ON THE ORDER OF GROWTH OF CONVERGENT SERIES OF INDEPENDENT RANDOM VARIABLES

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For independent random variables, the order of growth of the convergent series S_n is studied in this paper. More specifically, if the series S_n converges almost surely to a random variable, the tail series is a well-defined sequence of random variables and converges to 0 almost surely. For the almost surely convergent series S_n, a tail series strong law of large numbers (SLLN) is constructed by investigating the duality between the limiting behavior of partial sums and that of tail series.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and, as usual, their partial sums are denoted by $S_n = \sum_{j=1}^{n} X_j$, $n \geq 1$. The sequence of random variables $\{X_n, n \geq 1\}$ (such that the series S_n diverges almost surely (a.s.)) is said to obey the *strong law of large numbers* (SLLN) with positive norming constants $\{a_n, n \geq 1\}$ if

$$\frac{S_n}{a_n} \longrightarrow 0 \quad \text{a.s.} \tag{1.1}$$

If the series S_n converges a.s. to a random variable S, then (set $S_0 = X_0 = 0$) the *tail series*

$$T_n = S - S_{n-1} = \sum_{j=n}^{\infty} X_j, \quad n \geq 1, \tag{1.2}$$

is a well-defined sequence of random variables and converges to 0 a.s. In the same way, a sequence of random variables $\{X_n, n \geq 1\}$ is said to obey the *tail series SLLN* with norming constants $\{b_n, n \geq 1\}$ if the tail series T_n is well defined and for a given sequence of positive constants with $b_n = o(1)$,

$$\frac{T_n}{b_n} \longrightarrow 0 \quad \text{a.s.} \tag{1.3}$$

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In this paper, for independent random variables, we will be concerned with the rate in which S_n converges to a random variable S, or equivalently, in which the tail series T_n converges to 0. As will be seen, many results for partial sums S_n can be paired with analogous results for tail series T_n. Most of the limit laws for tail series have been developed by discovering and investigating this duality.

Pioneering work on the limiting behavior of $\{T_n,\ n \geq 1\}$ was conducted by Chow and Teicher [3], where they obtained for a tail series of suitably bounded and independent summands a counterpart to Kolmogorov's celebrated law of the iterated logarithm (LIL) (see, e.g., Chow and Teicher [4, Theorem 10.2.1, page 373] or Petrov [15, Theorem 7.1, page 239]). Barbour [1] then established a tail series analogue of the Lindeberg-Feller version of the central limit theorem. Other numerous investigations on the tail series LIL problem have followed; see Heyde [7], Wellner [20], Kesten [8], Budianu [2], Chow et al. [5], Klesov [9], Rosalsky [16], Mikosch [11], and Tomkins [19] for such work. The tail series SLLN problem was studied by Klesov [9, 10], Mikosch [11], and Nam and Rosalsky [12]. Also, Nam and Rosalsky [13] constructed a limit law which implies a tail series weak law of large numbers, and then this result was not only generalized by Sung and Volodin [18], but it was also generalized and simplified by Rosalsky and Rosenblatt [17].

For independent random variables, Petrov [14] proved a SLLN for partial sums, and Klesov [9, 10] then developed a tail series SLLN which provided a tail series counterpart to Petrov's [14] SLLN. It will be shown that Klesov's [10] tail series SLLN can be extended to a wider class of random variables in this paper. As will become apparent, the formulation and proof of the ensuing Theorem 3.1 owe much to the work of Klesov [9, 10].

2. Preliminary lemmas

Several lemmas are needed to establish the main results. Some of Klesov's [9, 10] work will now be described. Let Ψ^* be the class of positive and nondecreasing functions $\psi^*(x)$ such that the series $\sum_{n=1}^{\infty} 1/n\psi^*(n)$ converges and $x\psi^*(x^{-1})$ tends monotonically to 0 as $x \downarrow 0$.

Lemma 2.1 (Klesov [9, 10]). *Let $\{c_n,\ n \geq 1\}$ be a sequence of nonnegative constants such that $\sum_{n=1}^{\infty} c_n < \infty$. If $C_n \equiv \sum_{j=n}^{\infty} c_j > 0$, $n \geq 1$, then $\sum_{n=1}^{\infty}(c_n/C_n\psi^*(C_n^{-1})) < \infty$ obtains for each function $\psi^*(x) \in \Psi^*$.*

Lemma 2.2 (Heyde [7], Rosalsky [16], and Klesov [10]). *Let $\{x_n,\ n \geq 1\}$ be a sequence of constants and let $\{b_n,\ n \geq 1\}$ be a sequence of positive constants with $b_n \downarrow 0$. If the series $\sum_{n=1}^{\infty}(x_n/b_n)$ converges, then $(1/b_n)\sum_{j=n}^{\infty} x_j \longrightarrow 0$.*

Lemma 2.3 (Petrov [15, Theorem 6.4, page 207]). *Let $\{X_n,\ n \geq 1\}$ be independent random variables and let $\{g_n(x),\ n \geq 1\}$ be nondecreasing functions defined on $[0,\infty)$ such that*

$$g_n(0) = 0, \quad \lim_{x\to\infty} g_n(x) = \infty, \quad n \geq 1. \tag{2.1}$$

Suppose that one of the following three conditions prevails:

(i) *$x/g_n(x)$ is nondecreasing in $x > 0$ for each $n \geq 1$;*
(ii) *$g_n(x)/x$ and $x^2/g_n(x)$ are nondecreasing in $x > 0$, and also $E(X_n) = 0$, for each $n \geq 1$;*

(iii) $x^2/g_n(x)$ *is nondecreasing in* $x > 0$*, and* X_n *has a symmetric distribution, for each* $n \geq 1$.

Further, let $\{b_n,\ n \geq 1\}$ *be a sequence of positive constants. If the series*

$$\sum_{n=1}^{\infty} \frac{E(g_n(|X_n|))}{g_n(b_n)} < \infty, \tag{2.2}$$

then the series $\sum_{n=1}^{\infty} b_n^{-1} X_n$ *converges a.s.*

Under hypothesis (i) or (ii), Lemma 2.3 was proved, for the case $g_n \equiv g$, by Chung [6]. Using Lemmas 2.2 and 2.3, we obtain the following lemma. Not only does Lemma 2.4 imply Klesov [10, Assertion 3], but it also obtains a tail series analogue of Petrov [15, Theorem 6.5, page 209]. Note that the assumptions of the lemma entail (3.2) which, as will be demonstrated in the proof of Theorem 3.1, ensures that $\{T_n,\ n \geq 1\}$ is a well-defined sequence of random variables.

Lemma 2.4. *Let* $\{X_n,\ n \geq 1\}$ *be independent random variables and let* $\{g_n(x),\ n \geq 1\}$ *be nondecreasing functions defined on* $[0, \infty)$ *satisfying (2.1). Suppose that one of conditions (i), (ii), and (iii) of Lemma 2.3 holds. Let* $\{b_n,\ n \geq 1\}$ *be a sequence of positive constants with* $b_n \downarrow 0$*. If* $g_n(b_n) = O(1)$ *and (2.2) holds, then the tail series SLLN* $T_n/b_n \to 0$ *a.s. obtains.*

3. The main result

For independent random variables $\{X_n,\ n \geq 1\}$, a tail series SLLN, which implies the result of Klesov [10, Assertion 4] by taking $g_n \equiv g$ under hypothesis (i) or (ii), is obtained in Theorem 3.1. Assume that $\{X_n,\ n \geq 1\}$ are not eventually degenerate at 0.

Theorem 3.1. *Let* $\{X_n,\ n \geq 1\}$ *be independent random variables and let* $\{g_n(x),\ n \geq 1\}$ *be strictly increasing functions defined on* $[0, \infty)$ *such that (2.1) holds, and assume that*

$$g_n(x) \text{ is nondecreasing in } n \text{ for each fixed } x > 0. \tag{3.1}$$

Suppose that one of the following three conditions prevails:

(i) $x/g_n(x)$ *is nondecreasing in* $x > 0$ *for each* $n \geq 1$;

(ii) $g_n(x)/x$ *and* $x^2/g_n(x)$ *are nondecreasing in* $x > 0$*; and also* $E(X_n) = 0$*, for each* $n \geq 1$;

(iii) $x^2/g_n(x)$ *is nondecreasing in* $x > 0$*, and* X_n *has a symmetric distibution, for each* $n \geq 1$.

If the series

$$\sum_{n=1}^{\infty} E(g_n(|X_n|)) < \infty, \tag{3.2}$$

then setting $B_n = \sum_{j=n}^{\infty} E(g_j(|X_j|))$, $n \geq 1$*, the tail series SLLN*

$$\frac{T_n}{g_n^{-1}(B_n \psi^*(B_n^{-1}))} \longrightarrow 0 \quad \text{a.s. for each function } \psi^*(x) \in \Psi^*, \tag{3.3}$$

obtains, where g_n^{-1} *denotes the inverse function of* g_n.

Remark 3.2. Given that $g_n(x)$ is a nondecreasing function, each $g_n(x)$ is necessarily a continuous function for each of conditions (i), (ii), and (iii).

Proof of Theorem 3.1. First, it will be verified, by employing the Kolmogorov three-series criterion, that $\sum_{n=1}^{\infty} X_n$ converges a.s. For $n \geq 1$,

$$P\{|X_n| > 1\} = P\{g_n(|X_n|) > g_n(1)\} \leq \frac{E(g_n(|X_n|))}{g_n(1)} \tag{3.4}$$

by the Markov inequality, and so $\sum_{n=1}^{\infty} P\{|X_n| > 1\} < \infty$. Under hypothesis (i), note that $x^2 \leq {g_n}^2(x)/{g_n}^2(1) \leq g_n(x)/g_1(1)$, $x \leq 1$. Now, under hypothesis (ii) or (iii), also observe that $x^2/g_n(x) \leq 1/g_n(1)$ and so $x^2 \leq g_n(x)/g_1(1)$, $x \leq 1$. Thus

$$E(X_n^2 I_{[|X_n| \leq 1]}) \leq \frac{E(g_n(|X_n| I_{[|X_n| \leq 1]}))}{g_1(1)} \leq \frac{E(g_n(|X_n|))}{g_1(1)} \tag{3.5}$$

implying that $\sum_{n=1}^{\infty} \operatorname{Var}(X_n I_{[|X_n| \leq 1]}) < \infty$ for each case. Furthermore, under hypothesis (i), since $x/g_n(x) \leq 1/g_n(1)$, $x \leq 1$,

$$E(X_n I_{[|X_n| \leq 1]}) \leq \frac{E(g_n(|X_n| I_{[|X_n| \leq 1]}))}{g_n(1)} \leq \frac{E(g_n(|X_n|))}{g_1(1)}. \tag{3.6}$$

Also under hypothesis (ii), since $x/g_n(x) \leq 1/g_n(1)$, $x > 1$,

$$|E(X_n I_{[|X_n| \leq 1]})| = |E(X_n I_{[|X_n| > 1]})| \leq \frac{E(g_n(|X_n| I_{[|X_n| > 1]}))}{g_n(1)}. \tag{3.7}$$

In the case of (iii), note that $E(X_n I_{[|X_n| \leq 1]}) = 0$. Then $\sum_{n=1}^{\infty} E(X_n I_{[|X_n| \leq 1]}) < \infty$ is ensured for each case by (3.2). Therefore, the Kolmogorov three-series criterion guarantees that $\{T_n,\ n \geq 1\}$ is a well-defined sequence of random variables.

Next, let $c_n = E(g_n(|X_n|))$, $n \geq 1$, and note that $B_n = \sum_{j=n}^{\infty} c_j > 0$; then Lemma 2.1 ensures that for each function $\psi^*(x) \in \Psi^*$,

$$\sum_{n=1}^{\infty} \frac{E(g_n(|X_n|))}{g_n(g_n^{-1}(B_n \psi^*(B_n^{-1})))} = \sum_{n=1}^{\infty} \frac{E(g_n(|X_n|))}{B_n \psi^*(B_n^{-1})} < \infty. \tag{3.8}$$

Therefore, by setting $b_n = g_n^{-1}(B_n \psi^*(B_n^{-1}))$, the theorem is obtained by Lemma 2.4. □

The counterpart of Theorem 3.1 for partial sums, which generalizes Petrov [14, Theorem 5 or 15, Theorem 6.13, page 220], can be proved by the same argument as in Theorem 3.1. In the following corollaries, we obtain two truncated versions of Theorem 3.1, which improve the result of the theorem.

Corollary 3.3. *Let $\{X_n,\ n \geq 1\}$ be independent random variables and let $\{g_n(x),\ n \geq 1\}$ be strictly increasing functions defined on $[0, \infty)$ satisfying (2.1) and (3.1). Suppose that one* *of conditions (i), (ii), and (iii) of Theorem 3.1 holds. If*

$$\sum_{n=1}^{\infty} P\{|X_n| > C_n\} < \infty, \tag{3.9}$$

$$\sum_{n=1}^{\infty} E(g_n(|X_n I_{[|X_n| \le C_n]}|)) < \infty \tag{3.10}$$

are satisfied for some sequence of positive constants $\{C_n,\ n \ge 1\}$, *then, setting* $B_n^* = \sum_{j=n}^{\infty} E(g_j(|X_j I_{[|X_j| \le C_j]}|))$, $n \ge 1$, *the tail series SLLN*

$$\frac{T_n}{g_n^{-1}(B_n^* \psi^*(B_n^{*\,-1}))} \longrightarrow 0 \quad \text{a.s. for each function } \psi^*(x) \in \Psi^*, \tag{3.11}$$

obtains, where g_n^{-1} *denotes the inverse function of* g_n.

Remark 3.4. With the aid of the Borel-Cantelli lemma, a necessary condition for (3.11) to hold is given by (3.9) with $C_n = g_n^{-1}(B_n^* \psi^*(B_n^{*\,-1}))$.

Proof of Corollary 3.3. Set $Z_n = X_n I_{[|X_n| \le C_n]}$, $n \ge 1$. Then, by applying Theorem 3.1 to the random variables $\{Z_n,\ n \ge 1\}$, the tail series $T_n^* \equiv \sum_{j=n}^{\infty} Z_j$ is well defined and the tail series SLLN

$$\frac{T_n^*}{g_n^{-1}(B_n^* \psi^*(B_n^{*\,-1}))} \longrightarrow 0 \quad \text{a.s. for each function } \psi^*(x) \in \Psi^*, \tag{3.12}$$

obtains. Since $\{X_n, n \ge 1\}$ and $\{X_n I_{[|X_n| \le C_n]},\ n \ge 1\}$ are equivalent in the sense of Khintchine, $\{T_n,\ n \ge 1\}$ is well defined and the corollary follows. □

Corollary 3.5. *Let* $\{X_n,\ n \ge 1\}$ *be independent random variables and let* $\{g_n(x), n \ge 1\}$ *be strictly increasing functions defined on* $[0, \infty)$ *satisfying (2.1) and (3.1). Suppose that one of conditions (i), (ii), and (iii) of Theorem 3.1 holds. If (3.9) and*

$$\sum_{n=1}^{\infty} E(g_n(|X_n I_{[|X_n| \le C_n]} - E(X_n I_{[|X_n| \le C_n]})|)) < \infty \tag{3.13}$$

are satisfied for some sequence of positive constants $\{C_n,\ n \ge 1\}$, *then, setting* $\tilde{B}_n = \sum_{j=n}^{\infty} E(g_j(|X_j I_{[|X_j| \le C_j]} - E(X_j I_{[|X_j| \le C_j]})|))$ *and*

$$\tilde{T}_n = \sum_{j=n}^{\infty} \{X_j - E(X_j I_{[|X_j| \le C_j]})\}, \quad n \ge 1, \tag{3.14}$$

the tail series SLLN

$$\frac{\tilde{T}_n}{g_n^{-1}(\tilde{B}_n \psi^*(\tilde{B}_n^{-1}))} \longrightarrow 0 \quad \text{a.s. for each function } \psi^*(x) \in \Psi^*, \tag{3.15}$$

obtains, where g_n^{-1} *denotes the inverse function of* g_n.

Proof. Set

$$Z_n = X_n I_{[|X_n| \le C_n]} - E\left(X_n I_{[|X_n| \le C_n]}\right), \quad n \ge 1. \tag{3.16}$$

Then the corollary follows from (3.9) and (3.13) by the argument in the proof of Corollary 3.3 mutatis mutandis. □

4. Examples

Three examples are provided to illustrate the current results as well as to compare some of them with related results in the literature.

Example 4.1. Let $\{X_n, n \ge 1\}$ be independent random variables such that

$$P\left\{X_n = \frac{1}{n^\alpha}\right\} = P\left\{X_n = -\frac{1}{n^\alpha}\right\} = \frac{1}{2}, \quad \alpha > \frac{1}{2}. \tag{4.1}$$

Theorem 3.1 and the theorem of Klesov [10, Assertion 4] will be employed to determine the rate of almost sure convergence of the series $S_n = \sum_{j=1}^n X_j$. Define

$$g_n(x) = n^{\alpha - 1/2} x^2, \quad n \ge 1, \tag{4.2}$$

and $g(x) = g_1(x) = x^2$, $x \ge 0$. Then

$$E\left(g_n(|X_n|)\right) = n^{-(\alpha+1/2)}, \qquad E\left(g(|X_n|)\right) = n^{-2\alpha}, \tag{4.3}$$

implying that all the hypotheses of Theorem 3.1 as well as Klesov's [10] theorem are satisfied. Now, for $n \ge 1$,

$$B_n = \sum_{j=n}^{\infty} E\left(g_j(|X_j|)\right) \sim M_1 n^{-(\alpha-1/2)}, \tag{4.4}$$

$$\mathcal{B}_n = \sum_{j=n}^{\infty} E\left(g(|X_j|)\right) \sim M_2 n^{-(2\alpha-1)}, \tag{4.5}$$

where $M_1 = (\alpha - 1/2)^{-1}$ and $M_2 = M_1/2$. If $\psi^*(x)$ is taken to be the function

$$\psi^*(x) = \sqrt{x}, \tag{4.6}$$

then

$$B_n \psi^*(B_n^{-1}) \sim M_1^{1/2} n^{-(1/2)(\alpha-1/2)}, \qquad \mathcal{B}_n \psi^*(\mathcal{B}_n^{-1}) \sim M_2^{1/2} n^{-(\alpha-1/2)}, \tag{4.7}$$

and so, respectively,

$$\begin{aligned} g_n^{-1}\left(B_n \psi^*(B_n^{-1})\right) &\sim M_1^{1/4} n^{-(3/4)(\alpha-1/2)}, \\ g^{-1}\left(\mathcal{B}_n \psi^*(\mathcal{B}_n^{-1})\right) &\sim M_2^{1/4} n^{-(1/2)(\alpha-1/2)}. \end{aligned} \tag{4.8}$$

Thus, by applying Theorem 3.1 and the theorem of Klesov [10], the tail series SLLNs

$$n^{(3/4)(\alpha-1/2)}T_n \longrightarrow 0 \quad \text{a.s.}, \tag{4.9}$$

$$n^{(1/2)(\alpha-1/2)}T_n \longrightarrow 0 \quad \text{a.s.}, \tag{4.10}$$

obtain, respectively. Hence, recalling $\alpha > 1/2$, (4.9) dominates (4.10). Therefore, Theorem 3.1 gives us a sharper result than that which can be obtained by Klesov's [10] theorem with $\psi^*(x)$ as in (4.6).

Next, by taking

$$\psi^*(x) = \left(\log_1 x\right)^{1+\varepsilon}, \quad \varepsilon > 0, \tag{4.11}$$

where

$$\log_1 x = \begin{cases} \log x & \text{if } x \ge e, \\ e^{-1}x & \text{if } x < e, \end{cases} \quad \text{for the natural logarithm } \log x \ (x \ge e), \tag{4.12}$$

the two relations in (4.4) and (4.5) yield the asymptotic relations

$$\begin{aligned} B_n\psi^*\left(B_n^{-1}\right) &\sim M_3 n^{-(\alpha-1/2)}\left(\log_1 n\right)^{1+\varepsilon}, \\ \mathcal{B}_n\psi^*\left(\mathcal{B}_n^{-1}\right) &\sim M_4 n^{-(2\alpha-1)}\left(\log_1 n\right)^{1+\varepsilon}, \end{aligned} \tag{4.13}$$

where $M_3 = (\alpha - 1/2)^{\varepsilon}$ and $M_4 = 2^{\varepsilon}M_3$, and so, respectively,

$$\begin{aligned} g_n^{-1}\left(B_n\psi^*\left(B_n^{-1}\right)\right) &\sim M_3^{1/2} n^{-(\alpha-1/2)}\left(\log_1 n\right)^{(1+\varepsilon)/2}, \\ g^{-1}\left(\mathcal{B}_n\psi^*\left(\mathcal{B}_n^{-1}\right)\right) &\sim M_4^{1/2} n^{-(\alpha-1/2)}\left(\log_1 n\right)^{(1+\varepsilon)/2}. \end{aligned} \tag{4.14}$$

Hence, by either Theorem 3.1 or the theorem of Klesov [10], the tail series SLLN

$$\frac{n^{\alpha-1/2}}{\left(\log_1 n\right)^{(1+\varepsilon)/2}} T_n \longrightarrow 0 \quad \text{a.s.}, \tag{4.15}$$

obtains. Therefore, there is no advantage of Theorem 3.1 over Klesov's [10] theorem when $\psi^*(x)$ is as in (4.11).

In particular, by taking $\alpha = 1$ in Example 4.1, the *harmonic series* $S_n = \sum_{j=1}^{n} X_j$ with a random choice of signs is obtained in the following example.

Example 4.2. Let $\{X_n,\ n \ge 1\}$ be independent random variables such that

$$P\left\{X_n = \frac{1}{n}\right\} = P\left\{X_n = -\frac{1}{n}\right\} = \frac{1}{2}, \quad n \ge 1. \tag{4.16}$$

Let $0 < \beta < 1$ and define

$$g_n(x) = n^{1-\beta}x^2, \quad n \ge 1, \tag{4.17}$$

and $g(x) = g_1(x) = x^2$, $x \geq 0$. Then

$$E(g_n(|X_n|)) = n^{-(1+\beta)}, \qquad E(g(|X_n|)) = n^{-2}, \tag{4.18}$$

and so, all the hypotheses of both Theorem 3.1 and Klesov's [10] theorem are satisfied. Now, for $n \geq 1$,

$$B_n = \sum_{j=n}^{\infty} E(g_j(|X_j|)) \sim Mn^{-\beta}, \qquad \mathcal{B}_n = \sum_{j=n}^{\infty} E(g(|X_j|)) \sim n^{-1}, \tag{4.19}$$

where $M = \beta^{-1}$. If $\psi^*(x)$ is taken to be the function as in (4.6), then

$$B_n \psi^*(B_n^{-1}) \sim M^{1/2} n^{-\beta/2}, \qquad \mathcal{B}_n \psi^*(\mathcal{B}_n^{-1}) \sim n^{-1/2}, \tag{4.20}$$

and so, respectively,

$$g_n^{-1}(B_n \psi^*(B_n^{-1})) \sim M^{1/4} n^{-(1/2-\beta/4)}, \qquad g^{-1}(\mathcal{B}_n \psi^*(\mathcal{B}_n^{-1})) \sim n^{-1/4}. \tag{4.21}$$

Thus, by applying Theorem 3.1, we obtain the tail series SLLN

$$n^{1/2-\beta/4} T_n \longrightarrow 0 \quad \text{a.s.}, \tag{4.22}$$

which dominates the tail series SLLN

$$n^{1/4} T_n \longrightarrow 0 \quad \text{a.s.}, \tag{4.23}$$

given by the theorem of Klesov [10]. Hence, Theorem 3.1 gives us a sharper result than that which can be obtained by Klesov's [10] theorem. Also, if we include $\alpha = 1$ in (4.9) for the harmonic series, (4.22) with $\beta < 1/2$ dominates (4.9). Therefore, by taking $g_n(x)$ as in (4.17) instead of as in (4.2), this example gives us a better result than that which was obtained in Example 4.1.

Example 4.3. Let $\{X_n, n \geq 1\}$ be independent random variables such that

$$P\{X_n = 1\} = 1 - \frac{1}{n^2}, \quad P\{X_n = e^n\} = \frac{1}{n^2}, \quad n \geq 1. \tag{4.24}$$

Let $1/2 < p \leq 1$ and let $g_n(x) \equiv |x|^p$, $n \geq 1$. Then, by setting $C_n \equiv 1$ for all $n \geq 1$, all the assumptions of Corollary 3.5 are satisfied since

$$E(g_n(|X_n I_{[|X_n| \leq C_n]} - E(X_n I_{[|X_n| \leq C_n]})|)) = \frac{1}{n^{2p}}\left(1 - \frac{1}{n^2}\right) + \left(1 - \frac{1}{n^2}\right)^p \frac{1}{n^2}. \tag{4.25}$$

Next, for $n \geq 1$,

$$\tilde{B}_n = \sum_{j=n}^{\infty} E(g_n(|X_n I_{[|X_n| \leq C_n]} - E(X_n I_{[|X_n| \leq C_n]})|)) \sim M^{-1} n^{-(2p-1)}, \tag{4.26}$$

where $M = 2p - 1$. If $\psi^*(x)$ is taken to be the function as in (4.11), then

$$\tilde{B}_n \psi^*(\tilde{B}_n^{-1}) \sim M^{\varepsilon} n^{-(2p-1)} (\log_1 n)^{1+\varepsilon} = o(1), \quad \varepsilon > 0, \tag{4.27}$$

and so

$$g_n^{-1}(\tilde{B}_n \psi^*(\tilde{B}_n^{-1})) \sim M^{\varepsilon/p} n^{-(2-1/p)} (\log_1 n)^{(1+\varepsilon)/p}. \tag{4.28}$$

Thus, the tail series SLLN

$$\frac{n^{2-1/p}}{(\log_1 n)^{(1+\varepsilon)/p}} \tilde{T}_n \longrightarrow 0 \quad \text{a.s.}, \tag{4.29}$$

obtains by Corollary 3.5, where the tail series $\{\tilde{T}_n,\ n \geq 1\}$ are defined as in (3.14).

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