(t,\tilde{x},\lambda) \ge -M(x,\tilde{x})$$

whenever $v(t) \le \tilde{x} \le x \le w(t)$, $t \in J$ *and* $\alpha \le \lambda \le \beta$;

(*iii*) $g \in C[\Omega \times R, R]$, $g(x,\lambda)$ *is nondecreasing in* x *and for* $N \ge 0$ *such that*

$$g(x,\lambda) - g(x,\tilde{\lambda}) \ge -N(\lambda - \tilde{\lambda})$$

whenever $v(t) \le x \le w(t)$, $t \in J$, $\alpha \le \tilde{\lambda} \le \lambda \le \beta$.

Then, there exists monotone sequences $\{v_n, \alpha_n\}$, $\{w_n, \beta_n\}$ *which converge monotonically to minimal and maximal solutions* (ρ,k), (γ,l) *of* (2.1), *respectively, i.e.,*

$$v \le v_1 \le \ldots \le v_n \le \rho \le \gamma \le w_n \le \ldots \le w_1 \le w,$$

$$\alpha \le \alpha_1 \le \ldots \le \alpha_n \le k \le l \le \beta_n \le \ldots \le \beta_1 \le \beta, \quad \forall n.$$

Proof: For any $\eta \in \langle v, w \rangle$, where

$$\langle v, w \rangle = \{x \in C[J,R^n]: v(t) \le x(t) \le w(t), \quad t \in J,$$

and $e \in [\alpha,\beta]$, we consider the linear parametric equation

$$\begin{cases} x' + Mx = \sigma(t,e), \quad t \in J \\ 0 = g(\eta,e) - N(\lambda - e), \\ x(0) = A, \end{cases} \tag{2.3}$$

where $\sigma(t,e) = f(t,\eta(t),e) + M\eta(t)$.

It is not difficult to see that problem (2.3) possesses a unique solution $(x(t), \lambda)$ for any given pair $(\eta(t), e)$. Let (v_1, α_1), (w_1, β_1) be the solutions of (2.3) corresponding $(\eta, e) = (v, \alpha)$ and $(\eta, e) = (w, \beta)$, respectively.

Let $p = \alpha - \alpha_1$. We then get

$$0 = g(v, \alpha) - N(\alpha_1 - \alpha) \geq -N(\alpha_1 - \alpha) = Np,$$

which implies $p \leq 0$ and hence $\alpha \leq \alpha_1$. Similarly, we can show $\beta_1 \leq \beta$. Now let $p = \alpha_1 - \beta_1$. By condition *(iii)*, we get

$$0 = g(v, \alpha) - N(\alpha_1 - \alpha) = g(v, \alpha) - g(w, \beta) - N(\alpha_1 - \alpha) + N(\beta_1 - \beta)$$

$$\leq g(w, \alpha) - g(w, \beta) - N(\alpha_1 - \alpha) + N(\beta_1 - \beta)$$

$$N(\beta - \alpha) - N(\alpha_1 - \alpha) + N(\beta_1 - \beta) = -Np.$$

Hence $\alpha_1 \leq \beta_1$. Thus, we get

$$\alpha \leq \alpha_1 \leq \beta_1 \leq \beta. \tag{2.4}$$

Next, we shall show

$$v(t) \leq v_1(t) \leq w_1(t) \leq w(t), \quad t \in J. \tag{2.5}$$

Let $m(t) = v(t) - v_1(t)$, $t \in J$. Then

$$m' = v'(t) - v_1'(t) \leq f(t, v(t), \alpha) - f(t, v(t), \alpha) - M(v(t) - v_1(t))$$

$$= -Mm(t),$$

and $m(0) \leq A - A = 0$. Thus $m(t) \leq 0$ and $v(t) \leq v_1(t)$, $t \in J$. Similarly, we can show that $w_1(t) \leq w(t)$, $t \in J$. Now let $m(t) = v_1(t) - w_1(t)$. Then

$$m'(t) = v_1'(t) - w_1'(t) = f(t, v(t), \alpha) - M(v_1(t) - v(t))$$

$$- f(t, w(t), \beta) + M(w_1(t) - w(t))$$

$$\leq f(t, v(t), \beta) - f(t, w(t), \beta) - M(v_1(t) - w_1(t)) + M(v(t) - w(t))$$

$$\leq -Mm(t)$$

$m(0) = v_1(0) - w_1(0) = 0$. Thus we get $m(t) \leq 0$ and $v_1(t) \leq w(t)$, $t \in J$. Hence (2.5) is proved.

Now let $\eta_1, \eta_2 \in \langle v, w \rangle$, $e_1, e_2 \in [\alpha, \beta]$ such that

$$\eta_1(t) \leq \eta_2(t), \quad t \in J, \quad e_1 \leq e_2. \tag{2.6}$$

Let $(x_1(t), \lambda_1), (x_2(t), \lambda_2)$ be solutions of (2.3) corresponding to (η_1, e_1) and (η_2, e_2), respectively. We are going to show

$$x_1(t) \le x_2(t),\ t \in J \text{ and } \lambda_1 \le \lambda_2. \tag{2.7}$$

Let $m(t) = x_1(t) - x_2(t)$, $t \in J$. Then by condition (ii)

$$\begin{aligned} m'(t) &= x_1'(t) - x_2'(t) \\ &= f(t,\eta_1(t),e_1) - M(x_1(t) - \eta_1(t)) - f(t,\eta_2(t),e_2) + M(x_2(t) - \eta_2(t)) \\ &\le f(t,\eta_1(t),e_2) - f(t,\eta_2(t),e_2) - M(\eta_2(t) - \eta_1(t)) - M(x_1(t) - x_2(t)) \\ &\le -Mm(t), \end{aligned}$$

and $m(0) = x_1(0) - x_2(0) = 0$. Thus we have $m(t) \le 0$ and $x_1(x) \le x_2(t)$ on J. Set $p = \lambda_1 - \lambda_2$. Then by condition (iii)

$$\begin{aligned} 0 = g(\eta_1,e_1) - N(\lambda_1 - e_1) &= g(\eta_1,e_1) - g(\eta_2,e_2) - N(\lambda_1 - e_1) + N(\lambda_2 - e_2) \\ &\le N(e_2 - e_1) - N(\lambda_1 - e_1) + N(\lambda_2 - e_2) = -Np, \end{aligned}$$

which implies $\lambda_1 \le \lambda_2$. Thus (2.7) is established.

It is now easy to construct sequences $\{(v_n(t),\alpha_n)\}$ and $\{w_n(t),\beta_n\}$, where $(v_n(t),\alpha_n)$ and $(w_n(t),\beta_n)$ are solutions of (2.3) corresponding to $(\eta,e) = (v_{n-1},\alpha_{n-1})$ and $(\eta,e) = (w_{n-1},\beta_{n-1})$, respectively, with $v_0 = v$, $\alpha_0 = \alpha$, $w_0 = w$ and $\beta_0 = \beta$. We conclude from (2.4), (2.5) and (2.7)

$$v_0 \le v_1 \le v_2 \le \ldots \le v_n \le w_n \le \ldots \le w_2 \le w_1 \le w_0 \text{ on } J$$

and

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_2 \le \beta_1 \le \beta_0.$$

It then follows from the standard arguments that the sequences $\{(v_n(t),\alpha_n)\}$, $\{(w_n(t),\beta_n)\}$ converge uniformly and monotonically to $(\rho(t),k)$, $(\gamma(t),l)$, respectively, and $(\rho(t),k)$, $(\gamma(t),l)$ are solutions of the parametric equation (2.1).

To show that $(\rho(t),k)$, $(\gamma(t),l)$ are extremal solutions of (2.1), let $(x(t),\lambda)$ be any solution of (2.1) such that

$$v(t) \le x(t) \le w(t),\ t \in J \text{ and } \alpha \le \lambda \le \beta.$$

Suppose that for some n, we have

$$v_n(t) \le x(t) \le w_n(t),\ t \in J \text{ and } \alpha_n \le \lambda \le \beta_n. \tag{2.8}$$

Then, setting $m(t) = v_{n+1}(t) - x(t)$, we obtain by condition (ii), and (2.8)

$$\begin{aligned} m'(t) &= f(t,v_n(t),\alpha_n) - M(v_{n+1}(t) - v_n(t)) - f(t,x(t),\lambda) \\ &\le M(x(t) - v_n(t)) - M(v_{n+1}(t) - v_n(t) = -Mm(t), \end{aligned}$$

and $m(0) = 0$. Thus, we get $m(t) \le 0$ and $v_{n+1}(t) \le x(t)$, $t \in J$. Similarly, we can prove $w_{n+1}(t) \geq x(t), t \in J$.

Let $p = \alpha_n - \lambda$. Then, in view of condition (*iii*) and (2.8),

$$0 = g(v_n, \alpha_n) - N(\alpha_{n+1} - \alpha_n) \leq g(x, \alpha_n) - N(\alpha_{n+1} - \alpha_n)$$

$$= g(x, \alpha_n) - g(x, \lambda) - N(\alpha_{n+1} - \alpha_n)$$

$$\leq N(\lambda - a_n) - N(\alpha_{n+1} - \alpha_n) = -Np,$$

which implies $\alpha_{n+1} \leq \lambda$. Similarly, we can show $\lambda \leq \beta_{n+1}$. We conclude by induction that (2.8) is true for all n since $v_0(t) \leq x(t) \leq w_0(t)$, $t \in J$ and $\alpha_0 \leq \lambda \leq \beta_0$. Hence, it follows that

$$\rho(t) \leq x(t) \leq \gamma(t), \quad t \in J \text{ and } k \leq \lambda \leq l$$

by taking the limit as $n \rightarrow \infty$ and the proof is therefore complete. □

References

[1] Jankowski, T., Existence, uniqueness and approximate solutions of problems with a parameter, *Zeszyty Naukowe Politechniki Gdańskiej, Matematyka* **16** (1993), 3-167.

[2] Jankowski, T. and Lakshmikantham, V., Monotone iterations for differential equation with a parameter, *J. Appl. Math. Stoch. Anal.* **10** (1997), 273-278.

[3] Ladde, G.S., Lakshmikantham, V. and Vatsala, A.S., *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston 1985.

[4] Liu, X., Nonlinear boundary value problems for first order impulsive integro-differential equations, *J. Appl. Math. Simulation* **2** (1989), 185-198.

[5] Na, T.Y., *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York 1979.