WEAK LAWS OF LARGE NUMBERS FOR ARRAYS OF ROWWISE NEGATIVELY DEPENDENT RANDOM VARIABLES

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Weak laws of large numbers for arrays of rowwise negatively dependent random variables are obtained in this paper. The more general hypothesis of negative dependence relaxes the usual assumption of independence. The moment conditions are similar to previous results, and the stochastic bounded condition also provides a generalization of the usual distributional assumptions.

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1. Introduction

The history and literature on laws of large numbers is vast and rich as this concept is crucial in probability and statistical theory. The literature on concepts of negative dependence is much more limited but still very interesting. Lehmann [6] provides an extensive introductory overview of various concepts of positive and negative depen-

dence in the bivariate case. Negative dependence has been particularly useful in obtaining strong laws of large numbers (cf., Matula [7], Qi [8], Chandra and Ghosal [3], Bozorgnia, Patterson and Taylor [1, 2]). Weak laws of large numbers for negatively dependent random variables are obtained in this paper.

2. Preliminaries

Section 2 will contain some background materials on negative dependence which will be used in obtaining the major weak laws of large numbers (WLLNs) in Section 3.

Definition 2.1: Random variables X and Y are said to be *negatively dependent* (ND) if

$$P[X \leq x, Y \leq y] \leq P[X \leq x]P[Y \leq y] \tag{2.1}$$

for all $x, y \in R$. A collection of random variables is said to be *pairwise ND* if every pair of random variables in the collection satisfies (2.1).

It is important to note that (2.1) implies

$$P[X > x, Y > y] \leq P[X > x]P[Y > y] \tag{2.2}$$

for all $x, y \in R$. Moreover, it follows that (2.2) implies (2.1), and hence, (2.1) and (2.2) are equivalent. Ebrahimi and Ghosh [5] showed that (2.1) and (2.2) are not equivalent for a collection of 3 or more random variables. They considered random variables X_1, X_2 and X_3 where (X_1, X_2, X_3) assumed the values $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ and $(0, 0, 0)$ each with probability $\frac{1}{4}$. The random variables X_1, X_2 and X_3 are pairwise independent, and hence, satisfy both (2.1) and (2.2) for all pairs. However,

$$P[X_1 > x_1, X_2 > x_2, X_3 > x_3] \leq P[X_1 > x_1]P[X_2 > x_2]P[X_3 > x_3] \tag{2.3}$$

for all x_1, x_2 and x_3, but

$$P[X_1 \leq 0, X_2 \leq 0, X_3 \leq 0] = \frac{1}{4} > \frac{1}{8} = P[X_1 \leq 0]P[X_2 \leq 0]P[X_3 \leq 0]. \tag{2.4}$$

Placing probability $\frac{1}{4}$ on each of the other vertices $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$ provides the converse example of pairwise independent random variables which will not satisfy (2.3) with $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ but where the desired ' $\leq$ ' in (2.5) hold for all x_1, x_2 and x_3. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

Definition 2.2: The random variables $X_1, X_2, \ldots$ are said to be

(a) *lower negatively dependent* (LND) if for each $n \geq 2$

$$P[X_1 \leq x_1, \ldots, X_n \leq x_n] \leq \prod_{i=1}^{n} P[X_i \leq x_i] \tag{2.5}$$

for all $x_1, \ldots, x_n \in R$,

(b) *upper negative dependent* (UND) if for each $n \geq 2$

$$P[X_1 > x_1, \ldots, X_n > x_n] \leq \prod_{i=1}^{n} P[X_i > x_i] \tag{2.6}$$

for all $x_1, x_n \in R$,

(c) *negatively dependent* (ND) if both (2.5) and (2.6) hold.

Note that the example proved by Ebrahimi and Ghosh shows that UND can hold without LND and conversely. Any of the $\leq$'s or $>$'s can be consistently replaced by $<$'s or $\geq$'s. A simple example of ND for two variables is to let $Y = -X$ when X is a non-degenerate random variable. A second practical example is to let $X_1, \ldots, X_n$ denote items sampled without replacement from $\{1, 2, \ldots, N\}$ where $n \leq N$. First, for $a_1, \ldots, a_n \in R$ with $a_i \geq 1$ for all $1 \leq i \leq n$,

$$P[X_1 \leq a_1, \ldots, X_n \leq a_n] = \prod_{i=1}^{n} \frac{\beta_i - i + 1}{N - i + 1}$$

$$\leq \prod_{i=1}^{n} \frac{\beta_i}{N}$$

$$= \prod_{i=1}^{n} P[X_i \leq a_i] \qquad (2.7)$$

where $\beta_i = \min\{[a_{(i)}], N\}$, and where $a_{(1)} \leq \ldots \leq a_{(n)}$ denote the ordered values of $a_1, \ldots, a_n$ and [ ] denotes the greatest integer function. Thus, LND follows from (2.7) since (2.7) trivially hold if $a_i < 1$ for some i, $1 \leq i \leq n$. In a similar fashion

$$P[X_1 > a_1, \ldots, X_n > a_n] \leq \prod_{i=1}^{n} P[X_i > a_i]$$

follows for UND. Hence, ND is achieved for sampling without replacement from $\{1, 2, \ldots, N\}$. Several other stronger (more restrictive) definitions for forms of negative dependence are given in Lehmann [6] but will not be considered in this paper.

The following four properties are listed for reference in obtaining the main result in the next section. Detailed proofs can be found in the previously cited literature.

Lemma 2.1: *If $X_1, \ldots, X_n$ are pairwise ND random variables, then*

(a) $E(X_i X_j) \leq E(X_i)E(X_j)$ $i \neq j$

(b) $Cov(X_i, X_j) \leq 0$ $i \neq j$.

Lemma 2.2: (a) *If $\{X_n\}$ is a sequence of LND (UND) random variables and $\{f_n\}$ is a sequence of monotone increasing, Borel functions, then $\{f(X_n)\}$ is a sequence of LND (UND) random variables.*

(b) *If $\{X_n\}$ is a sequence of UND (LND) random variables and $\{f_n\}$ is a sequence of monotone decreasing, Borel functions, then $\{f_n(X_n)\}$ is a sequence of LND (UND) random variables.*

Corollary 2.1: *If $\{X_n\}$ is a sequence of ND random variables and $\{f_n\}$ is a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing), then $\{f_n(X_n)\}$ is a sequence of ND random variables.*

Corollary 2.2: *If $X_1, X_2, \ldots, X_n$ are LND (UND) random variables, then for any real numbers $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ such that $a_i < b_i$, $1 \leq i \leq n$,*

(a) $\{I_{(-\infty < X_i < b_i)}, 1 \leq i \leq n\}$ *are UND (LND)*,

(b) $\{Y_i, 1 \leq i \leq n\}$ *are LND (UND)*

where $Y_i = X_i I_{[a_i \leq X_i \leq b_i]} + b_i I_{[X_i > b_i]} + a_i I_{[X_i < a_i]}.$

For ND random variables, a major problem occurs with attempting to apply the usual methods of proof (i.e., the methods for independent random variables) to obtain laws of large numbers since truncated (and absolute values of) ND random

variables do not remain ND even when the random variables are identically distributed. For example, let $\Omega = \{a, b, c, d\}$, let $\mathcal{A}$ be all subsets of Ω and let P assign probability $1/4$ to each outcome. Then the random variables X and Y defined on the probability space $(\Omega, \mathcal{A}, P)$ by

ω	a	b	c	d
$X(\omega)$	2	1	0	-2
$Y(\omega)$	-2	1	0	2

are ND, but $|X(\omega)| \equiv |Y(\omega)|$ for all $\omega \in \Omega$ and $XI_{[|X| \le 1]}(\omega) \equiv YI_{[|Y| \le 1]}$ for all $\omega \in \Omega$. Hence, absolute values and truncation to compact subsets can transform ND random variables to positive (highly) dependent random variables. However, Corollary 2.2(b) provides a method of truncation which preserves ND and will be useful in obtaining laws of large numbers in the next section.

The next two lemmas will be needed in the proofs of the WLLN's in the next section. The lemmas will only be stated since they are well known.

Lemma 2.3: *For any random variable X and $r \ge 1$, $E|X|^r < \infty$ if and only if*

$$\sum_{n=1}^{\infty} n^{r-1} P[|X| > n] < \infty.$$

More precisely,

$$r 2^{-r+1} \sum_{n=2}^{\infty} n^{r-1} P[|X| > n] \le E|X|^r \le 1 + r 2^{r-1} \sum_{n=1}^{\infty} n^{r-1} P[|X| > n].$$

Lemma 2.4: *For any random variable X, $r \ge 1$ and $p > 0$,*

$$E(|X|^r I_{[|X| \le n^{1/p}]}) \le r \int_0^{n^{1/p}} t^{r-1} P[|X| > t] dt$$

and

$$E(|X| I_{[|X| > n^{1/p}]}) = n^{1/p} P[|X| > n^{1/p}] + \int_{n^{1/p}}^{\infty} P[|X| > t] dt.$$

A family of random variables (X_α) is said to be *stochastically bounded* by a random variable X if

$$\sup_\alpha P[|X_\alpha| > t] \le P[|X| > t] \text{ for all } t \in R. \tag{2.8}$$

3. Weak Law of Large Numbers for Arrays

In this section WLLNs are obtained for arrays of rowwise ND random variables. Many of the WLLNs for sequences can be obtained for arrays with similar hypotheses. For strong laws of large numbers, the array results typically require

stronger moment conditions than the results for sequences. The basic truncation technique for arrays (cf. (3.4) and (3.5) in the proof of Theorem 3.1) makes use of Corollary 2.2(a) and is the same for arrays or sequences. Theorem 3.1 extends Feller's WLLN for sequences of i.i.d. random variables (cf., Chow and Teicher [4, p. 126]) to arrays of random variables which are pairwise ND in each row.

Theorem 3.1: *Let* $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ *be an array of random variables which are pairwise ND in each row and which have distribution functions* $\{F_{ni}\}$ *and* $S_n = \sum_{i=1}^n X_{ni}.$ *Let* $\{b_n, n \geq 1\}$ *be a given sequence of real numbers increasing to* $\infty.$ *Suppose that*

(i)
$$\sum_{i=1}^n P[\,|X_{ni}| > b_n] = o(1) \tag{3.1}$$

and

(ii)
$$\frac{1}{b_n^2} \sum_{i=1}^n \int_{[\,|x| \leq b_n]} x^2 F_{ni}(x) = o(1). \tag{3.2}$$

Then setting

$$a_n = \sum_{i=1}^n \int_{[\,|x| \leq b_n]} x dF_{ni}(x), n \geq 1, \tag{3.3}$$

the WLLN

$$\frac{1}{b_n}(S_n - a_n) \xrightarrow{P} 0$$

obtains.

Proof: Define for $n \geq 1$ and $1 \leq i \leq n$

$$Y_{ni} = X_{ni}I_{[\,|X_{ni}| \leq b_n]} + b_n I_{[X_{ni} > b_n]} - b_n I_{[X_{ni} < -b_n]}$$

and $T_n = \sum_{i=1}^n Y_{ni}.$ By (3.1)

$$P[T_n \neq S_n] \leq \sum_{i=1}^n P[Y_{ni} \neq X_{ni}]$$

$$= \sum_{i=1}^n P[\,|X_{ni}| > b_n]$$

$$= o(1). \tag{3.4}$$

Next,

$$ET_n = \sum_{i=1}^n EY_{ni}$$

$$= \sum_{i=1}^n E(X_{ni}I_{[\,|X_{ni}| \leq b_n]}) + b_n \sum_{i=1}^n P[X_{ni} > b_n]$$

$$= \sum_{i=1}^n \int_{[\,|x| \leq b_n]} xdF_{ni}(x) + b_n \sum_{i=1}^n \int_{[x > b_n]} dF_{ni}(x)$$

$$-b_n \sum_{i=1}^n \int_{[x < -b_n]} dF_{ni}(x)$$

$$= a_n + o(b_n) \tag{3.5}$$

by (3.1). Corollary 2.2(b) and Lemma 2.1(b) provide

$$\text{Var}\left(\frac{T_n}{b_n}\right) \leq \sum_{i=1}^n \text{Var}\left(\frac{Y_{ni}}{b_n}\right)$$

$$\leq \sum_{i=1}^n E\left(\frac{Y_{ni}}{b_n}\right)^2$$

$$= \frac{1}{b_n^2} \sum_{i=1}^n EX_{ni}^2 I_{[\,|X_{ni}| \leq b_n]} + \sum_{i=1}^n P[\,|X_{ni}| > b_n]$$

$$= o(1) \tag{3.6}$$

by (3.1) and (3.2).

From (3.4) and (3.6), it follows that for arbitrary $\epsilon > 0$

$$P\left[\left|\frac{S_n - ET_n}{b_n}\right| > \epsilon\right] \leq P\left[\left|\frac{T_n - ET_n}{b_n}\right| > \epsilon, S_n = T_n\right]$$

$$+ P\left[\left|\frac{S_n - ET_n}{b_n}\right| > \epsilon, S_n \neq T_n\right]$$

$$\to 0 \text{ as } n \to \infty.$$

Hence,

$$\frac{S_n - ET_n}{b_n} \xrightarrow{P} 0,$$

and the conclusion follows by (3.5).   □

The next result is a WLLN that uses a pth moment, $1 < p < 2$, and a stochastic boundedness condition on the array of rowwise ND random variables. Certain aspects of Theorem 3.1 follow in Theorem 3.2, for example (3.7) implies (3.1) with $b_n = n^{1/p}$. However, the corresponding verification of (3.2) requires use of Lemmas 2.3 and 2.4, and the centering constants a_n in Theorem 3.1 are zero for $p > 1$ and $EX_{nk} = 0$ in Theorem 3.2.

Theorem 3.2: *Let* $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ *be an array of random variables which are pairwise ND in each row with* $EX_{nk} = 0$ *and which are stochastically bounded by a random variable* X *such that*

$$nP[\,|X|^p > n] \to 0 \text{ for some } 1 < p < 2. \tag{3.7}$$

Then

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \xrightarrow{P} 0.$$

Proof: First, (3.7) implies (3.1) with $b_n = n^{1/p}$. To establish (3.2) in Theorem 3.1, for arbitrary $\epsilon > 0$, choose $A(\epsilon)$ so that

$$P[\,|X| > t\,] \leq \frac{\epsilon(p-1)}{t^p} \tag{3.8}$$

for all $t \geq A(\epsilon) \equiv A$. Thus, from Lemma 2.4 and (3.8) for all $n \geq A^p$

$$\sum_{k=1}^{n} E(X_{nk}^2 I_{[\,|X_{nk}| \leq n^{1/p}]})$$

$$\leq 2 \sum_{k=1}^{n} \int_0^{n^{1/p}} t P[\,|X_{nk}| > t\,] dt$$

$$\leq 2n \int_0^A t P[\,|X| > t\,] dt + 2n \int_A^{n^{1/p}} t \frac{\epsilon(p-1)}{t^p} dt$$

$$\leq nA^2 + 2\frac{\epsilon(p-1)}{2-p} n[(n^{1/p})^{2-p} - (A)^{2-p}]$$

$$\leq nA^2 + \frac{2(p-1)}{2-p} \epsilon n^{2/p}. \tag{3.9}$$

Then (3.2) follows from (3.9) since ϵ is arbitrary and $2/p > 1$. Theorem 3.1 applies and

$$\frac{S_n - a_n}{n^{1/p}} \xrightarrow{P} 0$$

where

$$a_n = \sum_{k=1}^{n} E(X_{nk} I_{[\,|X_{nk}| \leq n^{1/p}]}).$$

Hence, the proof will be complete by showing that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} E(X_{nk} I_{[\,|X_{nk}| \leq n^{1/p}]}) \to 0 \tag{3.10}$$

as $n \to \infty$. Since $EX_{nk} = 0$,

$$|E(X_{nk} I_{[\,|X_{nk}| \leq n^{1/p}]})| = |E(X_{nk} I_{[\,|X_{nk}| > n^{1/p}]})|. \tag{3.11}$$

By Lemma 2.4

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} E(|X_{nk}| I_{[\,|X_{nk}| > n^{1/p}]})$$

$$= \frac{1}{n^{1/p}} \left(\sum_{k=1}^{n} n^{1/p} P[\,|X_{nk}| > n^{1/p}] + \sum_{k=1}^{n} \int_{n^{1/p}}^{\infty} P[\,|X_{nk}| > t\,] dt \right)$$

$$\leq nP[\ |\ X\ |\ > n^{1/p}] + \frac{n}{n^{1/p}} \int_{n^{1/p}}^{\infty} P[\ |\ X\ |\ > t]dt. \tag{3.12}$$

Since $nP[\ |\ X\ |\ > n^{1/p}] = nP[\ |\ X\ |^p > n] \to 0$ recalling (3.7), the first term of (3.12) goes to 0 as $n \to \infty$. Next, for arbitrary $\epsilon > 0$ and for all $n \geq A^p$, it follows from (3.8) that

$$\frac{n}{n^{1/p}} \int_{n^{1/p}}^{\infty} P[\ |\ X\ > t]dt$$

$$\leq \frac{n}{n^{1/p}} \int_{n^{1/p}}^{\infty} (p-1)\frac{\epsilon}{t^p}dt$$

$$= \frac{n}{n^{1/p}}\epsilon(p-1)\frac{(n^{1/p})^{1-p}}{p-1} = \epsilon \tag{3.13}$$

implying that the second term of (3.12) goes to 0 as $n \to \infty$. Hence, (3.10) follows from (3.11) and (3.12).   □

The exclusion of $p = 1$ (cf., (3.7)) in Theorem 3.2 is interesting and relates to the proof of the sequence of centering constants. Inequalities (3.8), (3.9) and (3.13) in the proof of Theorem 3.2 depend on $p > 1$. A second major consideration is that $nP[\ |\ X\ |\ > n] \to 0$ as $n \to \infty$ can occur without the existence of a first moment (which is assumed to be 0 in Theorem 3.2). However, a corresponding ($p = 1$) WLLN is available via a different proof and different centering conditions, and is given as Theorem 3.3. Again, (3.14) implies (3.1) of Theorem 3.1 (with $b_n = n$), and the major difficult in the proof of Theorem 3.3 is the corresponding verification of a truncated variance condition similar to (3.2).

Theorem 3.3:   *Let* $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ *be an array of random variables which are pairwise ND in each row and which are stochastically bounded by a random variable* X *such that*

$$nP[\ |\ X\ |\ > n] \to 0 \ as \ n \to \infty. \tag{3.14}$$

Then

$$\frac{1}{n} \sum_{i=1}^{n} (X_{ni} - c_{ni}) \to 0 \ in \ probability$$

where $c_{ni} = E(X_{ni}I_{[\ |\ X_{ni}|\ \leq n]})$.

Proof: As noted, (3.14) yields (3.1) of Theorem 3.1 with $b_n = n$. For (3.2) consider

$$\sum_{i=1}^{n} E(X_{ni}^2 I_{[\ |\ X_{ni}|\ \leq n]})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} E(X_{ni}^2 I_{[j-1 <\ |\ X_{ni}|\ \leq j]})$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} j^2 (P[\,|X_{ni}| \leq j] - P[\,|X_{ni}| \leq j-1])$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} j^2 (P[\,|X_{nk}| > j-1] - P[\,|X_{ni}| > j])$$

$$= \sum_{i=1}^{n} (P[\,|X_{ni}| > 0] - n^2 P[\,|X_{ni}| > n]$$

$$+ \sum_{j=1}^{n-1} ((j+1)^2 - j^2) P[\,|X_{ni}| > j])$$

$$\leq \sum_{i=1}^{n} (1 + \sum_{j=1}^{n-1} (2j+1) P[\,|X_{ni}| > j])$$

$$\leq n + 2n \sum_{j=1}^{n-1} j P[\,|X| > j] + n \sum_{j=1}^{n-1} P[\,|X| > j]$$

which is $o(n^2)$ by (3.14) since convergence implies Cesàro convergence. Hence, Theorem 3.1 yields the desired result. □

Corollary 3.4: *Let $\{X_n\}$ be a sequence of identically distributed, pairwise ND random variables. If*

$$nP[\,|X_1| > n] \to 0 \tag{3.15}$$

as $n \to \infty$, then

$$\frac{\sum_{i=1}^{n}(X_i - c_n)}{n} \to 0 \tag{3.16}$$

where $c_n = E(X_1 I_{[\,|X_1| \leq n]}) + o(1), n \geq 1$.

Proof: Identical distributions provide the stochastic boundedness condition. □

Remarks: (a) Via Corollary 3.4, it is easy to see that Theorem 3.3 is sharp since for the independent case, (3.15) and (3.16) are equivalent (Feller's i.i.d. WLLN).

(b) By examining the proof of Theorem 3.3, it can be seen that condition (3.14) and stochastic boundedness can be replaced by the weaker condition

$$n \left(\sup_{m \geq 1, 1 \leq i \leq m} P[\,|X_{mi}| > n] \right) \to 0.$$

(c) The case $0 < p < 1$ is interesting since the magnitude of the divisor $n^{1/p}$ allows $nP[\,|X|^p > n] \to 0$ to yield

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} X_{ni} \xrightarrow{P} 0$$

without any assumptions on the joint distribution of the random variables $\{X_{ni}; 1 \leq i \leq n\}$. The key step is that

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} E(\,|X_{ni}| I_{[\,|X_{ni}| \leq n^{1/p}]}) \to 0$$

can be obtained by using similar arguments as in (3.9).

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