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NONLINEAR ESSENTIAL MAPS OF MÖNCH, 1-SET CONTRACTIVE DEMICOMPACT AND MONOTONE $(S)_+$ TYPE

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In this paper the notion of an essential map is extended to a wider class of maps. Here we show if F is essential and $F \cong G$, then G has a fixed point.

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1. Introduction

In this paper we extend the notion of an essential map introduced by Granas in [4] to a larger class of maps. The notion of essential is more general than the notion of degree, and in [4] it was shown that if F is essential and $F \cong G$ then G is essential. However, to be essential is quite general and as a result, Granas was only able to show this homotopy property for particular classes of maps (usually compact or more generally condensing maps [7]). Precup in [10] extended this notion to other maps by introducing a "generalized topological transversality principle". However, from an application point of view, the authors in [4, 10] were asking too much. What one needs usually in applications is the following question to be answered: If F is essential and $F \cong G$, does G have a fixed point? In this paper we discuss this question and we show that for many classes of maps that arise in applications, this is in fact what happens. In particular, in Section 2, we discuss Mönch type maps, in Section 3, 1-set contractive demicompact maps and in Section 4, monotone maps of $(S)_+$ type to illustrate the ideas involved. It is worth remarking as well that the ideas presented in this paper are elementary (in fact they only rely on Urysohn's Lemma). This paper will only discuss single valued maps (the multivalued case will be discussed in a forthcoming paper).

2. Mönch Type Maps

Throughout this section, E is a Banach space, U is an open subset of E, with $0 \in U$ and $F: \bar{U} \to E$ is continuous (here $\bar{U}$ denotes the closure of U in E).

Definition 2.1: We let $M_{\partial U}(\bar{U}, E)$ denote the set of all continuous maps $F: \bar{U} \to E$, which satisfy Mönch's condition (i.e., if $C \subseteq \bar{U}$ is countable and $C \subseteq \overline{co}(\{0\} \cup F(C))$ then $\bar{C}$ is compact) and with $(I - F)(x) \neq 0$ for $x \in \partial U$; here I is the identity map and ∂U the boundary of U in E.

Remark 2.1: Mönch type maps were introduced in [6] (see also [3]).

Definition 2.2: A map $F \in M_{\partial U}(\bar{U}, E)$ is *essential* if for every $G \in M_{\partial U}(\bar{U}, E)$ with $G\mid_{\partial U} = F\mid_{\partial U}$, there exists $x \in U$, with $(I - G)(x) = 0$.

Theorem 2.1: *Let E be a Banach space, U an open subset of E, and $0 \in U$. Suppose $F \in M_{\partial U}(\bar{U}, E)$ is an essential map and $H: \bar{U} \times [0,1] \to E$ is a continuous map with the following properties:*

$$H(x,0) = F(x) \text{ for } x \in \bar{U} \tag{2.1}$$

$$(I - H_t)(x) \neq 0 \text{ for any } x \in \partial U \text{ and } t \in (0,1] \text{ (here } H_t(x) = H(x,t)) \tag{2.2}$$

and

$$\begin{cases} \text{for any continuous } \mu: \bar{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \text{ the map} \\ R_\mu: \bar{U} \to E \text{ defined by } R_\mu(x) = H(x, \mu(x)) \text{ satisfies Mönch's} \\ \text{condition (i.e. if } C \subseteq \bar{U} \text{ is countable and } C \subseteq \overline{co}(\{0\} \cup R_\mu(C)) \\ \text{then } \bar{C} \text{ is compact).} \end{cases} \tag{2.3}$$

Then H_1 has a fixed point in U.

Remark 2.2: It is possible to replace (2.3) in Theorem 2.1 with: *if $C \subseteq \bar{U}$ is countable and $C \subseteq \overline{co}(\{0\} \cup H(C \times [0,1]))$ then $\bar{C}$ is compact.*

Proof: Let

$$B = \{x \in \bar{U}: \ (I - H_t)(x) = 0 \text{ for some } t \in [0,1]\}.$$

When $t = 0$, $I - H_t = I - F$ and since $F \in M_{\partial U}(\bar{U}, E)$ is essential, there exists $x \in U$, with $(I - F)(x) = 0$. Thus $B \neq \emptyset$. The continuity of H implies that B is closed. In addition, (2.2) (together with $F \in M_{\partial U}(\bar{U}, E)$) implies $B \cap \partial U = \emptyset$. Thus there exists a continuous $\mu: \bar{U} \to [0,1]$, with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $R: \bar{U} \to E$ by

$$R(x) = H(x, \mu(x)).$$

Now, R is continuous and satisfies Mönch's condition (see (2.3)). Moreover, for $x \in \partial U$, $(I - R)(x) = (I - H_0)(x) = (I - F)(x) \neq 0$ and so $R \in M_{\partial U}(\bar{U}, E)$. Also notice

$$R\mid_{\partial U}=H_0\mid_{\partial U}=F\mid_{\partial U},$$

and since $F\in M_{\partial U}(\bar{U},E)$ is essential, there exists $x\in U$ with $(I-R)(x)=0$ (i.e., $(I-H_{\mu(x)})(x)=0$). Thus $x\in B$ and so $\mu(x)=1$. Consequently, $(I-H_1)(x)=0$ and we are finished. □

We now use Theorem 2.1 to obtain a nonlinear alternative of Leray-Schauder type for Mönch maps. To prove our result we need the following well known result from the literature [3].

Theorem 2.2: *Let E be a Banach space and D a closed, convex set of E, with $0\in D$. Suppose $J:D\to D$ is a continuous map, which satisfies Mönch's condition. Then J has a fixed point in D.*

Theorem 2.3: *Let E be a Banach space, U an open subset of E and $0\in U$. Suppose $G:\bar{U}\to E$ is a continuous map, which satisfies Mönch's condition and assume*

$$tG(x)\neq x \textit{ for } x\in\partial U \textit{ and } t\in(0,1). \tag{2.4}$$

Then G has a fixed point in $\bar{U}$.

Proof: We assume that $G(x)\neq x$ for $x\in\partial U$ (otherwise we are finished). Then

$$tG(x)\neq x \text{ for } x\in\partial U \text{ and } t\in[0,1]. \tag{2.5}$$

Let $H(x,t)=tG(x)$ for $(x,t)\in\bar{U}\times[0,1]$ and $F(x)=0$ for $x\in\bar{U}$. Clearly, (2.1) and (2.2) hold. To see the validity of (2.3), let $C\subseteq\bar{U}$ be countable and $C\subseteq\overline{co}(\{0\}\cup R_\mu(C))$. Now since $R_\mu(x)=\mu(x)G(x)$, we have $R_\mu(C)\subseteq co(G(C)\cup\{0\})$. In addition, since $co(G(C)\cup\{0\})$ is convex and $\{0\}\cup co(G(C)\cup\{0\})=co(G(C)\cup\{0\})$, we have

$$C\subseteq\overline{co}(\{0\}\cup R_\mu(C))\subseteq\overline{co}(co(G(C)\cup\{0\}))=\overline{co}(G(C)\cup\{0\}).$$

Since G satisfies Mönch's condition, we have $\bar{C}$ compact. Thus (2.3) holds. We can apply Theorem 2.1 if we show that F is essential. To see this, let $\theta\in M_{\partial U}(\bar{U},E)$ with $\theta\mid_{\partial U}=F\mid_{\partial U}=0$. We must show that there exists $x\in U$ with $\theta(x)=x$. Let $D=\overline{co}(\theta(\bar{U}))$ and let $J:D\to D$ be defined by

$$J(x)=\begin{cases}\theta(x), & x\in\bar{U}\\ 0, & x\notin\bar{U}.\end{cases}$$

Now $0\in D$ and $J:D\to D$ is continuous and satisfies Mönch's condition. To see this, let $C\subseteq D$ be countable with $C\subseteq\overline{co}(\{0\}\cup J(C))$. Then $C\subseteq\overline{co}(\{0\}\cup\theta(\bar{U}\cap C))$. Thus $C\cap\bar{U}\subseteq\bar{U}$ is countable and

$$C\cap\bar{U}(\subseteq C)\subseteq\overline{co}(\{0\}\cup\theta(\bar{U}\cap C)).$$

Now since $\theta:\bar{U}\to E$ satisfies Mönch's condition, we have $\overline{C\cap\bar{U}}$ compact. Thus since θ is continuous, $\theta(\overline{C\cap\bar{U}})$ is compact and Mazur's Theorem implies $\overline{co}(\{0\cup\theta(\overline{C\cap\bar{U}}))$ is compact. Now since $C\subseteq\overline{co}(\{0\}\cup\theta(\overline{C\cap\bar{U}}))$, we have $\bar{C}$ compact. Consequently, $J:D\to D$ is continuous and satisfies Mönch's condition. Theorem 2.2 implies that there exists $x\in D$ with $J(x)=x$. Now if $x\notin U$, we have $0=J(x)=x$, which is a contradiction, since $0 \in U$. Thus $x \in U$; so $x = J(x) = \theta(x)$. Hence, F is essential and we may apply Theorem 2.1 to deduce the result. $\square$

3. 1-Set Contractive, Demicompact Maps

Let E be a Banach space and U be an open, bounded subset of E, with $0 \in U$. In this section we are interested in maps $F: \bar{U} \to E$ which are continuous, 1-set contractive and demicompact. Recall that F is k-set contractive (here $k \geq 0$ is a constant) if $\alpha(F(\Omega)) \leq k\alpha(\Omega)$ for any $\Omega \subseteq \bar{U}$ (here α denotes the Kuratowskii measure of noncompactness). F is demicompact if each sequence $\{x_n\} \subseteq \bar{U}$ has a convergent subsequence $\{x_{n_k}\}$, whenever $\{x_n - F(x_n)\}$ is a convergent sequence in E.

Definition 3.1: We let $DM_{\partial U}(\bar{U}, E)$ denote the set of all continuous, 1-set contractive, demicompact maps $F: \bar{U} \to E$, with $(I - F)(x) \neq 0$ for $x \in \partial U$.

Remark 3.1: Demicompact 1-set contractive maps were discussed in detail in [8, 9].

Definition 3.2: A map $F \in DM_{\partial U}(\bar{U}, E)$ is *essential* if for every $G \in DM_{\partial U}(\bar{U}, E)$, with $G|_{\partial U} = F|_{\partial U}$, there exists $x \in U$, with $(I - G)(x) = 0$.

Theorem 3.1: *Let E be a Banach space and U be an open, bounded subset of E with $0 \in U$. Suppose $kF \in DM_{\partial U}(\bar{U}, E)$ is essential for every $k \in [0,1]$ (it is enough to assume this for $k \in [\epsilon, 1]$ for some fixed ϵ, $0 \leq \epsilon < 1$). Let $H: \bar{U} \times [0,1] \to E$ be continuous, 1-set contractive (i.e., $\alpha(H(A \times [0,1])) \leq \alpha(A)$ for any $A \subseteq \bar{U}$) map with the following properties:*

$$H(x,0) = F(x) \text{ for } x \in \bar{U} \tag{3.1}$$

$$\text{there exists } \delta > 0 \text{ with } |(I - H_t)(x)| \geq \delta \text{ for } x \in \partial U \text{ and } t \in [0,1] \tag{3.2}$$

and

$$H_1: \bar{U} \to E \text{ is a demicompact map.} \tag{3.3}$$

Then H_1 has a fixed point in U.

Remark 3.2: It is enough to assume (3.2) for $t \in (0,1]$, since $F: \bar{U} \to E$ is demicompact with $(I - F)(x) \neq 0$ for $x \in \partial U$; so there exists $\delta_0 > 0$ with $|(I - F)(x)| \geq \delta_0$ for $x \in \partial U$.

Proof: Now there exists an $M > 0$, with $|H_t(x)| \leq M$ for $(x,t) \in \bar{U} \times [0,1]$. Choose $M > 0$ so that $1 - \frac{\delta}{2M} > 0$. Fix $k \in \left(1 - \frac{\delta}{2M}, 1\right)$ and consider $H^k: \bar{U} \times [0,1] \to E$ defined by

$$H^k(x,t) = kH(x,t).$$

We first show that there exists an $x_k \in U$, with

$$x_k = H_1^k(x_k) \text{ (here } H_1^k = kH_1). \tag{3.4}$$

Let

$$B = \left\{x \in \bar{U} : (I - H_t^k)(x) = 0 \text{ for some } t \in [0,1]\right\}.$$

When $t=0$, $I-H_0^k=I-kF$ and since $kF\in DM_{\partial U}(\bar{U},E)$ is essential, there exists an $x\in U$, with $(I-kF)(x)=0$. Thus, $B\neq\emptyset$. Also B is closed. Next we **claim** $B\cap\partial U=\emptyset$. To see this, first notice that $|(H_t^k-H_t)(x)| = |(1-k)H_t(x)| \le (1-k)M$ for $x\in\bar{U}$ and $t\in[0,1]$. Thus for $x\in\partial U$ and $t\in[0,1]$, we have from (3.2) that

$$|(I-H_t^k)(x)| \ge |(I-H_t)(x)| - |(H_t^k-H_t)(x)| \ge \delta-(1-k)M \ge \frac{\delta}{2};$$

note that $k\in\left(1-\frac{\delta}{2M},1\right)$. Thus our claim is true. As a result, there exists a continuous $\mu:\bar{U}\to[0,1]$ with $\mu(\partial U)=0$ and $\mu(B)=1$. Define the map $R:\bar{U}\to E$ by

$$R(x)=H^k(x,\mu(x)).$$

It is easy to see that R is continuous and k-set contractive (so automatically, 1-set contractive and demicompact [11]). Moreover, for $x\in\partial U$,

$$|(I-R)(x)| = |(I-H_0^k)(x)| \ge \frac{\delta}{2}$$

and so $R\in DM_{\partial U}(\bar{U},E)$. Also notice that

$$R\,|_{\partial U} = H_0^k\,|_{\partial U} = kF\,|_{\partial U}$$

and since $kF\in DM_{\partial U}(\bar{U},E)$ is essential, there exists $x_k\in U$, with $(I-R)(x_k)=0$ (i.e., $(I-H^k_{\mu(x_k)})(x_k)=0$). Thus, $x_k\in B$ and so $\mu(x_k)=1$. Hence, (3.4) is true.

We can apply the above argument for any $k\in\left(1-\frac{\delta}{2M},1\right)$. Choose $n_0>1$, with $n_0\in\{1,2,\ldots\}$ so that $1-\frac{1}{n_0}>1-\frac{\delta}{2M}$. Let $N^+=\{n_0,n_0+1,\ldots\}$. For each $n\in N_0$, there exists $x_n\in U$ with

$$x_n=\left(1-\frac{1}{n}\right)H_1(x_n)$$

and so

$$x_n-H_1(x_n) = -\left(\frac{1}{n}\right)H_1(x_n). \tag{3.5}$$

Now since $|H_1(x)|\le M$ for $x\in\bar{U}$, $\{x_n-H_1(x_n)\}$ is a convergent sequence in E. Since H_1 is demicompact there exists a subsequence S of N^+ and $x\in\bar{U}$, with $x_n\to x$ as $n\to\infty$ in S. Let $n\to\infty$ in S in (3.5) (note H_1 is continuous) to deduce that $x-H_1(x)=0$. In fact, $x\in U$ from (3.2). $\square$

Remark 3.3: We may replace (3.3) in Theorem 3.1 with

$$(I-H_1)(\bar{U}) \text{ is closed.}$$

Also, in Theorem 3.1, the boundedness of U could be replaced by the boundedness of the maps.

We now use Theorem 3.1 to obtain a nonlinear alternative of Leray-Schauder type for 1-set contractive demicompact maps. We need the following result from the literature [8, pp. 326-327].

Theorem 3.2: *Let E be a Banach space and D be a nonempty, bounded, closed, convex subset of E. Suppose $J: D \to D$ is a continuous, 1-set contractive, and demicompact map. Then J has a fixed point in D.*

Theorem 3.3: *Let E be a Banach space and U be an open bounded subset of E with $0 \in U$. Suppose $G: \bar{U} \to E$ is a continuous, 1-set contractive, demicompact map, with*

$$tG(x) \neq x \text{ for } x \in \partial U \text{ and } t \in (0,1). \tag{3.6}$$

Then G has a fixed point in $\bar{U}$.

Proof: We assume $G(x) \neq x$ for $x \in \partial U$ (otherwise we are finished). Then

$$tG(x) \neq x \text{ for } x \in \partial U \text{ and } t \in [0,1]. \tag{3.7}$$

Let $H(x,t) = tG(x)$ for $(x,t) \in \bar{U} \times [0,1]$ and $F(x) = 0$ for $x \in \bar{U}$. Clearly, (3.1) and (3.3) hold. To see (3.2), suppose it is not true. Then there exist $\{x_n\} \subseteq \partial U$ and a sequence $\{t_n\} \subseteq [0,1]$, with $x_n - t_n G(x_n) \to 0$ as $n \to \infty$. Without loss of generality, assume $t_n \to t$. Then,

$$x_n - tG(x_n) = x_n - t_n G(x_n) + (t_n - t)G(x_n) \to 0 \text{ as } n \to \infty.$$

If $t = 0$, then $x_n \to 0$; so $0 \in \partial U$ and this is a contradiction. If $t \in (0,1]$, then tG is demicompact (if $t = 1$ then $tG = G$, whereas if $t \in (0,1)$ then tG is t-set contractive so demicompact [11]), so there exists a subsequence $\{x_{n_k}\}$ and a $x \in \partial U$, with $x_{n_k} \to x$. Also since G is continuous, $x - tG(x) = 0$. This contradicts (3.7). Thus (3.2) holds. We can apply Theorem 3.1 if we show kF is essential. To see this, let $k \in [0,1]$ be fixed and let $\theta \in DM_{\partial U}(\bar{U}, E)$ with $\theta|_{\partial U} = kF|_{\partial U} = 0$. We must show that there exists $x \in U$ with $\theta(x) = x$. Let $D = \overline{co}(\theta(\bar{U}))$ and let $J: D \to D$ be defined by

$$J(x) = \begin{cases} \theta(x), & x \in \bar{U} \\ 0, & x \notin \bar{U} \end{cases}.$$

It is easy to see that $J: D \to D$ is continuous, 1-set contractive and demicompact. To see demicompactness, suppose $\{x_n\} \subseteq D$ and let $\{x_n - J(x_n)\}$ be a convergent sequence in D. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, with $x_{n_k} \in \bar{U}$ for each n_k (in which case since θ is demicompact, $\{x_{n_k}\}$ has a convergent subsequence) or $x_{n_k} \notin \bar{U}$ for each n_k (in which case $\{x_{n_k}\} = \{x_{n_k} - J(x_{n_k})\}$ is convergent by the assumption). Theorem 3.2 implies that there exists $x \in D$, with $J(x) = x$. Now if $x \notin U$ we have $0 = J(x) = x$, which is a contradiction, since $0 \in U$. Thus, $x \in U$ so $x \in J(x) = \theta(x)$. Hence, kF is essential and we may apply Theorem 3.1 to deduce the result. □

4. Demicontinuous $(S)_+$ Maps

In this section, E will be a Banach space. E^* will denote the conjugate space of E and $(\cdot,\cdot)$ the duality between E^* and E. Let X be a subset of E. Now

(*i*) $f: X \to E^*$ is said to be *monotone* if $(f(x)-f(y), x-y) \geq 0$ for all $x, y \in X$,
(*ii*) $f: X \to E^*$ is said to be *of class* $(S)_+$ if for any sequence (x_j) in X, for which $x_j \overset{\text{weak}}{\to} x$ and $\limsup(f(x_j), x_j - x) \leq 0$, we have $x_j \to x$ (here $\overset{\text{weak}}{\to}$ denotes weak convergence),
(*iii*) $f: X \to E^*$ is said to be *maximal monotone* if it is monotone and maximal in the sense of graph inclusion among monotone maps from X to E^*,
(*iv*) $f: X \to E^*$ is called *hemicontinuous* if $f(x+ty) \overset{\text{weak}}{\to} f(x)$ as $t \to 0$,
(*v*) $f: X \to E^*$ is called *demicontinuous* if $y \to x$ implies $f(y) \overset{\text{weak}}{\to} f(x)$.

Throughout this section, E will be a reflexive Banach space. We assume that E is endowed with an equivalent norm, with respect to which, E and E^* are locally uniformly convex (this is always possible [1]). Then there exists a unique mapping (duality mapping) $J: E \to E^*$ such that $(J(x), x) = |x|^2 = |Jx|^2$ for all $x \in E$. Moreover, J is bijective, bicontinuous, monotone and of class $(S)_+$ (see [1, p. 20]).

Throughout this section, E, E^* and J will be as above. Also, U will be a nonempty, bounded, open subset of E and $T: E \to E^*$ will be a fixed monotone, hemicontinuous, locally bounded mapping.

Remark 4.1: From [5, p. 548], T is demicontinuous. Recall that $T: X \to E^*$ is locally bounded if $u_n \in X$, $u \in X$ and $u_n \to u$ imply that Tu_n is bounded.

Remark 4.2: Recall that any monotone hemicontinuous mapping is maximal monotone. Moreover, since $T: E \to E^*$ is maximal monotone, then $J+T$ is bijective and $(J+T)^{-1}: E^* \to E$ is demicontinuous.

Definition 4.1: We let $EM_{\partial U}(\bar{U}, E)$ denote the maps $f = (J+T)^{-1}(J-F)$: $\bar{U} \to E$, where $F: \bar{U} \to E^*$ is demicontinuous, bounded (i.e., maps bounded sets into bounded sets) of class $(S)_+$ with $(T+F)(x) \neq 0$ for $x \in \partial U$. In this case we say $f = (J+T)^{-1}(J-F) \in EM_{\partial U}(\bar{U}, E)$.

Definition 4.2: A map $f = (J+T)^{-1}(J-F) \in EM_{\partial U}(\bar{U}, E)$ is *essential* if for every $g = (J+T)^{-1}(J-G) \in EM_{\partial U}(\bar{U}, E)$, with $G|_{\partial U} = F|_{\partial U}$, there exists $x \in U$ with $(T+G)(x) = 0$.

Theorem 4.1: *Let E, E^*, U, J and T be as above. Suppose $f = (J+T)^{-1}(J-F) \in EM_{\partial U}(\bar{U}, E)$ is essential and $H: \bar{U} \times [0,1] \to E^*$ is bounded with the following properties:*

$$H(x,0) = F(x) \text{ for } x \in \bar{U} \tag{4.1}$$

$$\{x \in \bar{U}: (T+H_t)(x) = 0 \text{ for some } t \in (0,1]\} \text{ does not intersect } \partial U \tag{4.2}$$

and

$$\begin{cases} \text{for any sequence } \{x_j\} \subseteq \bar{U} \text{ with } x_j \overset{\text{weak}}{\to} x \text{ and any sequence } \{t_j\} \subseteq [0,1] \\ \text{with } t_j \to t \text{ for which } \limsup(H_{t_j}(x_j), x_j - x) \leq 0 \text{ we have} \\ x_j \overset{\text{weak}}{\to} x \text{ and } H_{t_j}(x_j) \to H_t(x) \text{ (here } H_t(x) = H(x,t)). \end{cases} \tag{4.3}$$

Then $T + H_1$ has a fixed point in U.

Proof: Let

$$B = \{x \in \bar{U}: (T+H_t)(x) = 0 \text{ for some } t \in [0,1]\}.$$

When $t = 0$, we have $T + H_0 = T + F$ and since $(J+T)^{-1}(J-F) \in EM_{\partial U}(\bar{U}, E)$ is

essential, there exists an $x \in U$ with $(T+F)(x)=0$. Thus, $B \neq \emptyset$. Next we show that B is closed. Let (x_j) be a sequence in B with $x_j \to x \in \bar{U}$ (in particular, $x_j \overset{\text{weak}}{\to} x$).
Thus, $(T+H_{t_j})(x_j)=0$ for some sequence (t_j) in $[0,1]$. Without loss of generality, assume $t_j \to t$. Now since T is monotone,

$$(H_{t_j}(x_j), x_j - x) = (-Tx_j + Tx, x_j - x) + (-Tx, x_j - x) \leq (-Tx, x_j - x)$$

and this, together with $x_j \to x$, gives

$$\limsup (H_{t_j}(x_j), x_j - x) \leq 0.$$

Now (4.3) implies $H_{t_j}(x_j) \overset{\text{weak}}{\to} H_t(x)$. This together with $H_{t_j}(x_j) + T(x_j) = 0$ and T demicontinuous gives $H_t(x) + T(x) = 0$. Consequently, B is closed. Also (4.3) and the fact that $(J+T)^{-1}(J-F) \in EM_{\partial U}(\bar{U}, E)$ implies $B \cap \partial U = \emptyset$. Thus there exists a continuous $\mu: \bar{U} \to [0,1]$, with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Let

$$R(x) = H(x, \mu(x)).$$

We first show R is demicontinuous. Let (x_j) be a sequence in $\bar{U}$ with $x_j \to x$. We can assume without loss of generality that $\mu(x_j) \to \lambda$. In fact, $\lambda = \mu(x)$, since μ is continuous. Since $x_j \to x$, we have $\limsup (H_{\mu(x_j)}(x_j), x_j - x) \leq 0$. Now (4.3) implies $R(x_j) = H_{\mu(x_j)}(x_j) \overset{\text{weak}}{\to} H_{\mu}(x) = R(x)$ and so R is demicontinuous. To show that R is of class $(S)_+$, let(x_j) be a sequence in $\bar{U}$ with $x_j \overset{\text{weak}}{\to} x$ and $\limsup (H_{\mu(x_j)}(x_j), x_j - x) \leq 0$. Now we may assume without loss of generality that $\mu(x_j) \to t$. Now (4.3) implies $x_j \to x$ and $H_{\mu(x_j)}(x_j) \overset{\text{weak}}{\to} H_t(x)$ [since $x_j \to x$ and μ is continuous, we have $t = \mu(x)$]. Thus, R is of class $(S)_+$. Also, if $x \in \partial U$ we have $(T+R)(x) = (T+H_0)(x) = (T+F)(x) \neq 0$. Thus $(J+T)^{-1}(J-R) \in EM_{\partial U}(\bar{U}, E)$. Next notice that

$$R\mid_{\partial U} = H_0 \mid_{\partial U} = F \mid_{\partial U},$$

and since $f = (J+T)^{-1}(J-F) \in EM_{\partial U}(\bar{U}, E)$ is essential, there exists $x \in U$ with $(T+R)(x) = 0$ (i.e., $(T+H_{\mu(x)})(x) = 0$). Thus, $x \in B$ and so, $\mu(x) = 1$. Consequently, $(T+H_1)(x) = 0$ and we are finished. □

Theorem 4.2: *Let E, E^*, U, J and T be as above and suppose $f = (J+T)^{-1}(J-F) \in EM_{\partial U}(\bar{U}, E)$ is essential. In addition, assume that the following are satisfied:*

$$G: \bar{U} \to E^* \text{ is demicontinuous, bounded and of class } (S)_+ \tag{4.4}$$

and

$$T(x) + (1-t)F(x) + tG(x) \neq 0 \text{ for all } t \in (0,1] \text{ and } x \in \partial U. \tag{4.5}$$

Then there exists $x \in U$ with $(T+G)(x) = 0$.

Proof: Let $H(x,t) = (1-t)F(x) + tG(x)$. Clearly (4.1), (4.2) and (4.3) (see [1, pp. 27, 28] hold. The result is immediate from Theorem 4.1. □

Theorem 4.3: *Let E, E^*, U, J and T be as above and let $G: \bar{U} \to E^*$ be demicontinuous, bounded and of class $(S)_+$. In addition, assume that*

$$0 \in U \text{ and } T(0) = 0 \tag{4.6}$$

and

$$T(x) + (1-t)J(x) + tG(x) \neq 0 \text{ for all } t \in [0,1] \text{ and } x \in \partial U$$

are satisfied. Then there exists $x \in U$ with $(T+G)(x) = 0$.

Proof: Let $F = J$ in Theorem 4.2. A standard result from the literature [10, p. 77] or [2]) immediately implies that $0 = (J+T)^{-1}(J-J) \in EM_{\partial U}(\bar{U}, E)$ is essential. □

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