RANDOM FIXED POINTS FOR *-NONEXPANSIVE RANDOM OPERATORS

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The notion of a *-nonexpansive multivalued map is different from that of a continuous map. In this paper we prove some fixed point theorems for *-nonexpansive multivalued random operators in the setup of Banach spaces and Fréchet spaces. Our work generalizes, refines and improves the earlier results of a number of authors.

Key words: Random Fixed Point, *-Nonexpansive Random Map, Banach Space, Fréchet Space, Weakly Inward Operator, Leray-Schauder Condition.

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1. Introduction

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operators lie at the heart of this discipline and their theory is needed for the study of various classes of random equations. The study of random fixed point theorems was initiated by the Prague school of probabilists in the 1950s. The generalization of these theorems from self maps to nonself maps has gained tremendous importance after the papers by Beg [2], Beg and Shahzad [3-5], Lin [12, 13], Sehgal and Singh [16], Shahzad [18] and Tan and Yuan [19, 20]. In particular, Lin [12], Shahzad [18] and Tan and Yuan [19] studied random fixed points of 1-set-contractive maps. The class of 1-set-contractive

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random maps includes condensing, nonexpansive and other interesting random maps such as locally almost nonexpansive (LANE) and semicontractive random maps. The purpose of this paper is to study the random fixed point theory of $*$-nonexpansive multivalued operators (which are not continuous) defined on convex and star-shaped subsets of Banach spaces as well as Fréchet spaces. Recent results of Beg [2], Shahzad [18] and Tan and Yuan [19] follow as a special case from our results. An error in Theorem 2.2 of Yi and Zhao [24] is pointed out and corrected.

2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{A})$ denotes a measurable space with $\mathcal{A}$ a σ-algebra of subsets of Ω unless stated otherwise. Let X be a normed space (or a Fréchet space), C a subset of X, 2^X the family of all subsets of X, $K(X)$ the family of all nonempty compact subsets of X, $CK(X)$ the family of all nonempty convex, compact subsets of X, $WK(X)$ the family of all nonempty weakly compact subsets of X and $CB(X)$ the family of all closed bounded subsets of X. A mapping $T:\Omega \to 2^X$ is called *measurable* if for any open subset B of X, $T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$. A mapping $\xi:\Omega \to X$ is said to be a *measurable selector* (cf. [7, 10]) of a measurable mapping $T:\Omega \to 2^X$ if ξ is measurable and for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega)$. A mapping $T:\Omega \times C \to 2^X$ is said to be a *random operator* if for any $x \in C$, $T(\,\cdot\,, x)$ is measurable. A mapping $\xi:\Omega \to C$ is said to be

  (i)   a *deterministic fixed point* of T if $\xi(\omega) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$ and
  (ii)   a *random fixed point* of T if ξ is a measurable map such that for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))$.

A mapping $T:C \to 2^X$ is said to be

  (i)   *upper (lower) semicontinuous* if for any closed (open) subset B of X, $T^{-1}(B)$ is closed (open); if T is both upper and lower semicontinuous, then T is called a *continuous* map,
  (ii)   *demiclosed* at 0 if the conditions $x_n \in C$, $x_n \to x$ weakly, $y_n \in Tx_n$, $y_n \to 0$ strongly imply $0 \in Tx$. A mapping $T:C \to CB(X)$ is a *contraction* if for any $x, y \in C$, $H(Tx, Ty) \leq k \,\| x - y \|$ where H is the Hausdorff metric on $CB(X)$ and $0 \leq k < 1$. If $k = 1$, then T is called a *nonexpansive* map.

A mapping $T:C \to X$ is called *condensing* if T is continuous and for any bounded subset B of C with $\alpha(B) > 0$, $\alpha(T(B)) < \alpha(B)$, where $\alpha(B) = \inf\{\epsilon > 0 : B$ can be covered by a finite number of sets of diameter $\leq \epsilon\}$. The number $\alpha(B)$ is called the *(set-) measure of noncompactness of B*. If there exists k, $0 \leq k \leq 1$, such that for each nonempty bounded subset B of C we have $\alpha(T(B)) \leq k\alpha(B)$, then a continuous map $T:C \to X$ is called a *k-set-contractive* map. In case C is a convex subset of X, the map $T:C \to X$ is *affine* if $T(\lambda x + (1 - \lambda)y) = \lambda Tx + (1 - \lambda)Ty$ for all $x, y \in C$ and $0 < \lambda < 1$.

Let (X, d) be a metrizable locally convex space. A ball $B_r(0) = \{z \in X : d(z, 0) \leq r\}$ with radius r and centered at 0 is said to be *compressible* if for every $\lambda > 1$ there is $t > r$ such that $B_t(0) \subset \lambda B_r(0)$. If every ball $B_r(0)$ in (X, d) is compressible (resp. convex), then we say that d is *compressible* (resp. *convex*) (see [21]).

A mapping $T:C \to 2^X$ is said to be

  (i)   *weakly nonexpansive* (cf. [8, 22]) if given $x \in C$ and $u_x \in Tx$ there is a

$u_y \in Ty$ for each $y \in C$ such that $d(u_x, u_y) \leq d(x, y)$,

(ii) *∗-nonexpansive* (cf. [8, 22]) if for all $x, y \in C$ and $u_x \in T_x$ with $d(x, u_x) = d(x, T_x) = \inf\{d(x, z): z \in T_x\}$, there exists $u_y \in T_y$ with $d(y, u_y) = d(y, T_y)$ such that $d(u_x, u_y) \leq d(x, y)$,

(iii) *hemicompact* if each sequence $\{x_n\}$ in C has a convergent subsequence whenever $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

For the above map T and each $x \in C$, we follow Xu [22] to define the set (possibly empty)

$$P_T(x) = \{u_x \in T_x: d(x, u_x) = d(x, T_x)\}.$$

A random operator $T: \Omega \times C \to 2^X$ is said to

(i) be *continuous (nonexpansive, hemicompact, ∗-nonexpansive, etc.)* if for each $\omega \in \Omega$, $T(\omega, \cdot)$ is continuous (nonexpansive, hemicompact, ∗-nonexpansive, etc.),

(ii) be *weakly inward* if for each $\omega \in \Omega$, $T(\omega, x) \subset \text{cl } I_C(x)$ for $x \in C$ where cl denotes closure and $I_C(x) = \{z \in X: z = x + a(y - x)$ for some $y \in C$ and $a \geq 0\}$,

(iii) satisfy the *Leray-Schauder condition* (in case C has a nonempty interior) if there is a point z in the interior of C (depending on ω) such that for each $u_y \in T(\omega, y)$,

$$u_y - z \neq m(y - z) \tag{1}$$

for all $y \in \partial C$ (the boundary of C) and $m > 1$.

A Banach space X satisfies *Opial's condition* if for each $x \in X$ and each sequence $\{x_n\}$ converging weakly to x, $\liminf_n \| x_n - y \| > \liminf_n \| x_n - x \|$ holds for all $y \neq x$ in X.

A ∗-nonexpansive multivalued mapping is different from a continuous mapping as is clear from the following example.

Example 2.1: Let $X = \mathbb{R}^2$ be equipped with Euclidean norm and $C = \{(a, 0): \frac{1}{\sqrt{2}} \leq a \leq 1\} \cup \{(0, 0)\}$. Define $T: C \to 2^X$ by

$$T(a, 0) = \begin{cases} (0, 1) & \text{if } a \neq 0 \\ L = \text{the line segment } [(0, 1), (1, 0)] & \text{if } a = 0. \end{cases}$$

Then $P_T(a, 0) = \{(0, 1)\}$ for all $(a, 0) \in C$ with $a \neq 0$ and $P_T(0, 0) = (\frac{1}{2}, \frac{1}{2})$.

Clearly T is a ∗-nonexpansive discontinuous multifunction (cf. [15, p. 537]). Moreover, for given $x = (0, 0)$ and $u_x = (1, 0) \in T_x$, there does not exist $y \neq x$ in C and $u_y \in T_y$ such that

$$| u_x - u_y | \leq | x - y |.$$

Recall that for $y \neq x$ in C, $u_y = (0, 1)$ and $| u_x - u_y | = | (1, 0) - (0, 1) | = \sqrt{2} > d(x, y)$. So T is not weakly nonexpansive.

Remarks 2.2: (i) In view of Example 2.1, the statement "each ∗-nonexpansive map is weakly nonexpansive" in [8, p. 389] is not valid.

(ii) It follows from the definition of Hausdorff metric that a weakly nonexpansive map is nonexpansive. The converse holds for compact-valued maps. For if $T: C \to 2^X$

is a compact-valued nonexpansive map, then for any $x \in C$ and $u_x \in T_x$, we can find some $u_y \in T_y$ for all y in C by compactness of T_y such that

$$d(u_x, u_y) \leq \sup\{d(u, T_y): u \in T_x\} \leq H(T_x, T_y) \leq d(x, y).$$

So T is weakly nonexpansive (also see Proposition 1 [22]).

   (iii)   *-nonexpansiveness and nonexpansiveness are two different concepts for multivalued mappings.

3.   Random Fixed Points in Banach Spaces

A general fixed point theorem for a class of discontinuous multivalued random operators is established in the following.

   Theorem 3.1: *Let C be a nonempty closed, bounded, convex, separable subset with nonempty interior of a strictly-convex, reflexive Banach space X satisfying Opial's condition. Suppose that $T:\Omega \times C \rightarrow 2^X$ is a closed, convex-valued, *-nonexpansive random operator that either*

   (i)   *is weakly inward*

or

   (ii)   *satisfies the Leray-Schauder condition.*

If P_T is a random operator, then T has a random fixed point.

   Proof: Suppose that assumption (i) is satisfied. As X is strictly convex so each $T(\omega, x)$ is a Chebyshev set. Therefore for all $\omega \in \Omega$ and all $x \in C$,

$$\{u_x\} = P_T(\omega, x) \in T(\omega, x).$$

Also for each $\omega \in \Omega$ and each $x, y \in C$,

$$d(P_T(\omega, x), P_T(\omega, y)) = d(u_x, u_y) \leq d(x, y).$$

This implies that $P_T:\Omega \times C \rightarrow X$ is a nonexpansive random operator. Further, $P_T(\omega, x) \in T(\omega, x) \subset \mathrm{cl}(I_C(x))$ for all $x \in C$ and any $\omega \in \Omega$ so it follows that P_T is weakly inward. We can easily show as in the proof of Theorems 3.1, 3.2 [11] that $I - P_T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$. By Theorem 3.1 [2], P_T has a random fixed point. That is, there is a measurable map $\xi:\Omega \rightarrow C$ such that

$$\xi(\omega) = P_T(\omega, \xi(\omega)) \text{ for all } \omega \in \Omega.$$

Since $P_T(\omega, x) \in T(\omega, x)$ for all $\omega \in \Omega$ and for all $x \in C$, it follows that $\xi(\omega) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$, as desired.

   Suppose that assumption (ii) holds. By the arguments used in case (i), $P_T: \Omega \times C \rightarrow X$ is a nonexpansive random operator such that $I - P_T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$. Since $P_T(\omega, y) \in T(\omega, y)$ for all $\omega \in \Omega$ and for all $y \in \partial C$ and T satisfies the Leray-Schauder condition, it follows from (1) that P_T also satisfies the Leray-Schauder condition. By Theorem 3.3 [2], P_T and hence T has a random fixed point.   □

   Remarks 3.2:   (i) Since *-nonexpansive multivalued maps are not necessarily continuous, Theorem 3.1 cannot be implied by the results of Beg and Shahzad in [3-5]

and Tan and Yuan in [20].

(*ii*) If $I - T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$ and P_T is weakly continuous in Theorem 3.1, then $I - P_T(\omega, \cdot)$ is demiclosed at 0 as follows:

Let $x_n \to x_0$ weakly and $I - P_T(\omega, \cdot)(x_n) \to 0$ strongly for any $\omega \in \Omega$. Now $I - P_T(\omega, \cdot)(x_n) \in I - T(\omega, \cdot)(x_n)$ and $I - T(\omega, \cdot)$ is demiclosed at 0 so $0 \in I - T(\omega, \cdot)(x)$. Since $P_T : \Omega \times C \to X$ is weakly continuous so $I - P_T(\omega, \cdot)(x_n) \to x_0 - P_T(\omega, x_0)$ weakly. Also $I - P_T(\omega, \cdot)(x_n) \to 0$ weakly. Hence by the Hausdorff property of weak topology, we have that $0 = I - P_T(\omega, \cdot)(x_0)$.

(*iii*) A continuous affine map is weakly continuous.

From Theorem 3.1, we now obtain:

Theorem 3.3: *Let C be a nonempty closed, bounded, convex, separable subset with nonempty interior of a strictly-convex, reflexive Banach space X. Suppose that T: $\Omega \times C \to 2^X$ is a closed, convex-valued, ∗-nonexpansive random operator that either*

    (*i*)   *is weakly inward*

or

    (*ii*)   *satisfies the Leray-Schauder condition.*

If $I - T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$ and P_T is an affine random operator, then T has a random fixed point.

We observe that if X is a uniformly convex space in Theorem 3.1, then the theorem holds (with the same proof) because in this case, the demiclosedness of $I - P_T(\omega, \cdot)$ follows as in Browder's Theorem 3 [6]. Consequently, we get the following corollary which extends Theorem 2.6 of Itoh [9], Theorem (6) (*ii*) of Lin [13] and Theorem 4 of Xu [23].

Corollary 3.4: [18, Corollary 3.4] *Let X be a uniformly-convex Banach space and let C (with nonempty interior) be a nonempty closed, bounded, convex, separable subset of X. If $T : \Omega \times C \to X$ is a nonexpansive random operator that either*

    (*i*)   *is weakly inward*

or

    (*ii*)   *satisfies the Leray-Schauder condition,*

then T has a random fixed point.

The following simple example contradicts the validity of Theorem 2.2 and hence Theorem 3.2 in [24].

Example 3.5: Let X be th set of real numbers with the usual metric and $C = \{0, 1\}$. Define $T : C \to X$ by $T(0) = 1$ and $T(1) = 0$. Then T is nonexpansive, weakly inward and $(I - T)(C) = \{-1, 1\}$ is a closed set. But T has no fixed point. Similarly we may obtain that T has no random fixed point where T is defined on a suitable subset of a complete σ-finite measure space $(\Omega, \mathcal{A}, \mu)$.

We assume in the remainder of this section that $(\Omega, \mathcal{A}, \mu)$ is a complete σ-finite measure space.

A combination of some sort of convexity in C, Theorem 2.1 and Lemma 3.1 (due to Yi and Zhao [24]) provide the following affirmative result.

Theorem 3.6: *Let C be a nonempty closed, star-shaped subset of a separable Banach space X and let $T : \Omega \times C \to K(X)$ be a weakly inward nonexpansive random operator. If for each $\omega \in \Omega$, $T(\omega, C)$ is bounded and $(I - T)(\omega, C)$ is closed, then T has a random fixed point.*

We will consider a weaker assumption on C in the next result to establish an extension of Theorem 3.6 for ∗-nonexpansive random operators.

Theorem 3.7: *Let C be a nonempty, weakly-closed, star-shaped subset of a separable Banach space X which satisfies Opial's condition and let $T : \Omega \times C \to K(X)$*

be a weakly inward *-nonexpansive random operator such that for each $\omega \in \Omega$, $T(\omega, C) \subseteq B$ for some weakly compact subset B of X. If P_T is a random operator, then T has a random fixed point.

Proof: As before, $P_T: \Omega \times C \to 2^X$ is a compact-valued, weakly inward nonexpansive random map (see also proof of Theorem 2 [22]). We shall show that for each $\omega \in \Omega$, $(I - P_T)(\omega, \cdot)(C)$ is closed. Let $\omega \in \Omega$ be fixed and y be a limit point of $I - P_T(\omega, \cdot)(C)$. Then there is a sequence $\{y_n\}$ with $y_n \in (I - P_T)(\omega, x_n)$ for some $x_n \in C$ and $y_n \to y$. Hence $x_n - y_n \in P_T(\omega, x_n)$ and $y_n \to y$. Since $\{x_n - y_n\} \in P_T(\omega, x_n) \subseteq T(\omega, x_n) \subseteq B$, there is $b \in B$ and a subsequence $\{x_m - y_m\}$ of $\{x_n - y_n\}$ such that $x_m - y_m \to b$ weakly. As $y_m \to y$ weakly, it follows that $x_m \to y - b$ weakly. Let $z = y - b$. As C is weakly-closed, $z \in C$. Without loss of generality, we may assume that x_n converges weakly to z. By Remarks 2.2 (ii), a compact-valued, nonexpansive map is weakly nonexpansive and so we obtain that P_T is weakly nonexpansive. Hence for each $x_n - y_n \in P_T(\omega, x_n)$, there is a $z_n \in P_T(\omega, z)$ such that

$$\| x_n - y_n - z_n \| \leq \| x_n - z \|. \tag{2}$$

The set $P_T(\omega, z)$ is compact so $z_n \to u \in P_T(\omega, z)$ and $y_n + z_n \to y + u$.

It now follows from (2) that

$$\underline{\lim} \| x_n - y - u \| \leq \underline{\lim} \| x_n - z \|.$$

Hence by Opial's condition, we have $y + u = z$ and so $y = z - u \in (I - P_T)(\omega, \cdot)(C)$ for each $\omega \in \Omega$, as desired. Thus by Theorem 3.6, P_T and hence T has a random fixed point.   □

For single-valued maps, the concepts of *-nonexpansive and nonexpansive coincide. Hence the following two results generalize Corollary 3.5 of Tan and Yuan [19] to weakly-closed and star-shaped sets in the context of Opial spaces.

Corollary 3.8: *Let C be a nonempty, weakly-closed, star-shaped subset of a separable Banach space X satisfying Opial's condition and let $T: \Omega \times C \to X$ be a nonexpansive random operator such that for each $\omega \in \Omega$, $T(\omega, C) \subseteq B$ for some weakly compact subset B of X. If T is weakly inward, then T has a random fixed point.*

Corollary 3.9: *Let C be a nonempty, weakly-compact, star-shaped subset of a separable Banach space X satisfying Opial's condition and let $T: \Omega \times C \to C$ be a nonexpansive random operator. Then T has a random fixed point.*

We remark that in Corollary 3.9, the conditions (the fixed point property of C, the convexity of C and strict convexity of X needed in Theorem 1 by Xu [23]) are relaxed.

4. Random Fixed Points in Fréchet Spaces

Fixed point results in the context of Fréchet spaces have been studied in [14, 17]. In this section, we prove fixed point theorems for *-nonexpansive operators defined on a subset of a Fréchet space.

We shall need the following results.

Theorem A: [17, Theorem 3.3] *Let C be a nonempty, weakly-compact, convex subset of a separable Fréchet space X and let $T: \Omega \times C \to X$ be 1-set contractive*

random operator that either

    (i)   is weakly inward

or

    (ii)   satisfies the Leray-Schauder condition.

If for any $\omega \in \Omega$, $T(\omega, C)$ is bounded and $I - T(\omega, \cdot)$ is demiclosed at 0, then T has a random fixed point.

Theorem B: [21, Theorem 2.1] *Let (X, d) be a locally convex, metrizable topological vector space with d as convex and compressible metric. Then every weak sequentially compact subset K of X is proximinal.*

Theorem C: [1, Theorem 2] *Every convex proximinal set in a strictly-convex metric linear space is Chebyshev.*

The following result extends Theorem 3.1 from Banach spaces to Fréchet spaces.

Theorem 4.1: *Let C be a nonempty, closed, bounded, convex subset with nonempty interior of a uniformly-convex separable Fréchet space X and let $T: \Omega \times C \to 2^X$ be a closed, convex-valued, ∗-nonexpansive random operator that either*

    (i)   is weakly inward

or

    (ii)   satisfies the Leray-Schauder condition.

If for each $\omega \in \Omega$, $I - T(\omega, \cdot)$ is demiclosed at 0 and P_T is an affine random operator, then T has a random fixed point.

Proof: It is well known that a closed, convex subset of a uniformly-convex Fréchet space is Chebyshev (see [1, Corollary 3]). So we may obtain as in the proof of Theorem 3.1 that $P_T: \Omega \times C \to X$ is a nonexpansive random operator and $I - P_T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$. Moreover, the class of 1-set contractive operators includes nonexpansive operators. By Theorem A, P_T and hence T has a random fixed point. □

Corollary 4.2: [19, Corollary 3.5] *Let C be a nonempty, weakly-compact, convex subset of a separable, uniformly-convex Banach space X and let $T: \Omega \times C \to X$ be a nonexpansive random operator. If T is weakly inward, then T has a random fixed point.*

An extension of Theorem 4.1 to the case of strictly-convex Fréchet spaces is obtained in the following.

Theorem 4.3: *Let C be a nonempty, weakly-compact, convex subset with nonempty interior of a strictly-convex separable Fréchet space X with convex and compressible metric d and let $T: \Omega \times C \to 2^X$ be a weakly sequentially compact, convex-valued, ∗-nonexpansive random operator that either*

    (i)   is weakly inward

or

    (ii)   satisfies the Leray-Schauder condition.

If for each $\omega \in \Omega$, $I - T(\omega, \cdot)$ is demiclosed at 0 and P_T is affine random operator, then T has a random fixed point.

Proof: Theorems B and C imply that each $T(\omega, x)$ is Chebyshev. Thus, as before, $P_T: \Omega \times C \to X$ is a nonexpansive random operator that either

    (i)   is weakly inward

or

    *(ii)   satisfies the Leray-Schauder condition.

Moreover, as in Remarks 3.2 *(ii)*, we obtain that $I - P_T(\omega, \cdot)$ is demiclosed at 0 for each $\omega \in \Omega$. The conclusion follows from Theorem A. □

A related random fixed point theorem for hemicompact operators is given below

(see Theorems 4.8 and 4.9 in [19]).

Theorem 4.4: *Let C be a nonempty, closed, separable subset of a Fréchet space X with convex and compressible metric d. Suppose $T: \Omega \times C \to 2^X$ is a weakly sequentially, compact-valued, $*$-nonexpansive hemicompact random operator. If for each $\omega \in \Omega$, $G(\omega) = \{x \in C : x \in T(\omega, x)\}$ is nonempty and P_T is a random operator, then T has a random fixed point.*

Proof: As before, $P_T: \Omega \times C \to 2^X$ is a nonexpansive random operator. We show that P_T is hemicompact. Let $\{x_n\}$ be any sequence in C such that $d(x_n, P_T(x_n)) \to 0$ as $n \to \infty$. By definition of P_T, we have

$$d(x_n, P_T(\omega, x_n)) \leq d(x_n, u_{x_n}) = d(x_n, Tx_n) \leq d(x_n, P_T(\omega, x_n)). \tag{3}$$

So $d(x_n, T_{x_n}) \to 0$ as $n \to \infty$. The hemicompactness of T implies that $\{x_n\}$ has a convergent subsequence. Hence P_T is hemicompact. From (3) it follows that x is a fixed point of $T(\omega, \cdot)$ iff x is a fixed point of $P_T(\omega, \cdot)$. Thus $F(\omega) = \{x \in C : x \in P_T(\omega, x)\} \neq \emptyset$ for each $\omega \in \Omega$. By Theorem 3.1 [3] or Theorem 2.3 [20], P_T and therefore T has a random fixed point.   □

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