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NONOSCILLATION OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Necessary and sufficient conditions are found for existence of at least one bounded nonoscillatory solution of a class of impulsive differential equations of third order and fixed moments of impulse effect. Some asymptotic properties of the nonoscillating solutions are investigated.

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1. Introduction

The impulsive differential equations with deviating argument are adequate mathematical models of numerous processes and phenomena in physics, biology and electrical engineering. In spite of wide possibilities for their application, the theory of these equations is developing rather slowly because of considerable difficulties in technical and theoretical character related to their study.

In the recent twenty years, the number of investigations devoted to the oscillatory and nonoscillatory behavior of the solutions of functional differential equations has considerably increased. A large part of the works of this subject published in 1977 is presented in [5]. In monographs [2] and [3], published in 1987 and 1991, respectively, the oscillatory and asymptotic properties of the solutions of various classes of functional differential equations were systematically studied. A pioneering work devoted to the investigation of the oscillatory properties of the solutions of impulsive differential equations with deviating argument was rendered by Gopalsamy and Zhang [1].

In the present paper, necessary and sufficient conditions are found for existence of at least one bounded nonoscillatory solution of a class of impulsive differential equations of third order and fixed moments of impulse effect. Some asymptotic properties of the nonoscillating solutions are investigated.

2. Preliminary Notes

Consider the impulsive differential equation

$$y'''(t) + f(t, y(t)) = 0, \quad t \neq \tau_k, k \in N,$$
$$\Delta y''(\tau_k) + f_k(y(\tau_k)) = 0, \quad k \in N, \tag{1}$$
$$\Delta y'(\tau_k) = \Delta y(\tau_k) = 0, \quad k \in N$$

with initial conditions

$$y^{(i)}(0) = y_i, \tag{2}$$

where $y_i \in R$, $i = 0, 1, 2$.

Here $\Delta y''(\tau_k) = y''(\tau_k + 0) - y''(\tau_k - 0)$. We suppose that $y(\tau_k - 0) = y(\tau_k)$; $y'(\tau_k - 0) = y'(\tau_k)$; $R_+ = (0, +\infty)$; $\tau_1, \tau_2, \ldots$ are the moments of impulse effect.

Introduce the following conditions:

H1. $0 < \tau_1 < \tau_2 < \ldots$, $\lim_{k \to +\infty} \tau_k = +\infty$.

H2. $f \in C(R_+ \times R, R)$, $uf(t, u) > 0$ for $u \neq 0$, $t \in R_+$; $|f(t, u_1)| \leq |f(t, u_2)|$ for $|u_1| \leq |u_2|$, $u_1, u_2 \in R$, $t \in R_+$.

H3. $f_k \in C(R, R)$, $uf_k(u) > 0$ for $u \neq 0$ and $|f_k(u_1)| \leq |f_k(u_2)|$ for $|u_1| \leq |u_2|$, $u_1, u_2 \in R$, $k \in N$.

Definition 1: A function $y \in C(R_+, R)$ is called a *solution* of the equation (1) with initial conditions (2) if it satisfies the following conditions:

(*a*) If $0 = \tau_0 \leq t \leq \tau_1$, then the function y coincides with the solution of the equation

$$y'''(t) + f(t, y(t)) = 0$$

with initial conditions (2).

(*b*) If $\tau_k < t \leq \tau_{k+1}$, then the function y coincides with the solution of the equation

$$y'''(t) + f(t, y(t)) = 0$$

with initial conditions

$$y''(\tau_k + 0) = y''(\tau_k - 0) - f_k(y(\tau_k)),$$
$$y'(\tau_k + 0) = y'(\tau_k - 0) = y'(\tau_k),$$
$$y(\tau_k + 0) = y(\tau_k - 0) = y(\tau_k).$$

Definition 2: The nonzero solution $y(t)$ of the problem (1), (2) is said to be *nonoscillatory* if there exists a point $t_0 \geq 0$ such that $y(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $y(t)$ is said to *oscillate*.

Definition 3: ([4]) A set Ω of real-valued functions defined on the interval $[t_0, +\infty)$ is said to be equiconvergent at ∞ if all functions in Ω are convergent in R as $t \to \infty$ and for any $\varepsilon > 0$ there exists $t_0' \geq t_0$ such that for each function $f \in \Omega$, the inequality $|f(t) - \lim_{s \to \infty} f(s)| < \varepsilon$ is valid for $t \geq t_0'$.

Lemma 1: ([4]) *Let Ω be uniformly bounded and an equicontinuous subset of the Banach space $B([t_0, \infty))$, and let Ω be equiconvergent at ∞. then the set Ω is relatively compact.*

3. Main Results

Theorem 1: *Let the following conditions hold:*

(*a*) *Conditions* ***H*1-*H*3** *are met.*

(*b*) *There exists a point* $T \geq 0$ *such that*

$$\int_T^{\infty} (u-T)^2 \mid f(u,c) \mid du + \sum_{T < \tau_k} (\tau_k - T)^2 \mid f_k(c) \mid \ = +\infty.$$

for some constant $c \neq 0$.

Then every bounded solution $y(t)$ *of the equation* (1) *either oscillates or*

$$\lim_{t \to +\infty} y(t) = \lim_{t \to +\infty} y'(t) = \lim_{t \to +\infty} y''(t) = 0.$$

Proof: Let $y(t)$ be a positive and bounded solution of the equation (1) for $t \geq t_1 \geq 0$. From condition **H3**, $f_k(y(\tau_k)) > 0$ for $\tau_k \geq t_1$. Then

$$\Delta y''(\tau_k) < 0, \ \ \tau_k \geq t_1. \tag{3}$$

From $y(t) > 0$, $t \geq t_1$ and condition ***H*2** implies $f(t, y(t)) > 0$, $t > t_1$. Therefore,

$$y'''(t) < 0, \quad t \geq t_1 \tag{4}$$

From (3) and (4) it follows that $y''(t)$ is a decreasing function for $t \geq t_1$.

The following two cases are possible:

Case 1: $y''(t) > 0$ for $t \geq t_1$. Then $y'(t)$ is an increasing function for $t \geq t_1$.

1.1: If $y'(t) > 0$ for $t \geq t_2 \geq t_1$, then $y'(t) \geq y'(t_2) > 0$. We integrate the last inequality from t_2 to $t (t \geq t_2)$ and conclude

$$y(t) \geq y'(t_2)(t - t_2) + y(t_2).$$

It follows from the above inequality as $t \to +\infty$, that $\lim_{t \to \infty} y(t) = +\infty$ which contradicts the assumption that y is a bounded solution of the equation (1).

1.2: If $y'(t) < 0$ for $t \geq t_2 \geq t_1$. Then $y(t)$ is decreasing and bounded, so there exists a limit, $\lim_{t \to +\infty} y(t) = c_1 \geq 0$. From $y''(t) > 0$, $\Delta y'(\tau_k) = 0$ for $t, \tau_k \geq t_1$ to see that $y'(t)$ is an increasing negative function. Therefore, $\lim_{t \to +\infty} y'(t) = c_2 \leq 0$.

Let us suppose $c_2 < 0$. Then there exists a constant $c_3 < 0$ and a point $t_3 \geq t_2$ such that $y'(t) \leq c_3$ for $t \geq t_3$. Now, we integrate the above inequality from t_3 to t, $(t \geq t_3)$ and arrive at the inequality $y(t) \leq c_3 t + y(t_3)$. It follows from the above inequality after taking the limit as $t \to +\infty$, that $\lim_{t \to +\infty} y(t) = -\infty$, which contradicts the assumption that y is a positive bounded solution of the equation (1). Therefore,

$$\lim_{t \to +\infty} y'(t) = 0.$$

From $y'''(t) < 0$, $\Delta y''(\tau_k) < 0$ for t, $\tau_k \geq t_1$ we see that $y''(t)$ is a decreasing positive function. Therefore, $\lim_{t \to +\infty} y''(t) = c_4 \geq 0$. We want to prove that $c_4 = 0$. Assume that $c_4 > 0$. Then there exists a constant $c_5 > 0$ and a point $t_4 \geq t_1$ such that $y''(t) \geq c_5$ for $t \geq t_4$. Now, we integrate the above inequality from t_4 to t,

$(t \geq t_4)$ and arrive at the inequality

$$y'(t) \geq c_5(t - t_4) + y'(t_4). \tag{5}$$

It follows from (5) and after taking the limit as $t \to +\infty$, that $\lim_{t\to+\infty} y'(t) = +\infty$, which contradicts that $\lim_{t\to+\infty} y'(t) = 0$. Therefore,

$$\lim_{t\to+\infty} y''(t) = 0.$$

Let us suppose $\lim_{t\to+\infty} y(t) = c_1 > 0$. But y is a bounded, continuous, decreasing and positive function. Thus, there exists constants $c > 0$, $c_6 > 0$ and point $t_5 \geq t_1$ such that $c \leq y(t) \leq c_6$ for $t \geq t_5$.

We integrate (1) from t to $+\infty$, $(t \geq t_5)$ and arrive at the equality

$$\lim_{t\to+\infty} y''(t) - y''(t) - \sum_{\tau_k \geq t} \Delta y''(\tau_k) + \int_t^\infty f(u, y(u))du = 0.$$

From $y(t) \geq c$ for $t \geq t_5$ and conditions **H2** and **H3**, we have

$$y''(t) \geq \int_t^\infty f(u,c)du + \sum_{\tau_k \geq t} f_k(c).$$

We integrate the last inequality from s to ∞ and conclude

$$\lim_{t\to+\infty} y'(t) - y'(s) \geq \int_s^\infty [\int_t^\infty f(u,c)du + \sum_{\tau_k \geq t} f_k(c)]dt$$

$$= \int_t^\infty (u-t)f(u,c)du + \sum_{\tau_k \geq t} (\tau_k - t)f_k(c)$$

i.e.

$$-y'(t) \geq \int_t^\infty (u-t)f(u,c)du + \sum_{\tau_k \geq t} (\tau_k - t)f_k(c). \tag{6}$$

Integrating (6) from T to t, $(t \geq T)$, we obtain

$$y(T) - y(t) \geq \int_T^t [\int_s^\infty (u-s)f(u,c)du + \sum_{\tau_k \geq s} (\tau_k - s)f_k(c)]ds,$$

or

$$y(T) - y(t) \geq \frac{1}{2}\int_T^t (u-T)^2 f(u,c)du + \frac{1}{2} \sum_{T \leq \tau_k \leq t} (\tau_k - T)^2 f_k(c)ds,$$

$$-\lim_{t\to+\infty} y(t) \geq \frac{1}{2}\int_T^\infty (u-T)^2 f(u,c)du + \frac{1}{2} \sum_{T \leq \tau_k} (\tau_k - T)^2 f_k(c).$$

The last inequality contradicts condition 2 of Theorem 1. Therefore,

$$\lim_{t \to +\infty} y(t) = 0.$$

Case 2: $y''(t) < 0$ for $t \geq t_1$.

It is easy to see that $\lim_{t \to +\infty} y(t) = -\infty$. This contradicts the assumption $y(t) > 0$. The proof is complete.

Theorem 2: *Let the following conditions hold:*

(1) *Conditions* **H1-H3** *are met.*

(2)

$$\int^{\infty} t^2 \mid f(t,c) \mid dt + \sum_{k=1}^{\infty} \tau_k^2 \mid f_k(c) \mid \ < +\infty$$

for some constant $c \neq 0$.

Then the equation (1) *has a bounded nonoscillatory solution.*

Proof: From condition (2), one can find a sufficiently large $T \geq 0$ such that

$$\int_T^{\infty} t^2 \mid f(u,c) \mid dt + \sum_{\tau \geq T} \tau_k^2 \mid f_k(c) \mid \ \leq \ \mid c \mid . \tag{7}$$

Let X be the space of bounded continuous functions on $[T,\infty)$. Let $Y \subset X$ be defined by

$$Y = \{ y \in X ; \frac{\mid c \mid}{2} \leq y \quad \text{sign} c \leq \ \mid c \mid \}.$$

Then Y is a bounded convex closed subset of X.

Define the operator $S: Y \to X$ as follows:

$$(Sy)(t) = \begin{cases} \frac{c}{2} + \frac{1}{2}\int_t^{\infty} (s-t)^2 f(s,y(s))ds + \frac{1}{2}\sum_{\tau_k \geq t} (\tau_k - t)^2 f_k(y(\tau_k)), & t \geq T \\ \frac{c}{2} + \frac{1}{2}\int_T^{\infty} (s-T)^2 f(s,y(s))ds + \frac{1}{2}\sum_{\tau_k \geq T} (\tau_k - T)^2 f_k(y(\tau_k)) & 0 \leq t \leq T. \end{cases}$$

(*a*) S maps Y into itself. In fact

$$\begin{aligned} \frac{\mid c \mid}{2} \leq (Sy)(t) \text{sign} c &\leq \frac{\mid c \mid}{2} + \frac{1}{2}\int_t^{\infty} (s-t)^2 \mid f(s,c) \mid ds + \frac{1}{2}\sum_{\tau_k \geq t} (\tau_k - t)^2 \mid f_k(c) \mid \\ &\leq \frac{\mid c \mid}{2} + \frac{1}{2}\int_t^{\infty} s^2 \mid f(s,c) \mid ds + \frac{1}{2}\sum_{\tau_k \geq t} \tau_k^2 \mid f_k(c) \mid \ \leq \frac{\mid c \mid}{2} + \frac{\mid c \mid}{2} = \ \mid c \mid , \end{aligned}$$

due to conditions **H2**, **H3** and (7).

(*b*) S is continuous. To prove this, let $\{y_n\}$ be a Cauchy sequence in Y, and let $\lim_{t \to +\infty} \| y_n - y \| = 0$. Because Y is closed, $y \in Y$. To prove the continuity of S, we see that

$$\mid (Sy_n)(t) - (Sy)(t) \mid \tag{8}$$

$$\leq \frac{1}{2}\int_t^\infty (s-t)^2 \mid f(s,y_n(s)) - f(s,y(s)) \mid ds + \frac{1}{2}\sum_{\tau_k \geq t} (\tau_k - t)^2 \mid f_k(y_n(t_k)) - f_k(y(\tau_k)) \mid .$$

Set

$$F_n(s) = s^2 \mid f(s,y_n(s)) - f(s,y(s)) \mid$$

$$L_n(\tau_k) = \tau_k^2 \mid f_k(y_n(\tau_k)) - f_k(y(\tau_k)) \mid .$$

Then (8) reduces to

$$\mid (Sy_n)(t) - (Sy)(t) \mid \leq \frac{1}{2}\int_t^\infty F_n(s)ds + \frac{1}{2}\sum_{\tau_k \geq t} L_n(\tau_k), \tag{9}$$

noting the fact that $(s-t)^2 \leq s^2$ for $s \geq t \geq 0$. It is obvious that

$$F_n(s) \leq 2s^2 \mid f(s,c) \mid , \quad L_n(\tau_k) \leq 2\tau_k^2 \mid f_k(c) \mid . \tag{10}$$

From the definition of G_n, L_n and conditions **H2**, **H3**, we obtain

$$\lim_{n \to +\infty} F_n(s) = 0, \quad \lim_{n \to \infty} L_n(\tau_k) = 0. \tag{11}$$

From (9), (10), (11) and the Lebesgue convergence theorem, we have

$$\lim_{n \to \infty} \| Sy_n - Sy \| = 0,$$

which means that S is continuous.

(c) To show SY is precompact, we see that $(Sy)(t)$, $y \in Y$, is uniformly bounded. Now we will prove that SY is an equicontinuous family of functions on R_+.

For $y \in Y$ and $t_2 > t_1 > 0$, we have

$$\mid (Sy)(t_2) - (Sy)(t_1) \mid$$

$$\leq \frac{1}{2} \mid \int_{t_2}^\infty (s-t_2)^2 f(s,y(s))ds - \int_{t_1}^\infty (s-t_1)^2 f(s,y(s))ds \mid$$

$$+ \frac{1}{2} \mid \sum_{\tau_k \geq t_2} (\tau_k - t_2)^2 f_k(y(\tau_k)) - \sum_{\tau_k \geq t} (\tau_k - t_1)^2 f_k(y(\tau_k)) \mid$$

$$\leq \int_{t_1}^\infty s^2 \mid f(s,y(s)) \mid ds + \sum_{\tau_k \geq t_1} \tau_k^2 \mid f_k(y(\tau_k)) \mid$$

$$\leq \int_{t_1}^\infty s^2 \mid f(s,c) \mid ds + \sum_{\tau_k \geq t_1} \tau_k^2 \mid f_k(c) \mid .$$

For any given $\varepsilon > 0$, there exists $T_1 > T$ such that

$$\int_{T_1}^{\infty} s^2 \mid f(s,c) \mid ds + \sum_{\tau_k \geq T_1} \tau_k^2 \mid f_k(c) \mid \; < \varepsilon.$$

Hence, for any $t_2 > t_1 \geq T_1$, from (8), we have $\mid (Sy)(t_2) - (Sy)(t_1) \mid \; < \; \varepsilon$ for all $y \in Y$.

For $T \leq t_1 < t_2 \leq T_1$,

$$\mid (Sy)(t_2) - (Sy)(t_1) \mid$$

$$\leq \frac{1}{2} \mid \int_{t_2}^{\infty} (s-t_2)^2 f(s,y(s))ds - \int_{t_1}^{\infty} (s-t_1)^2 f(s,y(s))ds \mid$$

$$+ \frac{1}{2} \mid \sum_{\tau_k \geq t_2} (\tau_k - t_2)^2 f_k(y(\tau_k)) - \sum_{\tau_k \geq t_1} (\tau_k - t_1)^2 f_k(y(\tau_k)) \mid$$

$$= \frac{1}{2} \mid \int_{t_2}^{\infty} (s-t_2)^2 f(s,y(s))ds - \int_{t_1}^{t_2} (s-t_1)^2 f(s,y(s))ds - \int_{t_1}^{\infty} (s-t_1)^2 f(s,y(s))ds \mid$$

$$+ \frac{1}{2} \mid \sum_{\tau_k \geq t_2} (\tau_k - t_2)^2 f_k(y(\tau_k)) - \sum_{t_1 \leq \tau_k < t_2} (\tau_k - t_1)^2 f_k(y(\tau_k)) - \sum_{t_2 \leq \tau_k} (\tau_k - t_1)^2 f_k(y(\tau_k)) \mid$$

$$\leq \frac{1}{2} \mid \int_{t_2}^{\infty} [(s-t_2)^2 - (s-t_1)^2] f(s,y(s))ds \mid + \frac{1}{2} \int_{t_1}^{t_2} (s-t_1)^2 \mid f(s,y(s)) \mid ds$$

$$+ \frac{1}{2} \mid \sum_{\tau_k \geq t_2} [(\tau_k - t_2)^2 - (\tau_k - t_1)^2] f_k(y(\tau_k)) \mid + \sum_{t_1 \leq \tau_k \leq t_2} (\tau_k - t_1)^2 \mid f_k(y(\tau_k)) \mid$$

$$\leq \mid t_2 - t_1 \mid [\int_{t_2}^{\infty} s^2 \mid f(s,c) \mid \mid ds + \sum_{\tau_k \geq t_2} \tau_k^2 \mid f_k(c) \mid]$$

$$+ \int_{t_1}^{t_2} s^2 \mid f(s,c) \mid ds + \sum_{t_1 \leq \tau_k < t_2} \tau_k^2 \mid f_k(c) \mid$$

$$\leq M \mid t_2 - t_1 \mid + \int_{t_1}^{t_2} s^2 \mid f(s,c) \mid ds + \sum_{t_1 \leq \tau_k < t_2} \tau_k^2 \mid f_k(c) \mid .$$

Hence, for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\mid (Sy)(t_2) - (Sy)(t_1) \mid \; < \varepsilon, \quad \mid t_2 - t_1 \mid \; < \delta,$$

for all $y \in Y$.

That is, the interval $[T, +\infty)$ can be divided into a finite number of subintervals on which every $(Sy)(t)$, $y \in Y$, has oscillation less than ε.

Therefore, SY is an equicontinuous family on $[T, \infty)$.

We prove that the set SY is equiconvergent to ∞.

The definition of the operator S implies

$$| (Sy)(t) - \tfrac{c}{2} |$$

$$\leq \tfrac{1}{2}\Big[\int_t^\infty s^2 \, | f(s, y(s)) | \, ds + \sum_{\tau_k \geq t} \tau_k^2 \, | f_k(y(\tau_k)) | \Big] \tag{12}$$

$$\leq \tfrac{1}{2}\Big[\int_t^\infty s^2 \, | f(s, c) | \, ds + \sum_{\tau_k \geq t} \tau_k^2 \, | f_k(c) | \Big].$$

Hence, for any given $\varepsilon > 0$, there exists a point $T_\varepsilon > T$ such that

$$\int_{T_\varepsilon}^\infty s^2 \, | f(s, c) | \, ds + \sum_{\tau_k \geq T_\varepsilon} \tau_k^2 \, | f_k(c) | < 2\varepsilon. \tag{13}$$

From (12) and (13), for $t \geq T_\varepsilon$ we get $| (Sy)(t) - \frac{c}{2} | < \varepsilon$ for all $y \in Y$. Therefore SY is equiconvergent at ∞. Lemma 1 implies that the set SY is relatively compact.

According to the Schauder fixed point theorem, there exists a $y \in Y$ such that $y = SY$. This y is a bounded nonoscillatory solution of (1). The proof is complete.

Theorem 3: *Assume that conditions* **H1-H3** *hold. Then condition*

$$\int_0^\infty t^2 \, | f(t, c) | \, dt + \sum_{k=1}^\infty \tau_k^2 \, | f_k(c) | < +\infty$$

for some constant $c \neq 0$ is necessary and sufficient for the existence of a bounded oscillatory solution y such that $\lim_{t \to \infty} y(t) = d$, $d \neq 0$.

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