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Research Article

Mild Solutions of Neutral Stochastic Partial Functional Differential Equations

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This paper studies the existence and uniqueness of a mild solution for a neutral stochastic partial functional differential equation using a local Lipschitz condition. When the neutral term is zero and even in the deterministic special case, the result obtained here appears to be new. An example is included to illustrate the theory.

1. Introduction

In this paper, a neutral stochastic partial functional differential equation is considered in a real separable Hilbert space of the form

$$d\left[x(t)+f(t,x_t)\right]=\left[Ax(t)+a(t,x_t)\right]dt+b(t,x_t)dw(t),\quad t>0, \tag{1.1}$$

$$x(t)=\varphi(t),\quad t\in[-r,0]\ (0\le r<\infty), \tag{1.2}$$

where $x_t(s)=x(t+s)$, $-r\le s\le 0$.

A study of such class of (1.1) was initiated recently in Govindan [1]. Equation (1.1) when $f\equiv 0$ has been well-studied; see Taniguchi et al. [2], Govindan [3], and the references cited therein. Using a global Lipschitz condition on the nonlinear terms $f(t,u)$, $a(t,u)$ and $b(t,u)$, existence and stability problems were addressed in [1] and also in [4] but with a different iteration procedure. For a motivation and details, we refer to [1].

In this note, our goal is to study the existence and uniqueness of the mild solution of (1.1) using a local Lipschitz condition. Even in the special case (when $f=0$), the result obtained here appears to be new. Taniguchi et al. [2] discussed this special case in $C([-r,T],L^p(\Omega,X))$ when $p>2$.

Adimy and Ezzinbi [5] studied the following neutral partial functional differential equation:

$$\frac{dD(x_t)}{dt} = AD(x_t) + F(x_t), \quad t > 0, \tag{1.3}$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \tag{1.4}$$

where D is a bounded linear operator from $C := C([-r,0], X)$ into X (a Hilbert space) defined by $D(\phi) = \phi(0) - D_0(\phi)$, for $\phi \in C$, where the operator D_0 is given by

$$D_0(\phi) = \int_{-r}^{0} d\kappa(\theta)\phi(\theta), \quad \phi \in C, \tag{1.5}$$

and $\kappa : [-r, 0] \to L(X)$ is of bounded variation and nonatomic at zero. Equation (1.3) has been well-studied, see Wu [6]. The authors from [5] developed a basic theory on such equations and studied an existence result, among others, using a global Lipschitz condition on $F(u)$. For details, we refer to [5].

Clearly, (1.1) (when $b = 0$) is more general than (1.3). Our main result (when $b = 0$) proved using a local Lipschitz condition, therefore, is new in this case as well. We also refer to Ezzinbi et al. [7] for yet another class of deterministic neutral equations which is again a particular case of (1.1) wherein the authors study existence and regularity problems using global Lipschitz conditions.

The paper is organized as follows. In Section 2, we consider the formulation of the problem under study from [1] and the references therein. Section 3 is devoted to the main result on the existence and uniqueness of a mild solution of (1.1). An example is given in Section 4.

2. Mathematical Formulation

Let X, Y be real separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators mapping Y into X. For convenience, we will use the same notation $|\cdot|$ to denote the norms in X, Y, and $L(Y, X)$ and use $(\cdot,\cdot)$ to denote innerproduct of X and Y without any confusion. Let $(\Omega, B, P, \{B_t\}_{t\geq 0})$ be a complete probability space with an increasing right continuous family $\{B_t\}_{t\geq 0}$ of complete sub-σ-algebras of B. Let $\beta_n(t)(n = 1, 2, 3, \ldots)$ be a sequence of real-valued standard Brownian motions mutually independent defined on this probability space. Set

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \quad t \geq 0, \tag{2.1}$$

where $\lambda_n \geq 0$ $(n = 1, 2, 3, \ldots)$ are nonnegative real numbers and $\{e_n\}(n = 1, 2, 3, \ldots)$ is a complete orthonormal basis in Y. Let $Q \in L(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$. The above Y-valued stochastic process $w(\mathrm{t})$ is called a Q-Wiener process. Now, we define the stochastic integral of a Y-valued B_t-adapted process $h(t)$ with respect to the Q-Wiener process $w(t)$.

Definition 2.1. Let $h(t)$ be a Y-valued B_t-adapted process such that $E\int_0^t |h(t)|^2 dt < \infty$ for any $t \in [0,\infty)$. Then, we define the real-valued stochastic integral $\int_0^t \langle h(s), dw(s)\rangle$ by

$$\int_0^t \langle h(s), dw(s)\rangle = \sum_{n=1}^{\infty} \int_0^t (h(s), e_n) dw(s) e_n, \tag{2.2}$$

where $w(s)e_n = (w(s), e_n) = \sqrt{\lambda_n}\beta_n(s)$.

Definition 2.2. Let $h(t)$ be an $L(Y,X)$-valued function and let λ be a sequence $\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots\}$. Then we define

$$|h(t)|_\lambda = \left\{ \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n} h(t) e_n \right|^2 \right\}^{1/2}. \tag{2.3}$$

If $|h(t)|_\lambda^2 < \infty$, then $h(t)$ is called λ-Hilbert-Schmidt operator and let $\sigma(\lambda)(Y,X)$ denote the space of all λ-Hilbert-Schmidt operators from Y to X.

Next, we define the X-valued stochastic integral with respect to the Y-valued Q-Wiener process $w(t)$. See [1] and the references therein.

Definition 2.3. Let $\Phi : [0,\infty) \to \sigma(\lambda)(Y,X)$ be a B_t-adapted process satisfying $\int_0^t E|\Phi(s)|_\lambda^2 ds < \infty$. Then we define the X-valued stochastic integral $\int_0^t \Phi(s) dw(s) \in X$ with respect to $w(t)$ by

$$\left(\int_0^t \Phi(s) dw(s), h \right) = \int_0^t \langle \Phi^*(s) h, dw(s)\rangle, \quad h \in X, \tag{2.4}$$

where Φ^* is the adjoint operator of Φ.

A semigroup $\{S(t), t \geq 0\}$ is said to be exponentially stable if there exist positive constants M and a such that $\|S(t)\| \leq M\exp(-at)$, $t \geq 0$, where $\|\cdot\|$ denotes the operator norm in $L(X,X)$. If $M = 1$, the semigroup is said to be a contraction. If $\{S(t), t \geq 0\}$ is an analytic semigroup, see Pazy [8, page 60] with infinitesimal generator A such that $0 \in \rho(A)$ (the resolvent set of A) then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$ as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha))$ is dense in X and the expression

$$\|x\|_\alpha = \left|(-A)^\alpha x\right|, \quad x \in D\left((-A)^\alpha\right) \tag{2.5}$$

defines a norm on $X_\alpha = D((-A)^\alpha)$. Let C be the space of continuous functions $x : [-r, 0] \to X$ with the norm $\|x\|_C = \sup_{-r\leq s\leq 0} |x(s)|$.

For convenience of the reader, we will state the following lemmas that will be used in the sequel.

Lemma 2.4 (see [8]). *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$. If $0 \in \rho(A)$ then,*

(a) $S(t) : X \to X_\alpha$ *for every* $t > 0$ *and* $\alpha \geq 0$.

(b) *For every* $x \in X_\alpha$ *one has*

$$S(t)(-A)^\alpha x = (-A)^\alpha S(t)x. \tag{2.6}$$

(c) *For every* $t > 0$ *the operator* $(-A)^\alpha S(t)$ *is bounded and*

$$\left\|(-A)^\alpha S(t)\right\| \leq \mu_\alpha t^{-\alpha} e^{-at}, \quad a > 0. \tag{2.7}$$

(d) *Let* $0 < \alpha \leq 1$ *and* $x \in X_\alpha$ *then*

$$|S(t)x - x| \leq \gamma_\alpha t^\alpha \left|(-A)^\alpha x\right|. \tag{2.8}$$

Lemma 2.5 (see [4]). *Let $-A$ be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X. Then, for any stochastic process $F : [0, \infty) \to X$ which is strongly measurable with $\int_0^T E|(-A)^\alpha F(t)|^p dt < \infty$, $p \geq 2$ and $0 < T \leq \infty$, the following inequality holds for $0 < t \leq T$:*

$$E\left|\int_0^t (-A)S(t-s)F(s)ds\right|^p \leq k(p, a, \alpha) \int_0^t E\left|(-A)^\alpha F(s)\right|^p ds, \tag{2.9}$$

provided $1/p < \alpha < 1$, *where*

$$k(p, a, \alpha) = M_{1-\alpha}^p \frac{(p-1)^{p\alpha-1}\left[\Gamma((p\alpha-1)/(p-1))\right]^{p-1}}{(pa)^{p\alpha-1}}, \tag{2.10}$$

and $\Gamma(\cdot)$ *is the Gamma function.*

3. Existence and Uniqueness of a Solution

In this section, we establish the existence and uniqueness of a mild solution of (1.1) using local Lipschitz conditions.

We now make (1.1) precise: let $-A : D(-A) \subseteq X \to X$ be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ defined on X. Let the functions $a(t, u)$, $f(t, u)$, and $b(t, u)$ be defined as follows: $a : R^+ \times C \to X$, where $R^+ = [0, \infty)$, $f : R^+ \times C \to X_\alpha$ and $b : R^+ \times C \to L(Y, X)$ are Borel measurable; and for each (t, u) are measurable with respect to the σ-algebra B_t.

Let the following assumptions hold a.s.:

(H1) $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X and that the semigroup is a contraction, (H2) The functions $a(t,u)$ and $b(t,u)$ are continuous and that there exist positive constants $C_i = C_i(T), i = 1,2$ such that

$$|a(t,u) - a(t,v)| \le C_1 \|u - v\|_C,$$
$$|b(t,u) - b(t,v)|_\lambda \le C_2 \|u - v\|_C, \tag{3.1}$$

for all $t \in [0,T]$ and $u, v \in C$.
Under this assumption, we may suppose that there exists a positive constant $C_3 = C_3(T)$ such that

$$|a(t,u)|^2 + |b(t,u)|_\lambda^2 \le C_3^2\left(1 + \|u\|_C^2\right). \tag{3.2}$$

(H3) The function $f(t,u)$ is continuous and that there exists a positive constant $C_4 = C_4(T)$ such that

$$\left\|f(t,u) - f(t,v)\right\|_\alpha \le C_4 \|u - v\|_C, \tag{3.3}$$

for all $t \in [0,T]$ and $u, v \in C$.
Under this assumption, we may suppose that there exists a positive constant $C_5 = C_5(T)$ such that

$$\left\|f(t,u)\right\|_\alpha \le C_5(1 + \|u\|_C). \tag{3.4}$$

(H4) $f(t,u)$ is continuous in the quadratic mean sense:

$$\lim_{(t,u)\to(s,v)} E\left\|f(t,u) - f(s,v)\right\|_\alpha^2 \longrightarrow 0. \tag{3.5}$$

We now introduce the concept of a mild solution of the problem (1.1)-(1.2).

Definition 3.1. An X-valued stochastic process $\{x(t), t \in [-r,T]\}(0 < T < \infty)$ is called a mild solution of the problem (1.1)-(1.2) if

(i) $x(t)$ is B_t-adapted with $\int_0^T |x(t)|^2 dt < \infty$, a.s.,
(ii) $x(t) = \varphi(t), t \in [-r,0]$ a.s.,
(iii) $x(t)$ satisfies the integral equation

$$x(t) = S(t)\left[\varphi(0) + f(0,\varphi)\right] - f(t,x_t) - \int_0^t AS(t-s)f(s,x_s)ds$$
$$+ \int_0^t S(t-s)a(s,x_s)ds + \int_0^t S(t-s)b(s,x_s)dw(s), \quad \text{a.s.}, \ t \in [0,T]. \tag{3.6}$$

Next, assume that $T > 0$ is a fixed time. Let Γ_T be the subspace of all continuous processes x which belong to the space $C([-r,T], L^2(\Omega,X))$ with the norm $\|x\|_{\Gamma_T} < \infty$, where $\|x\|_{\Gamma_T} := \sup_{0\le t\le T}(E\|x_t\|_C^2)^{1/2}$. See [2].

Let the past process $\varphi \in C([-r,\mathrm{T}], L^2(\Omega, X))$, such that $E\|\varphi\|_C^2 < \infty$.

In the rest of the paper, we will restrict α to the interval $1/2 < \alpha < 1$.

Theorem 3.2. *Suppose that the assumptions (H1)–(H4) are satisfied. Then, there exists a time* $0 < t_m = t_{\max} \le \infty$ *such that* (1.1) *has a unique mild solution. Further, if* $t_m < \infty$*, then* $\lim_{t\uparrow t_m} E|x(t)|^2 = \infty$.

Define a map G on Γ_T:

$$\begin{aligned}(Gx)(t) &= S(t)\left[\varphi(0) + f(0,\varphi)\right] - f(t,x_t) - \int_0^t AS(t-s)f(s,x_s)ds \\ &\quad + \int_0^t S(t-s)a(s,x_s)ds + \int_0^t S(t-s)b(s,x_s)dw(s), \quad t > 0, \\ (Gx)(t) &= \varphi(t), \quad t \in [-r,0].\end{aligned} \tag{3.7}$$

To prove this theorem, we need some lemmas. The first one establishes the continuity of the map G defined on $[0,T]$ and taking values in $L^2(\Omega, X)$ thereby showing that it is a well-defined map in the space $C([-r,T], L^2(\Omega, X))$. The second one then shows that G maps Γ_T into itself. See [2, 9, 10].

Lemma 3.3. *For arbitrary* $x \in \Gamma_T$, $(Gx)(t)$ *is continuous on* $[0,T]$ *in the* $L^2(\Omega, X)$ *sense.*

Proof. Let $0 \le t \le T$, $h > 0$ and $t + h \in [0,T]$. Consider

$$\begin{aligned}(Gx)(t) &= S(t)\varphi(0) + S(t)f(0,\varphi) - f(t,x_t) + \int_0^t (-A)S(t-s)f(s,x_s)ds \\ &\quad + \int_0^t S(t-s)a(s,x_s)ds + \int_0^t S(t-s)b(s,x_s)dw(s) \\ &=: \sum_{j=1}^{6} I_j(t).\end{aligned} \tag{3.8}$$

Hence,

$$E|(Gx)(t+h) - (Gx)(t)|^2 \le 6\sum_{j=1}^{6} E\left|I_j(t+h) - I_j(t)\right|^2. \tag{3.9}$$

By virtue of closedness of $(-A)^\alpha$ and the fact that $S(t)$ commutes with $(-A)^\alpha$ on X_α, we have by Lemma 2.4 and the assumption (H3) that

$$\begin{aligned}E|I_1(t+h) - I_1(t)|^2 &= E\left|(S(t+h) - S(t))\varphi(0)\right|^2 \\ &= E\left|(S(h) - I)S(t)\varphi(0)\right|^2 \\ &\le \gamma_\alpha^2 \mu_\alpha^2 t^{-2\alpha} h^{2\alpha} e^{-2at} E\|\varphi\|_C^2,\end{aligned}$$

$$
\begin{aligned}
E|I_2(t+h)-I_2(t)|^2 &= E\left|(S(h)-I)S(t)(-A)^{-\alpha}(-A)^{\alpha}f(0,\varphi)\right|^2 \\
&\le 2\gamma_\alpha^2 h^{2\alpha}\mu_\alpha^2 t^{-2\alpha}e^{-2at}C_5^2\|(-A)^{-\alpha}\|^2\left(1+E\|\varphi\|_C^2\right), \\
E|I_3(t+h)-I_3(t)|^2 &\le \|(-A)^{-\alpha}\|^2 E\left|(-A)^{\alpha}f(t+h,x_{t+h})-(-A)^{\alpha}f(t,x_t)\right|^2.
\end{aligned}
\tag{3.10}
$$

Next, using Lemmas 2.4 and 2.5 and assumption (H3), we obtain

$$
\begin{aligned}
E|I_4(t+h)-I_4(t)|^2 &= E\left|\int_0^t(-A)S(t-s)(S(h)-I)f(s,x_s)ds\right. \\
&\qquad\left.+\int_t^{t+h}(-A)S(t+h-s)f(s,x_s)ds\right|^2 \\
&\le 2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\left\{\int_0^t E\left|(-A)^{\alpha}(S(h)-I)f(s,x_s)\right|^2ds\right. \\
&\qquad\left.+\int_t^{t+h}E\left|(-A)^{\alpha}S(h)f(s,x_s)\right|^2ds\right\} \\
&\le 2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\left\{\gamma_\beta^2h^{2\beta}\int_0^t E\left|(-A)^{\beta+\alpha}f(s,x_s)\right|^2ds\right. \\
&\qquad\left.+e^{-2ah}\int_t^{t+h}E\left|(-A)^{\alpha}f(s,x_s)\right|^2ds\right\} \\
&\le 4C_5^2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\left[\gamma_\beta^2h^{2\beta}t+he^{-2ah}\right]\left(1+\sup_{0\le t\le T}E\|x_t\|_C^2\right),
\end{aligned}
\tag{3.11}
$$

where we chose $\beta > 0$, such that $1/2 < \alpha + \beta < 1$. Next, by assumption (H2):

$$
\begin{aligned}
E|I_5(t+h)-I_5(t)|^2 &= E\left|\int_0^t(S(h)-I)S(t-s)a(s,x_s)ds+\int_t^{t+h}S(t+h-s)a(s,x_s)ds\right|^2 \\
&\le 2\gamma_\alpha^2h^{2\alpha}\mu_\alpha^2\int_0^t(t-s)^{-2\alpha}e^{-2a(t-s)}E|a(s,x_s)|^2ds \\
&\quad+2\int_t^{t+h}e^{-2a(t+h-s)}E|a(s,x_s)|^2ds.
\end{aligned}
\tag{3.12}
$$

Hence, using similar arguments as in Ahmed [9, Theorem 6.3.2, pages 206–209], one can find constants K_1 and $K_2 > 0$ depending on the parameters $\alpha, \beta, \gamma_\alpha, \mu_\alpha, C_3, T$ such that

$$
E|I_5(t+h)-I_5(t)|^2 \le 2\gamma_\alpha^2\mu_\alpha^2\left[K_1h^{2\alpha}+K_2h\right]\left(1+\sup_{0\le t\le T}E\|x_t\|_C^2\right),
\tag{3.13}
$$

for $t \in [0,T]$.

Lastly, for the stochastic integral term $I_6(t)$, again exploiting Lemma 2.4 and assumption (H2), we obtain

$$E|I_6(t+h)-I_6(t)|^2 = E\left|\int_0^t (S(h)-I)S(t-s)b(s,x_s)dw(s) + \int_t^{t+h} S(t+h-s)b(s,x_s)dw(s)\right|^2 \leq 2\kappa\left\{E\left|\int_0^t (S(h)-I)S(t-s)b(s,x_s)ds\right|^2 + E\left|\int_t^{t+h} S(t+h-s)b(s,x_s)ds\right|^2\right\}, \quad \kappa>0, \tag{3.14}$$

wherein we used Da Prato and Zabczyk [11, Theorem 6.10, page 160] or Lemma 2.4 [1]. Arguing as before, we find constants K_3 and $K_4 > 0$ such that

$$E|I_6(t+h)-I_6(t)|^2 \leq 2\kappa\gamma_\alpha^2\mu_\alpha^2\left[K_3h^{2\alpha}+K_4h\right]\left(1+E\sup_{0\leq t\leq T}\|x_t\|_C^2\right). \tag{3.15}$$

Similar estimates hold for

$$E|(Gx)(t-h)-(Gx)(t)|^2 \quad \text{for } t\geq h>0. \tag{3.16}$$

Thus letting $h \to 0$, thanks to (H4) the desired continuity follows from all the foregoing estimates. □

Lemma 3.4. *G maps Γ_T into itself, that is, $G(\Gamma_T)\subset\Gamma_T$.*

Proof. Let $x\in\Gamma_T$ and assume that $t\geq r$. Then

$$E\|(Gx)_t\|_C^2 \leq 6\left\{E\sup_{-r\leq\theta\leq 0}\left|S(t+\theta)\varphi(0)\right|^2 + E\sup_{-r\leq\theta\leq 0}\left|S(t+\theta)f(0,\varphi)\right|^2 + E\sup_{-r\leq\theta\leq 0}\left|f(t+\theta,x_{t+\theta})\right|^2 + E\sup_{-r\leq\theta\leq 0}\left|\int_0^{t+\theta}(-A)S(t+\theta-s)f(s,x_s)ds\right|^2 + E\sup_{-r\leq\theta\leq 0}\left|\int_0^{t+\theta}S(t+\theta-s)a(s,x_s)ds\right|^2 + E\sup_{-r\leq\theta\leq 0}\left|\int_0^{t+\theta}S(t+\theta-s)b(s,x_s)dw(s)\right|^2\right\} =: \sum_{i=1}^{6}J_i. \tag{3.17}$$

We now estimate each term on the R.H.S. of (3.17):

$$J_1 \leq 6e^{2ar}e^{-2at}E\|\varphi\|_C^2. \tag{3.18}$$

By Lemma 2.4 and assumption (H3), we have

$$\begin{aligned} J2 &\leq 6\|(-A)^{-\alpha}\|^2 e^{-2at}C_5^2\left(1+E\|\varphi\|_C^2\right), \\ J_3 &\leq 6\|(-A)^{-\alpha}\|^2 C_5^2\left(1+\sup_{0\leq t\leq T}E\|x_t\|_C^2\right). \end{aligned} \tag{3.19}$$

Next, using assumption (H3) and Lemma 2.5, we have

$$\begin{aligned} J_4 &\leq 6\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\int_0^t E\left|(-A)^{\alpha}f(s,x_s)\right|^2 ds \\ &\leq 12TC_5^2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\left(1+\sup_{0\leq t\leq T}E\|x_t\|_C^2\right), \end{aligned} \tag{3.20}$$

and by assumption (H2) and Lemma 2.4, we get

$$\begin{aligned} J_5 &\leq 6E\sup_{-r\leq\theta\leq 0}\int_0^{t+\theta} e^{-2a(t+\theta-s)}|a(s,x_s)|^2 ds \\ &\leq 6TC_3^2\left(1+\sup_{0\leq t\leq T}E\|x_t\|_C^2\right). \end{aligned} \tag{3.21}$$

Lastly, by [11, Theorem 6.10] and assumption (H2), we have

$$\begin{aligned} J_6 &\leq 6\kappa E\int_0^t |b(s,x_s)|^2 ds \\ &\leq 6\kappa TC_3^2\left(1+\sup_{0\leq t\leq T}E\|x_t\|_C^2\right). \end{aligned} \tag{3.22}$$

Consequently, $E\|(Gx)_t\|_C^2 < \infty$, implying that G maps Γ_T into itself.

Next, assume that $t+\theta \leq 0$, that is, $t+\theta \in [t-r, 0]$. Then $(Gx)_t(\theta) = \varphi(\theta)$ and therefore

$$E\sup_{-r\leq\theta\leq -t}|(Gx)_t(\theta)|^2 = E\sup_{t-r\leq s\leq 0}\left|\varphi(s)\right|^2 = E\|\varphi\|_C^2 < \infty. \tag{3.23}$$

□

Proof of Theorem 3.2. Let $x, y \in \Gamma_T$ and assume that $t \geq r$. Then for any fixed $t \in [0, T]$, we have

$$\begin{aligned}
E\left\|(Gx)_t - (Gy)_t\right\|_C^2 &= E \sup_{-r\leq\theta\leq 0} \left|(Gx)(t+\theta) - (Gy)(t+\theta)\right|^2 \\
&\leq 4\Bigg\{ E \sup_{-r\leq\theta\leq 0} \left|f(t+\theta, x_{t+\theta}) - f(t+\theta, y_{t+\theta})\right|^2 \\
&\quad + E \sup_{-r\leq\theta\leq 0} \left|\int_0^{t+\theta} (-A)S(t+\theta-s)\left[f(s,x_s) - f(s,y_s)\right]ds\right|^2 \\
&\quad + E \sup_{-r\leq\theta\leq 0} \left|\int_0^{t+\theta} S(t+\theta-s)\left[a(s,x_s) - a(s,y_s)\right]ds\right|^2 \\
&\quad + E \sup_{-r\leq\theta\leq 0} \left|\int_0^{t+\theta} S(t+\theta-s)\left[b(s,x_s) - b(s,y_s)\right]dw(s)\right|^2 \Bigg\} \\
&\leq 4C_4^2\left\|(-A)^{-\alpha}\right\|^2 \sup_{0\leq t\leq T} E\left\|x_t - y_t\right\|_C^2 \\
&\quad + 4\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}} T C_4^2 \sup_{0\leq t\leq T} E\left\|x_t - y_t\right\|_C^2 \\
&\quad + 4TC_1^2 \sup_{0\leq t\leq T} E\left\|x_t - y_t\right\|_C^2 \\
&\quad + 4\kappa T C_2^2 \sup_{0\leq t\leq T} E\left\|x_t - y_t\right\|_C^2.
\end{aligned} \tag{3.24}$$

Now choosing $T > 0$ sufficiently small, we can find a positive number $K(T) \in [0, 1)$ such that

$$\left\|Gx - Gy\right\|_{\Gamma_T} \leq K(T)\left\|x - y\right\|_{\Gamma_T}, \tag{3.25}$$

for any $x, y \in \Gamma_T$. Hence, by the Banach fixed point theorem, G has a unique fixed point $x \in \Gamma_T$ and this fixed point is the unique mild solution of (1.1) on $[0, T]$. Next, we continue the solution for $t \geq T$, see Ahmed [9] and Govindan [10]. For notational convenience, set $T = t_1$. For $t \in [t_1, t_2]$, where $t_1 < t_2$, we say that a function $\hat{x}(t)$ is a continuation of $x(t)$ to the interval

$[t_1, t_2]$ if

(a) $\hat{x} \in C([-r, t_2],\ L^2(\Omega, X))$, and

(b) $\hat{x}(t) = S(t-t_1)[\varphi(t_1)+f(t_1,\varphi)]-f(t,\hat{x}_t)-\int_{t_1}^{t} AS(t-s)f(s,\hat{x}_s)ds+\int_{t_1}^{t} S(t-s)a(s,\hat{x}_s)ds+\int_{t_1}^{t} S(t-s)b(s,\hat{x}_s)dw(s)$, a.s.

The terminology mild continuation applied to $\hat{x}(t)$ is justified by the observation that if we define a new function $v(t)$ on $[0, t_2]$ by setting

$$v(t) = \begin{cases} x(t) & \text{if } 0 \le t \le t_1, \\ \hat{x}(t) & \text{if } t_1 \le t \le t_2, \end{cases} \tag{3.26}$$

and $v(t) = \varphi(t), t \in [-r, 0]$, then $v(t)$ is a mild solution of (1.1) on $[0, t_2]$. The existence and uniqueness of the mild continuation $\hat{x}(t)$ is demonstrated exactly as above with only some minor changes. The details are therefore omitted. Repeating this procedure, one continues the solution till the time $t_m = t_{\max}$ where $[0, t_m]$ is the maximum interval of the existence and uniqueness of a solution. For t_m finite, $\lim E|x(t)|^2 = \infty$ as $t \uparrow t_m$. If not, then there exists a sequence $\{\tau_n\}$ converging to t_m and a finite positive number δ such that $E|x(\tau_n)|^2 \le \delta$ for all n. Taking n sufficiently large so that τ_n is infinitesimally close to t_m, one can use the previous arguments to extend the solution beyond t_m, which is a contradiction.

Next, assume that $t + \theta \le 0$. In that case,

$$E\left\|(Gx)_t - (Gy)_t\right\|_C^2 = 0. \tag{3.27}$$

This completes the proof. □

4. An Example

Consider the neutral stochastic partial functional differential equation with finite delays r_1, r_2, and r_3 $(r > r_i \ge 0, i = 1, 2, 3)$:

$$d\left[z(t,x) + \frac{\ell_3(t)}{\left\|(-A)^{3/4}\right\|}\int_{-r_3}^{0} z(t+u,x)du\right] = \left[\frac{\partial^2}{\partial x^2}z(t,x) + \ell_1(t)\int_{-r_1}^{0} z(t+u,x)du\right]dt + \ell_2(t)z(t-r_2,x)d\beta(t), \quad t > 0, \tag{4.1}$$

$$\begin{gathered} \ell_i : R^+ \longrightarrow R^+, \quad i = 1,2,3; \qquad z(t,0) = z(t,\pi) = 0, \quad t > 0, \\ z(s,x) = \varphi(s,x), \quad \varphi(\cdot,x) \in C \quad \text{a.s.}, \\ \varphi(s,\cdot) \in L^2[0,\pi], \quad -r \le s \le 0,\ 0 \le x \le \pi, \end{gathered} \tag{4.2}$$

where $\beta(t)$ is a standard one-dimensional Wiener process, $\ell_i(t)$, $i = 1, 2, 3$ are continuous functions and $E\|\varphi\|_C^2 < \infty$.

Take $X = L^2[0,\pi]$, $Y = R^1$. Define $-A : X \to X$ by $-A = \partial^2/\partial x^2$ with domain $D(-A) = \{w \in X : w, \partial w/\partial x$ are absolutely continuous, $\partial^2 w/\partial x^2 \in X$, $w(0) = w(\pi) = 0\}$. Then

$$-Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(-A), \tag{4.3}$$

where $w_n(x) = \sqrt{2/\pi} \sin nx$, $n = 1, 2, 3, \ldots$, is the orthonormal set of eigenvectors of $-A$.

It is well known that $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$ in X and is given by

$$S(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} (w, w_n) w_n, \quad w \in X, \tag{4.4}$$

that satisfies $\|S(t)\| \leq \exp(-\pi^2 t), t \geq 0$, and hence is a contraction semigroup.

Define now

$$\begin{aligned} f(t, z_t) &= \frac{\ell_3(t)}{\left\|(-A)^{3/4}\right\|} \int_{-r_3}^{0} z(t+u, x)\,du, \\ a(t, z_t) &= \ell_1(t) \int_{-r_1}^{0} z(t+u, x)\,du, \\ b(t, z_t) &= \ell_2(t) z(t - r_2, x). \end{aligned} \tag{4.5}$$

Next,

$$\begin{aligned} \|f(t, z_t)\|_{3/4} &= \frac{\ell_3(t)}{\left\|(-A)^{3/4}\right\|} \left|(-A)^{3/4} \int_{-r_3}^{0} z(t+u, x)\,du\right| \\ &\leq \ell_3(T) r_3 \|z\|_C, \quad \text{a.s.} \end{aligned} \tag{4.6}$$

This shows that $f : R^+ \times C \to D((-A)^{3/4})$ with $C_4(T) = \ell_3(T) r_3$. Similarly, $a : R^+ \times C \to X$ and $b : R^+ \times C \to L(R, X)$. Thus, (4.1) can be expressed as (1.1) with $-A$, f, a and b as defined above. Hence, there exists a unique mild solution by Theorem 3.1.

The existence results from [1, 4] are not applicable to (4.1); and the one from [5] is also not applicable to the deterministic case of (4.1) as they all employ global Lipschitz conditions.

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