SECOND-ORDER NEUTRAL STOCHASTIC EVOLUTION EQUATIONS WITH HEREDITY

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Existence, continuous dependence, and approximation results are established for a class of abstract second-order neutral stochastic evolution equations with heredity in a real separable Hilbert space. A related integro-differential equation is also mentioned, as well as an example illustrating the theory.

1. Introduction

The focus of this investigation is the class of abstract neutral semilinear stochastic evolution equations with heredity of the form

$$\begin{gathered} d\left[x'(t)-f_1(t,x_t)\right]=Ax(t)dt+f_2(t,x_t)dt+g(t,x_t)dW(t), \quad 0\le t\le T,\\ x(t)=\phi(t), \quad -r\le t\le 0,\\ x'(0)=\varsigma, \end{gathered} \tag{1.1}$$

in a real separable Hilbert space H, where the linear (possibly multivalued) operator $A:D(A)\subset H\to H$ is the infinitesimal generator of a strongly continuous cosine family on H, W is a K-valued Wiener process with incremental covariance given by the nuclear operator Q defined on a complete probability space $(\Omega,\mathfrak{I},P)$ equipped with a normal filtration $(\mathfrak{I}_t)_{t\ge 0}$, $\phi\in L^p(\Omega;C_r)$, and ς is an $\mathfrak{I}_0$-measurable H-valued random variable independent of W. We develop existence and approximation results by imposing various Lipschitz and Carathéodory-type conditions on the mappings $f:[0,T]\times C_r\to H$ $(i=1,2)$ and $g:[0,T]\times C_r\to \mathrm{BL}(K;H)$, where K is another real separable Hilbert space and $\mathrm{BL}(K;H)$ is the space of bounded linear operators from K into H.

Stochastic partial differential equations (SPDEs) with finite delay arise naturally in the mathematical modeling of various phenomena in the natural and social sciences [19, 20, 22]. As such, researchers have devoted considerable attention to such equations. Just as in the first-order case, where many SPDEs can be described by a single abstract evolution equation and investigated in a unified setting using various methods (e.g., semigroup methods, approximation schemes, and compactness methods), the same is true for the

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relationship between (1.1) and a wide class of second-order SPDEs. The deterministic form of (1.1), and variants thereof, has been thoroughly investigated, while the stochastic version (namely, (1.1)) has not yet been treated. In fact, abstract second-order stochastic evolution equations have only recently been investigated (cf. [15]). The motivation of the present work therefore lies primarily in formulating a theory which extends the results in [9, 12, 18] to equations of the abstract form (1.1).

Following a brief review of notation and preliminaries, Section 3 of the paper is devoted to a discussion of the well-posedness of (1.1), as well as an approximation result, under Lipschitz assumptions (see [11]). An analogous, but more general, existence result formulated under weaker Carathéodory-type conditions in the spirit of those used in [9, 12] is presented in Section 4. An example illustrating the abstract theory is then provided in Section 5.

2. Preliminaries

For details of this section, the reader is referred to [4, 5, 7, 8, 12, 13, 16, 17, 19, 21, 22]. Throughout the paper, H and K denote real separable Hilbert spaces. For $r > 0$, $C_r = C([-r,0];H)$ is the space of continuous functions from $[-r,0]$ into H equipped with the norm

$$\|z\|_{C_r}^2 = \sup_{-r\le\theta\le 0} ||z_t(\theta)||_H^2, \tag{2.1}$$

where $z_t(s) = z(t+s)$ for all $t \ge 0$, $-r \le s \le 0$. Next, $L^p(\Omega;H)$, $p \ge 2$, represents the space of all strongly measurable, p-integrable H-valued random variables equipped with the norm

$$||X(\cdot)||_{L^p}^p = E||X(\omega)||_H^p, \tag{2.2}$$

where E stands for expectation, $E(g) = \int_\Omega g(w)\,dP$. Similarly, $L^p([0,T];H)$ is the space of all $\Im_t$-measurable random variables such that $\int_0^T \|X(t,\cdot)\|_{L^p}^p dt < \infty$.

Finally, we let

$$X_{T,p} = \{Z \in C([-r,T];L^p(\Omega;H)) : Z \text{ is } \Im_t\text{-adapted and } \|Z\|_{X_{T,p}} < \infty\}, \tag{2.3}$$

where

$$\|Z\|_{X_{T,p}}^p = \sup_{0\le t\le T} E||Z_t||_{C_r}^p. \tag{2.4}$$

Next, we recall some facts about cosine families of operators.

Definition 2.1. (i) The one-parameter family $\{C(t) : t \in \mathbb{R}\} \subset \mathrm{BL}(H)$ satisfying

(a) $C(0) = I$,
(b) $C(t)x$ is continuous in t on $\mathbb{R}$ for all $x \in H$,
(c) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t,s \in \mathbb{R}$,

is called a *strongly continuous cosine family*.

(ii) The corresponding *strongly continuous sine family* $\{S(t) : t \in \mathbb{R}\} \subset \mathrm{BL}(H)$ is defined by $S(t)x = \int_0^t C(s)x\,ds$, for all $t \in \mathbb{R}$, for all $x \in H$.

(iii) The (*infinitesimal*) *generator* $A : H \to H$ of $\{C(t) : t \in \mathbb{R}\}$ is given by

$$Ax = \frac{d^2}{dt^2}C(t)x\bigg|_{t=0}, \quad \forall x \in D(A) = \{x \in H : C(\cdot)x \in C^2(\mathbb{R};H)\}. \tag{2.5}$$

It is known that the infinitesimal generator A is a closed, densely defined operator on H (see [10, 21]). Such cosine, and corresponding sine, families and their generators satisfy the following properties.

Proposition 2.2. *Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t) : t \in \mathbb{R}\}$ (cf. Definition 2.1). Then, the following hold:*

(i) *there exist $M_A \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq M_A e^{\omega|t|}$ and hence, $\|S(t)\| \leq M_A e^{\omega|t|}$;*

(ii) *$A\int_s^r S(u)x\,du = [C(r) - C(s)]x$, for all $0 \leq s \leq r < \infty$;*

(iii) *there exists $N \geq 1$ such that $\|S(s) - S(r)\| \leq N|\int_s^r e^{\omega|s|}ds|$, for all $0 \leq s \leq r < \infty$.*

The uniform boundedness principle, together with (i) above, implies that both $\{C(t) : t \in [0,T]\}$ and $\{S(t) : t \in [0,T]\}$ are uniformly bounded by $M_* = M_A e^{\omega|T|}$.

Proposition 1.9 in [14], and variations thereof, is used throughout this paper. We recall it here for convenience.

Proposition 2.3. *Let $G : [0,T] \times \Omega \to \mathrm{BL}(K;H)$ be strongly measurable such that $\int_0^T \|G(t)\|_{L^p}^p dt < \infty$. Then, for all $0 \leq t \leq T$ and $p \geq 2$,*

$$E\left\|\int_0^t G(s)dW(s)\right\|^p \leq \left[\frac{1}{2}p(p-1)\right]^{p/2}(TrQ)^{p/2}t^{p/2-1}\int_0^t E\|G(s)\|_{\mathrm{BL}(K;H)}^p ds. \tag{2.6}$$

Finally, in addition to the familiar Young, Hölder, and Minkowski inequalities, the following inequality (which follows from the convexity of x^m, $m \geq 1$) is important:

$$\left(\sum_{i=1}^n a_i\right)^m \leq n^{m-1}\sum_{i=1}^n a_i^m, \tag{2.7}$$

where a_i is a nonnegative constant ($i = 1,\dots,m$).

3. Lipschitz case

Throughout this section, we consider (1.1) under the following assumptions:

(H_A) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ on H;

(H_{f_i}) $f_i : [0,T] \times C_r \to H$ ($i = 1,2$) satisfies

(i) $f_i(0,0) = 0$,

(ii) $\|f_i(t,x_t) - f_i(t,y_t)\| \leq M_{f_i}\|x_t - y_t\|_{C_r}$, for some $M_{f_i} > 0$, globally on $[0,T] \times C_r$,

(iii) $\|f_i(t,x_t)\| \leq \overline{M}_{f_i}(1 + \|x_t\|_{C_r})$, for some $\overline{M}_{f_i} > 0$, globally on $[0,T] \times C_r$;

(H_g) $g : [0,T] \times C_r \to \mathrm{BL}(K;H)$ satisfies
(i) $\|g(t,x_t) - g(t,y_t)\|_{\mathrm{BL}} \le M_g \|x_t - y_t\|_{C_r}$, for some $M_g > 0$, globally on $[0,T] \times C_r$;
(ii) $\|g(t,x_t)\|_{\mathrm{BL}} \le \overline{M_g}(1 + \|x_t\|_{C_r})$, for some $\overline{M_g} > 0$, globally on $[0,T] \times C_r$.

We begin by establishing the existence and uniqueness of a global *mild solution* of (1.1) under these assumptions in the sense of the following definition.

Definition 3.1. A continuous stochastic process $x : [-r,T] \to H$ is a *mild solution* of (1.1) on $[0,T]$ if

(i) $x(t)$ is measurable and $\Im_t$-adapted, for all $-r \le t \le T$,
(ii) $\int_0^T \|x(s)\|^2 ds < \infty$, a.s. $[P]$,
(iii) $x(t) = C(t)\phi(0) + S(t)(\varsigma - f_1(0,\phi)) + \int_0^t C(t-s) f_1(s,x_s)ds + \int_0^t S(t-s) f_2(s,x_s)ds + \int_0^t S(t-s)g(s,x_s)dW(s)$, $0 \le t \le T$,
(iv) $x(t) = \phi(t)$, $-r \le t \le 0$.

Our first result is the following theorem.

Theorem 3.2. *Let $\phi \in L^p(\Omega, C_r)$ $(p \ge 2)$ be an $\Im_0$-measurable random variable independent of W. If (H_a), (H_{f_i}), and (H_g) hold, then (1.1) has a unique mild solution $x \in X_{T,p}$ (in the sense of Definition 3.1).*

Proof. Define the solution operator $\Phi : X_{T,p} \to X_{T,p}$ by

$$(\Phi x)(t) = \begin{cases} C(t)\phi(0) + S(t)(\varsigma - f_1(0,\phi)) + \displaystyle\int_0^t C(t-s) f_1(s,x_s)ds & \\ + \displaystyle\int_0^t S(t-s) f_2(s,x_s)ds + \int_0^t S(t-s)g(s,x_s)dW(s), & 0 \le t \le T, \\ C(t)\phi(0) + S(t)(\varsigma - f_1(0,\phi)) + \displaystyle\sum_{i=1}^{3} I_i^x(t), & 0 \le t \le T, \\ \phi(t), & -r \le t \le 0. \end{cases} \tag{3.1}$$

We begin by verifying the L^p-continuity on $[0,T]$. To this end, observe that for $x \in X_{T,p}$ and $0 \le t_1 < t_2 \le T$, applying (2.7) yields

$$\begin{aligned} &E\|(\Phi x)(t_2) - (\Phi x)(t_1)\|^p \\ &\quad \le 5^{p-1}\Bigg[E\|(C(t_2) - C(t_1))\phi(0)\|^p + E\|(S(t_2) - S(t_1))(\varsigma - f_1(0,\phi))\|^p \\ &\qquad + \sum_{i=1}^{3} E\|I_i^x(t_2) - I_i^x(t_1)\|^p\Bigg]. \end{aligned} \tag{3.2}$$

The strong continuity of $C(t)$ and $S(t)$ ensures that the first two terms on the right-hand side of (3.2) go to zero as $t_2 - t_1 \to 0$. Next, an application of the Hölder inequality to

(H_{f_i}) yields

$$\begin{aligned}
&\sum_{i=1}^{2} E\left|\left|I_i^x(t_2)-I_i^x(t_1)\right|\right|^p\\
&\quad\le E\left|\left|\int_0^{t_1}\left[C(t_2-s)-C(t_1-s)\right]f_1(s,x_s)\,ds+\int_{t_1}^{t_2}C(t_2-s)f_1(s,x_s)\,ds\right|\right|^p\\
&\qquad+E\left|\left|\int_0^{t_1}\left[S(t_2-s)-S(t_1-s)\right]f_2(s,x_s)\,ds+\int_{t_1}^{t_2}S(t_2-s)f_2(s,x_s)\,ds\right|\right|^p\\
&\quad\le 2^{p-1}t_1\left(\overline{M_{f_1}}^p+\overline{M_{f_2}}^p\right)\left(\int_0^{t_1}\left[||C(t_2-s)-C(t_1-s)||+||S(t_2-s)-S(t_1-s)||\right]^q ds\right)^{p/q}\\
&\qquad\times\left(1+\|x\|_{X_{T,p}}^p\right)+2^{p-1}(t_2-t_1)^{p/q}\left(\overline{M_{f_1}}^p+\overline{M_{f_2}}^p\right)\left(M_C^p+M_S^p\right)\left(1+\|x\|_{X_{T,p}}^p\right),
\end{aligned}\tag{3.3}$$

where $1\le p,\ q<\infty$ are conjugate indices. Hence, using the continuity of $S(t)$ and $C(t)$, together with Lebesgue's dominated convergence theorem, we conclude that the right-hand side of (3.3) tends to zero as $t_2-t_1\to 0$. Similarly, using Hölder's inequality, Ito's formula, and Proposition 2.3 yields

$$\begin{aligned}
E||I_3^x(t_2)-I_3^x(t_1)||^p &\le 2^{p-1}\overline{M_g}^pL_g^pt_1^{p/2-1}\int_0^{t_1}||S(t_2-s)-S(t_1-s)||^p ds\times\left(1+||x||_{X_{T,p}}^p\right)\\
&\quad+2^{p-1}(t_2-t_1)^{p/2-1}\overline{M_g}^pM_S^p\left(1+\|x\|_{X_{T,p}}^p\right),
\end{aligned}\tag{3.4}$$

which also tends to zero as $t_2-t_1\to 0$. So, we conclude that Φ is L^p-continuous on $[0,T]$.

Next, we show that $\Phi(X_{T,p})\subset X_{T,p}$. To this end, let $x\in X_{T,p}$ and $t\in[0,T]$. Since $\phi\in L^p(\Omega;C_r)$, it is the case that

$$\sup_{-r\le\theta\le 0}\left\{E||(\Phi x)(t+\theta)||^p : -r\le T+\theta\le 0\right\}<\infty.\tag{3.5}$$

Further, for all $-r\le\theta\le 0$ for which $t+\theta>0$, standard computations involving Hölder's inequality, (H_A), (H_{f_i}), (H_g), and Proposition 2.3 yield

$$\begin{aligned}
&E\left(\sup_{-r\le\theta\le 0}||C(t+\theta)\phi(0)||^p+||S(t+\theta)(\varsigma-f_1(0,\phi))||^p\right)\\
&\quad\le M_C^p\|\phi\|_{C_r}^p+M_S^p\left(\|\varsigma\|_{L^p}+M_{f_1}^p\|\phi\|_{C_r}^p\right)E\left(\sup_{-r\le\theta\le 0}\sum_{i=1}^{3}||I_i^x(t+\theta)||^p\right)\\
&\quad\le 3^{p-1}\left[T^{p/q}\left(M_C^p\overline{M_{f_1}}^p+M_S^p\overline{M_{f_2}}^p\right)+T^{p/2-1}M_S^pM_g^pL_g^p\right]\times\left(1+\|x\|_{X_{T,p}}^p\right).
\end{aligned}\tag{3.6}$$

Consequently, we conclude that (cf. (3.1))

$$\sup_{-r\le\theta\le 0}\left\{E||(\phi x)(t+\theta)||^p : 0\le t+\theta\le T\right\}<\infty.\tag{3.7}$$

Hence, (3.5) and (3.7) together imply that $E\|(\Phi x)_t\|_{C_r}^p < \infty$ for all $0 \le t \le T$, so that $\Phi(x) \in X_{T,p}$. Since the $\Im_t$-measurability of $(\Phi x)(t)$ is easily verified, we can conclude that Φ is well defined.

It remains to show that Φ has a unique fixed point. Let $x, y \in X_{T,p}$ and note that (3.1) implies that

$$E\|(\Phi x)_t - (\Phi y)_t\| \le 3^{p-1} E\left[\sup_{-r\le\theta\le 0} \sum_{i=1}^{3} \|I_i^x(t+\theta) - I_i^y(t+\theta)\|^p\right]$$
$$\le 3^{p-1}\left[T^{p/q}\left(M_C^p M_{f_1}^p + M_s^p M_{f_2}^p\right) + T^{p/2-1} M_s^p M_g^p L_g^p\right] \cdot \int_0^t E\|x_\theta - y_\theta\|_{C_r}^p d\theta, \quad 0 \le t \le T. \tag{3.8}$$

Successive iterations of (3.8) yield, for $n \ge 1$,

$$\|(\Phi^n x)_t - (\Phi^n y)_t\|_{X_{T,p}}^p \le \xi_n \|x_t - y_t\|_{X_{T,p}}^p, \tag{3.9}$$

where ξ_n is a positive constant depending on growth conditions, T, and n. For sufficiently large n, $\xi_n < 1$, so that the Banach contraction mapping principle implies that Φ^n, and hence Φ, have a unique fixed point in $X_{T,p}$ which coincides with a mild solution of (1.1). □

Next, we establish results concerning the continuous dependence of mild solutions of (1.1) on the initial data, as well as the boundedness of the pth moments.

Proposition 3.3. *Let $\phi, \psi \in L^2(\Omega; C_r)$, ς_1, and ς_2 be given $\Im_0$-measurable H-valued random variables independent of W, and denote the corresponding mild solutions of (1.1) by x_ϕ, x_ψ. Then*

$$E\left\|(x_\phi)_t - (x_\psi)_t\right\|_{C_r}^2 \le \beta_1\left(\|\varsigma_1 - \varsigma_2\|_{L^2}^2 + \|\phi - \psi\|_{C_r}^2\right) \exp(\beta_2 t), \quad 0 \le t \le T, \tag{3.10}$$

for some positive constants β_i.

Proof. Using the computations that led to (3.8), followed by an application of Gronwall's lemma, yields the result. □

A more general estimate involving $\phi, \psi \in L^p(\Omega; C_r)$ $(p \ge 2)$ can be established similarly using an integral inequality due to Pachpatte [16] in place of Gronwall's lemma. A related result concerning the boundedness of pth moments is as follows.

Proposition 3.4. *Let x be a mild solution of (1.1), as guaranteed by Theorem 3.2. Then, for all $p \ge 2$, there exists a positive constant ξ_p such that*

$$\sup_{0\le t\le T} E\|x_t\|_{C_r}^{2p} \le \xi_p\left(1 + E\|\phi(0)\|_{C_r}^{2p} + E\|\eta\|^{2p}\right). \tag{3.11}$$

Proof. Observe that

$$E\|x_t\|_{C_r}^{2p} \le 5^{2(p-1)}\left[\sup_{-r\le\theta\le 0} E\|C(t+\theta)\phi(0)\|^{2p} + \sup_{-r\le\theta\le 0} E\|S(t+\theta)(\varsigma-g(0,\phi))\|^{2p} + \sup_{-r\le\theta\le 0}\sum_{i=1}^{3} E\|I_i^x(t+\theta)\|^{2p}\right]. \tag{3.12}$$

An application of Hölder's inequality then yields the result. □

We now formulate a result in which a related deterministic second-order evolution equation is approximated by a sequence of stochastic equations of the form (1.1) (cf. [15] for a related result). Precisely, consider the deterministic initial-value problem

$$\begin{gathered}\left[y'(t)+f_1(t,y_t)\right]' = Ay(t)+f_2(t,y_t), \quad 0\le t\le T,\\ y(t)=\phi(t), \quad -r\le t\le 0,\\ y'(0)=\varsigma.\end{gathered} \tag{3.13}$$

For every $\varepsilon>0$, consider the stochastic initial-value problem

$$\begin{gathered}d\left[x_\varepsilon'(t)+f_{1\varepsilon}(t,(x_\varepsilon)_t)\right] = A_\varepsilon x_\varepsilon(t)dt+f_{2\varepsilon}(t,(x_\varepsilon)_t)dt+g_e(t,(x_e)_t)dW(t), \quad 0\le t\le T,\\ x_\varepsilon(t)=\phi(t), \quad -r\le t\le 0,\\ x_\varepsilon'(0)=\varsigma.\end{gathered} \tag{3.14}$$

Assume that A and f_i $(i=1,2)$ satisfy (H_A) and (H_{f_i}), respectively, so that the results in [4, 21] can be adapted to guarantee the existence of a unique mild solution y of (3.13). Regarding (3.14), we assume that, for each $\varepsilon>0$,

- $(\mathrm{H}_{A_\varepsilon})$ $A_\varepsilon : D(A)\subset H\to H$ generates a cosine family $\{C_\varepsilon(t):t\ge 0\}$ and associated sine family $\{S_\varepsilon(t):t\ge 0\}$ satisfying $C_\varepsilon(t)\to C(t)$ and $S_\varepsilon(t)\to S(t)$ strongly as $\varepsilon\to 0^+$, uniformly in $t\in[0,T]$. Also, $\{C_\varepsilon(t):0\le t\le T\}\cup\{S_\varepsilon(t):0\le t\le T\}$ are uniformly bounded by $M_* = M_A\exp(\omega T)$ (the same growth bound for the cosine and sine family generated by A),
- $(\mathrm{H}_{f_{i\varepsilon}})$ $f_{i\varepsilon}:[0,T]\times C_r\to H$ $(i=1,2)$ is Lipschitz in the second variable (with the same Lipschitz constant M_{f_i} as in (H_{f_i})) and $f_{i\varepsilon}(t,z)\to f_i(t,z)$ as $\varepsilon\to 0^+$, for all $z\in C_r$, uniformly in $t\in[0,T]$,
- $(\mathrm{H}_{g_\varepsilon})$ $g_\varepsilon:[0,T]\times C_r\to \mathrm{BL}(K;H)$ is Lipschitz in the second variable (with the same Lipschitz constant M_g as in (H_g)) and $g_\varepsilon(t,z)\to 0$ as $\varepsilon\to 0^+$, for all $z\in C_r$, uniformly in $t\in[0,T]$.

Under the assumptions, Theorem 3.2 ensures the existence of a unique mild solution of (3.14), for every $\varepsilon>0$. Now, we have the following convergence result.

Theorem 3.5. *Let y, x_ε be mild solutions to (3.13) and (3.14), respectively. Then, there exist $\xi>0$ and a positive function $\Psi(\varepsilon)$ which decreases to 0 as $\varepsilon\to 0^+$ such that*

$$E\|(x_\varepsilon)_t-y_t\|_{C_r}^{p} \le \xi\cdot\psi(\varepsilon)\exp(\xi t) \quad \forall 0\le t\le T. \tag{3.15}$$

Proof. We proceed by estimating each term of the representation formula for $E\|(x_\varepsilon)_t - y_t\|_{C_r}^p$ separately (cf. (3.1)). Note that for $-r \le t \le 0$, $E\|(x_\varepsilon)_t - y_t\|_{C_r}^p = 0$. As such, we need to focus only on $0 \le t \le T$. Throughout the proof, C_i denotes a positive constant and $\beta_i(\varepsilon)$ are positive functions which decrease to 0 as $\varepsilon \to 0^+$. First, $(\mathrm{H}_{A_\varepsilon})$ guarantees the existence of C_i and $\beta_i(\varepsilon)$ $(i = 1,2)$ such that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} E\|C_\varepsilon(t)\phi(0) - C(t)\phi(0)\|_H^p &\le C_1\beta_1(\varepsilon), \\ E\|S_\varepsilon(t)\varsigma - S(t)\varsigma\|_H^p &\le C_2\beta_2(\varepsilon). \end{aligned} \tag{3.16}$$

Next, since $(\mathrm{H}_{f_{i\varepsilon}})$ implies that for every $\varepsilon > 0$,

$$\begin{aligned} \|f_{1\varepsilon}(0,\phi)\| &\le M_{f_1}\|\phi\|_{C_r}, \\ E\|f_{1\varepsilon}(0,\phi) - f_1(0,\phi)\|^p &\le \beta_3(\varepsilon), \end{aligned} \tag{3.17}$$

for some $\beta_3(\varepsilon)$, an application of Minkowski's inequality subsequently yields

$$\begin{aligned} E\|S_\varepsilon(t)f_{1\varepsilon}(0,\phi) - S(t)f_1(0,\phi)\|^p &\le 2^p\left[M_{f_1}^p\|\phi\|_{C_r}^p\beta_4(\varepsilon) + C_3\beta_\varepsilon(\varepsilon)\right] \\ &= C_3\beta_3(\varepsilon) + C_4\beta_4(\varepsilon). \end{aligned} \tag{3.18}$$

Regarding the term $E\|\int_0^t[C_\varepsilon(t-s)f_{1\varepsilon}(s,(x_\varepsilon)_s) - C(t-s)f_1(s,y_s)]ds\|^p$, the continuity of $f_{1\varepsilon}$, together with $(\mathrm{H}_{A_\varepsilon})$, ensures the existence of C_5 and $\beta_5(\varepsilon)$ such that for sufficiently small $\varepsilon > 0$,

$$\int_0^t E\|[C_\varepsilon(t-s) - C(t-s)]f_1(s,y_s)\|^p ds \le C_5\beta_5(\varepsilon), \tag{3.19}$$

for all $0 \le t \le T$. Also, observe that Young's inequality and $(\mathrm{H}_{f_{i\varepsilon}})$ together yield

$$\begin{aligned} &\int_0^t E\left\|C_\varepsilon(t-s)\left[f_{1\varepsilon}\left(s,(x_\varepsilon)_s\right) - f_1(s,y_s)\right]\right\|^p ds \\ &\quad\le M_C^p\int_0^t E\left\|f_{1\varepsilon}\left(s,(x_\varepsilon)_s\right) - f_{1\varepsilon}(s,y_s) + f_{1\varepsilon}(s,y_s) - f_1(s,y_s)\right\|^p ds \\ &\quad\le 2^pM_C^p\int_0^t\left[M_{f_1}^pE\left\|(x_\varepsilon)_s - y_s\right\|_{C_r}^p + E\left\|f_{1\varepsilon}(s,y_s) - f_1(s,y_s)\right\|^p\right]ds. \end{aligned} \tag{3.20}$$

Note that $(\mathrm{H}_{f_{i\varepsilon}})$ guarantees the existence of C_6 and $\beta_6(\varepsilon)$ such that for small enough $\varepsilon > 0$, $E\|f_{1\varepsilon}(s,y_s) - f_1(s,y_s)\|^p \le C_6\beta_6(\varepsilon)$, for all $0 \le t \le T$, so that we can conclude from (3.20) that

$$\int_0^t E\left\|C_\varepsilon(t-s)\left[f_{1\varepsilon}\left(s,(x_\varepsilon)_s\right) - f_1(s,y_s)\right]\right\|^p ds \le 2^pM_C^p\int_0^t E\|(x_\varepsilon)_s - y_s\|^p ds + 2^pTC_6\beta_6(\varepsilon), \tag{3.21}$$

for all $0 \le t \le T$. Using (3.19) and (3.21), together with the Hölder, Minkowski, and Young inequalities, yields

$$E\left\|\int_0^t \left[C_\varepsilon(t-s)f_{1\varepsilon}\left(s,(x_\varepsilon)_s\right) - C(t-s)f_1(s,y_s)\right]ds\right\|^p \\ \le 2^pT^{1/q}\left[C_5\beta_5(\varepsilon) + 2^pTC_6\beta_6(\varepsilon) + 2^pM_C^pM_{f_1}^p\int_0^t E\|(x_\varepsilon)_s - y_s\|_{C_r}^p ds\right]. \tag{3.22}$$

Similarly, we can argue that

$$E\left\|\int_0^t \left[S_\varepsilon(t-s)f_{2\varepsilon}\left(s,(x_\varepsilon)_s\right) - S(t-s)f_2(s,y_s)\right]ds\right\|^p \\ \le 2^pT^{1/q}\left[C_7\beta_7(\varepsilon) + 2^pTC_8\beta_8(\varepsilon) + 2^pM_s^pM_{f_2}^p\int_0^t E\|(x_\varepsilon)_s - y_s\|_{C_r}^p ds\right]. \tag{3.23}$$

It remains to estimate $E\|\int_0^t S_\varepsilon(t-s)g_\varepsilon(s,(x_\varepsilon)_s)dW(s)\|^p$. Computations similar to those leading to (3.22), together with Proposition 2.3, yield

$$E\left\|\int_0^t S_\varepsilon(t-s)g_\varepsilon\left(s,(x_\varepsilon)_s\right)dW(s)\right\|^p \\ \le 2^pM_S^pL_g^pT^{1/q}\left[\int_0^t M_g^pE\left\|(x_\varepsilon)_s - y_s\right\|_{C_r}^p ds + \int_0^t E\|g_\varepsilon(s,y_s)\|^p ds\right]. \tag{3.24}$$

Note that $(\mathrm{H}_{g_\varepsilon})$ guarantees the existence of C_9 and $\beta_9(\varepsilon)$ such that for small enough $\varepsilon > 0$,

$$E\|g_\varepsilon(s,y_s)\|^p \le C_9\beta_9(\varepsilon), \tag{3.25}$$

for all $0 \le s \le T$. Using (3.25) in (3.24), in conjunction with (3.16), (3.18), (3.19), (3.20), (3.21), (3.22), (3.23), and (3.24), we conclude that for all $\varepsilon > 0$ sufficiently small to ensure that (3.16), (3.18), (3.19), (3.20), (3.21), (3.22), (3.23), and (3.24) all hold simultaneously, we have, for an appropriate constant $\eta > 0$,

$$E\|(x_\varepsilon)_t - y_t\|_{C_r}^p \le \sum_{i=1}^9 C_i\beta_i(\varepsilon) + \eta\int_0^t E\|(x_\varepsilon)_s - y_s\|_{C_r}^p ds, \tag{3.26}$$

so that an application of Gronwall's lemma yields

$$E\|(x_\varepsilon)_t - y_t\|_{C_r}^p \le \Psi(\varepsilon)\exp(\xi t), \tag{3.27}$$

for all $0 \le t \le T$, where $\Psi(\varepsilon) = \sum_{i=1}^9 C_i\beta_i(\varepsilon)$. This completes the proof. □

Remarks 3.6. (1) The case in which the classical initial condition $x(t) = \phi(t)$, $-r \le t \le 0$, in (1.1) is replaced by a so-called nonlocal term of the form

$$x(t) + h(x_{t_1},\dots,x_{t_m})(t) = \phi(t), \quad -r \le t \le 0, \tag{3.28}$$

where $0 < t < \cdots < t_m \leq T$ are fixed, $\phi \in L^p(\Omega; C_r)$, and $h : (C_r)^m \to C_r$ is a continuous function satisfying

$$\left\| h(x_{t_1}, \ldots, x_{t_m})(s) - h(\overline{x_{t_1}}, \ldots, \overline{x_{t_m}})(s) \right\| \leq M_h \|x - \overline{x}\|_{C_r}, \tag{3.29}$$

for all $x, \overline{x} \in C_r$, for some positive constant M_h, can be easily handled by making slight modifications to the above proofs. The utility of studying nonlocal initial-value problems lies in the fact that more information can be taken into account at the beginning of the experiment so as to reduce initial error due to measurement. Further discussion can be found in [1, 2, 6] and the references therein.

(2) A related integro-differential initial-value problem obtained by replacing $f_2(t, x_t)$ by $\int_0^t H(t,s) f_2(s, x_s)\,ds$, where $H : [0,T]^2 \to [0, \infty)$ is a continuous kernel, can be handled in a similar manner in the sense that the computations are similar, with the exception that a more general integral inequality [16] is needed to obtain the key estimates (e.g., (3.28)). (See [3] for applications of such equations.)

4. Carathéodory conditions

We now consider (1.1) under conditions which are more general than Lipschitz conditions but for which uniqueness is still guaranteed. The theory we present here is an extension of the results presented in [9, 12, 18] to a class of abstract second-order delay evolution equations. We replace assumptions (H_{f_i}) and (H_g) by the following.

(H1) $f_i : [0, T] \times C_r \to H$ $(i = 1, 2)$ and $g : [0, T] \times C_r \to \mathrm{BL}(K; H)$ are $\Im_t$-measurable, for each (t, μ) in $[0, T] \times C_r$, continuous in the second variable, and satisfying the following conditions.

(i) There exists $H : [0, T] \times [0, \infty) \to (0, \infty)$ such that

$$E\left\| f_1(t, x_t) \right\|^p + E\left\| f_2(t, x_t) \right\|^p + E\left\| g(t, x_t) \right\|_{\mathrm{BL}}^p \leq H\left(t, E\left\| x_t \right\|^p\right), \tag{4.1}$$

for all $0 \leq t \leq T$ and $x_t \in C_r$. Here, $H(t, \mu)$ is locally integrable in t, for every $\mu \in [0, \infty)$, and is continuous, monotone nondecreasing, and concave in μ, for every $t \in [0, T]$.

(ii) There exists $G : [0, T] \times (0, \infty) \to (0, \infty)$ which is locally integrable in the first variable, for every $\mu \in (0, \infty)$ and is continuous, monotone nondecreasing, and concave in μ, for every $t \in [0, T]$, and which further satisfies

(a) $G(t, 0) = 0$, for every $0 \leq t \leq T$,

(b) $E(\| f_1(t, x_t) - f_1(t, y_t)\|^p + \| f_2(t, x_t) - f_2(t, y_t)\|^p + \|g(t, x_t) - g(t, y_t)\|_{\mathrm{BL}}^p)$ $\leq G(t, E\|x_t - y_t\|^p)$, for all $0 \leq t \leq T$ and $x_t, y_t \in C_r$.

(H2) If $z : [0, \overline{T}] \to [0, \infty)$ is a nondecreasing, continuous function such that $z(0) = 0$ and $z(t) \leq \alpha \int_0^t G(s, z(s))\,ds$, for all $0 \leq t \leq \overline{T}$, for an appropriate constant α and $0 < \overline{T} \leq T$, then $z = 0$ on $[0, \overline{T}]$.

Examples of functions G satisfying (H1)(ii) and (H2) can be found in [9]. We have the following local existence and uniqueness result.

Theorem 4.1. *If (H_A), (H1), and (H2) are satisfied, then there exists $0 < \overline{T} \leq T$ such that (1.1) has a unique mild solution x in $X_{\overline{T}, p}$.*

Proof. Consider the sequence of successive approximations defined as follows:

$$\begin{aligned}
x_0(t) &= C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)], \quad 0 \le t \le T, \\
x_n(t) &= C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)] \\
&\quad + \int_0^t C(t-s) f_1\left(s,(x_{n-1})_s\right) ds + \int_0^t S(t-s) f_2\left(s,(x_{n-1})_s\right) ds \\
&\quad + \int_0^t S(t-s) g\left(s,(x_{n-1})_s\right) dW(s), \quad 0 \le t \le T,\ n \ge 1, \\
x_n(t) &= \phi(t), \quad -r \le t \le 0.
\end{aligned} \tag{4.2}$$

Consider the initial-value problem

$$\begin{aligned}
\mu'(t) &= CH(t,\mu(t)), \quad 0 \le t \le T, \\
\mu(0) &= \mu_0,
\end{aligned} \tag{4.3}$$

where $\mu_0 > \xi_1^* + E\|x_0\|_{C_r}^p$ and $C = \xi_2^*$ (ξ_1^* and ξ_2^* are positive constants prescribed below). Using (H1), we can deduce that there exists $0 < \overline{T} \le T$ such that (4.3) has a unique solution $\mu(\cdot;\mu_0) = \mu(\cdot)$ on $[0,\overline{T}]$. Next, we claim that

$$\sup_{n\ge 1} E\left\|(x_n)_t\right\|^p \le \max\left\{\|\phi\|_{L^p(C_r)}^p, \mu(t)\right\}, \quad 0 \le t \le \overline{T}, \tag{4.4}$$

$$\sup_{n\ge 1}\ \sup_{t\in[0,T^*]} E\left\|(x_n)_t - \left(C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)]\right)\right\|^p \le R, \tag{4.5}$$

for an appropriate constant R and $T^* \leqq \overline{T}$. Indeed, observe that using (H1)(i) and standard computations yields, with the help of Proposition 2.3,

$$\begin{aligned}
E\left\|(x_1)_t\right\|^p &\le 4^{p-1}\Big[E\|C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)]\|^p \\
&\qquad + T^{1/q}\int_0^t E\left\|C(t-s) f_1\left(s,(x_0)_s\right)\right\|^p ds \\
&\qquad + T^{1/q}\int_0^t E\left\|S(t-s) f_2\left(s,(x_0)_s\right)\right\|^p ds \\
&\qquad + E\left\|\int_0^t S(t-s) g\left(s,(x_0)_s\right) dW(s)\right\|^p\Big] \\
&\le 4^{p-1}\left[M_C^p\|\phi(0)\|_{L^p}^p + M_S^p\left[\|\varsigma\|_{L^p}^p + \|f_1(0,\phi)\|_{L^p}^p\right]\right] \\
&\quad + 4^{p-1}\left[T^{1/q}M_C^p + T^{1/q}M_S^p + M_S^p L_g^p\right] \\
&\quad \times \int_0^t \left[E\left\|f_1\left(s,(x_0)_s\right)\right\|^p + E\left\|f_2\left(s,(x_0)_s\right)\right\|^p + E\left\|g\left(s,(x_0)_s\right)\right\|_{\mathrm{BL}}^p\right] ds \\
&\le \xi_1^* + \xi_2^* \int_0^t H\left(s, E\left\|(x_0)_s\right\|^p\right) ds \le \xi_1^* + \xi_2^* \int_0^t H\left(s, E\|x_0\|_{C_r}^p\right) ds \\
&\le \xi_1^* + \xi_2^* \int_0^t H(s,\mu(s))\, ds = \mu(t) + (\xi_1^* - \mu_0) \le \mu(t),
\end{aligned} \tag{4.6}$$

where we have used the choice of μ_0 prescribed earlier and the monotonicity of H, as well as the form of the solution to (4.3). If $t+\theta \in [-r,0]$, then $E\|(x_1)_t\|^p = E\|\phi(t)\|^p$. Combining this observation with (4.6) yields

$$E\left\|(x_1)_t\right\|^p \leq \max\left\{E\|\phi(t)\|^p, \mu(t)\right\}, \quad 0 \leq t \leq \overline{T}. \tag{4.7}$$

Next, observe that for all $0 \leq t \leq \overline{T}$ for which $t+\theta \in [0,\overline{T}]$,

$$\begin{aligned} &E\|(x_1)_t - (C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)])\|^p \\ &\quad \leq \xi_2^* \int_0^t H\left(s, E\left\|(x_0)_s\right\|^p\right) ds \leq \xi_2^* \int_0^t H(s,\mu(s))\,ds. \end{aligned} \tag{4.8}$$

If $0 \leq t \leq \overline{T}$ is such that $t+\theta \in [-r,0]$, then there exists a positive constant η^* such that

$$\begin{aligned} &E\|(x_1)_t - (C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)])\|^p \\ &\quad = E\|\phi(t) - C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)]\|^p \leq \eta^*. \end{aligned} \tag{4.9}$$

Hence, using (4.8) and (4.9), we see that for all $0 \leq t \leq \overline{T}$,

$$E\|(x_1)_t - (C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)])\|^p \leq \eta^* + \xi_2^* \int_0^t H(s,\mu(s))\,ds. \tag{4.10}$$

For a given $R > \eta^*$, the continuity of μ and H guarantees the existence of $T^* \in [0,\overline{T}]$ such that

$$\eta^* + \xi_2^* \int_0^t H(s,\mu(s))\,ds \leq R, \quad 0 \leq t \leq T^*, \tag{4.11}$$

so that

$$E\|(x_1)_t - (C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)])\|^p \leq R, \quad 0 \leq t \leq T^*. \tag{4.12}$$

One can now argue inductively to verify that for all $n \geq 1$,

$$\begin{gathered} E\|(x_n)_t\|^p \leq \max\left\{E\|\phi(t)\|^p, \mu(t)\right\}, \\ E\|(x_n)_t - (C(t)\phi(0) + S(t)[\varsigma - f_1(0,\phi)])\|^p \leq R, \end{gathered} \tag{4.13}$$

for all $0 \leq t \leq T^*$, thereby establishing (4.4) and (4.5).

Now, we assert that

$$E\left\|(x_{n+m})_t - (x_n)_t\right\|^p \leq \xi_3^* \int_0^t G(s, 2^{p+1}R)\,ds, \tag{4.14}$$

for all $0 \leq t \leq T^*$ and $n, m \geq 1$, where $\xi_3^* = 3^p[T^{1/q}(M_C^p + M_S^p) + L_g^p M_S^p]$. Indeed, observe that routine calculations, together with the monotonicity of G, yield

$$\begin{aligned} E\|(x_{n+m})_t - (x_n)_t\|^p &\leq 3^p\Big[E\Big\|\int_0^t C(t-s)\big[f_1\big(s,(x_{n+m-1})_s\big) - f_1\big(s,(x_{n-1})_s\big)\big]ds\Big\|^p \\ &\quad + E\Big\|\int_0^t S(t-s)\big[f_2\big(s,(x_{n+m-1})_s\big) - f_2\big(s,(x_{n-1})_s\big)\big]ds\Big\|^p \\ &\quad + E\Big\|\int_0^t S(t-s)\big[g\big(s,(x_{n+m-1})_s\big) - g\big(s,(x_{n-1})_s\big)\big]dW(s)\Big\|^p\Big] \\ &\leq 3^p\big[T^{1/q}(M_C^p + M_S^p) + L_g^p M_S^p\big]\int_0^t G\Big(s, E\big\|(x_{n+m-1})_s - (x_{n-1})_s\big\|^p\Big)ds \\ &\leq \xi_3^* \int_0^t G(s, 2^{p+1}R)\,ds, \end{aligned} \tag{4.15}$$

where we have used (4.5) in the last step. This establishes (4.14).

Now, define the following two sequences on $[0, T^*]$:

$$\omega_1(t) = \xi_3^* \int_0^t G(s, 2^{p+1}R)\,ds, \tag{4.16}$$

$$\omega_{n+1}(t) = \xi_3^* \int_0^t G(s, \omega_n(s))\,ds, \quad n \geq 1, \tag{4.17}$$

$$\vartheta_{mn}(t) = E\big\|(x_{n+m})_t - (x_n)_t\big\|^p, \quad n, m \geq 1. \tag{4.18}$$

The continuity of G ensures the existence of $T^{**} \in [0, T^*]$ such that

$$\omega_1(t) \leq 2^{p+1}R, \quad 0 \leq t \leq T^{**}. \tag{4.19}$$

We claim that the string of inequalities (for $m, n \geq 1$) given by

$$\vartheta_{mn}(t) \leq \omega_n(t) \leq \omega_{n-1}(t) \leq \cdots \leq \omega_1(t), \quad 0 \leq t \leq T^{**}, \tag{4.20}$$

follows easily by induction, taking into account (H1)(ii). As such, we deduce that $\{\omega_n(\cdot)\}$ is a decreasing sequence in n, and that for each $n \geq 1$, $\omega_n(t)$ is an increasing function of t.

Finally, we prove that (1.1) has a mild solution x on $[0, T^{**}]$ under assumptions (H1) and (H2). To this end, define

$$\omega(t) = \inf_{n \geq 1} \omega_n(t), \quad 0 \leq t \leq T^{**}. \tag{4.21}$$

Observe that ω is nonnegative and continuous, $\omega(0) = 0$, and

$$\omega(t) \leq \xi_3^* \int_0^t G(s, \omega(s))\,ds, \quad 0 \leq t \leq T^{**}. \tag{4.22}$$

Thus, (H2) implies that $\omega(t) = 0$ on $[0, T^{**}]$. Now, (4.20) implies that

$$\sup_{t \in [0, T^{**}]} \vartheta_{mn}(t) \leq \sup_{t \in [0, T^{**}]} \omega_n(t) \leq \omega_n(T^{**}), \tag{4.23}$$

where the right-hand side of (4.23) tends to 0 as $n \to \infty$. Hence, we deduce from (4.18) that $\{x_n\}$ is a Cauchy sequence in $X_{T**,p}$. From completeness, it follows that there exists $x \in X_{T**,p}$ such that

$$\sup_{t\in[0,T**]} E\left\|(x_n)_t - x_t\right\|^p \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.24}$$

Observe that

$$\begin{aligned} E&\left\|\int_0^t C(t-s)\left[f_1\left(s,(x_n)_s\right) - f_1(s,x_s)\right]ds\right\|^p + E\left\|\int_0^t S(t-s)\left[f_2\left(s,(x_n)_s\right) - f_2(s,x_s)\right]ds\right\|^p \\ &\quad + E\left\|\int_0^t S(t-s)\left[g\left(s,(x_n)_s\right) - g(s,x_s)\right]dW(s)\right\|^p \\ &\quad \le \xi_4^* \int_0^t G\left(s,E\left\|(x_n)_s - x_s\right\|_{C_r}^p\right)ds. \end{aligned} \tag{4.25}$$

By the continuity of G, (4.24) implies that $G(s,E\|(x_n)_s - x_s\|_{C_r}^p) \to 0$ as $n \to \infty$. Since $G(t,0) = 0$, for all $0 \le t \le T$, we conclude that the left-hand side of (4.25) tends to 0 as $n \to \infty$. Thus, x is indeed a mild solution of (1.1), as desired. Uniqueness easily follows from (H2).

A standard argument can be employed to prove that the above solution can be extended to the entire interval $[0,T]$. □

5. Example

Let D be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D. Consider the following initial-boundary value problem:

$$\begin{gathered} \partial\left[\frac{\partial x(t,\vec{z})}{\partial t} - f_1(t,x(t-r,\vec{z}))\right] + \sum_{j=1}^n\sum_{k=1}^n \frac{\partial}{\partial z_j}\left[a_{jk}(\vec{z})\frac{\partial x(t,\vec{z})}{\partial z_j}\right]\partial t + C(\vec{z})x(t,\vec{z})\partial t \\ = f_2(t,x(t-r,\vec{z}))\partial t + g(t,x(t-r,\vec{z}))dW(t), \quad \text{a.e. on } (0,T)\times D, \\ x(t,\vec{z}) = \phi(t,z), \quad -r \le t \le 0, \text{ a.e. on } D, \\ \frac{\partial x(0,\vec{z})}{\partial t} = x_1(\vec{z}), \quad \text{a.e. on } D, \\ x(t,\vec{z}) = 0, \quad \text{a.e. on } (0,T)\times\partial\Omega, \end{gathered} \tag{5.1}$$

where $\vec{z} = \langle z_1,\dots,z_n\rangle \in D$, $x_1 \in L_0^2(\Omega;L^2(D))$, $\phi \in C_r(L^2(D))$, and W is an $L^2(D)$-valued Wiener process, while $f_i : [0,T]\times\mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ and $g : [0,T]\times\mathbb{R} \to \mathrm{BL}(L^2(D))$ satisfy (H_{f_i}) and (H_g) (or (H1) and (H2)). Further, we assume that

(H3) $a_{jk} : D \to \mathbb{R}$ $(1 \le j,k \le n)$ and $C : D \to \mathbb{R}$ are bounded, measurable mappings.

Let $H = K = L^2(D)$ and define $A : H \to H$ by

$$Ax(t,\cdot) = \sum_{j=1}^n\sum_{k=1}^n \frac{\partial}{\partial z_j}\left(a_{jk}(\cdot)\frac{\partial x(t,\cdot)}{\partial z_j}\right) + C(\cdot)x(t,\cdot). \tag{5.2}$$

It is known that A is a uniformly elliptic, densely defined, symmetric, selfadjoint operator which generates a strongly continuous cosine family on H (see [10]). As such, (5.1) can be written in the abstract form (1.1) in H so that an application of Theorem 3.2 (or Theorem 4.1) yields the following result.

Theorem 5.1. *If either* (H_{f_i}) *and* (H_g), *or* $(H1)$ *and* $(H2)$, *are satisfied, then (5.1) has a unique mild solution* $x \in X_{T,2}$.

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