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WEIGHTED NORMS AND VOLTERRA INTEGRAL EQUATIONS IN L^p SPACES[1]

JAROSLAW KWAPISZ
Institute of Mathematics, Warsaw University
00-901 Warsaw, PKiN, Poland

ABSTRACT

A new simple proof of existence and uniqueness of solutions of the Volterra integral equation in Lebesque spaces is given. It is shown that the weighted norm technique and the Banach contraction mapping principle can be applied (as in the case of continuous functions space).

Key words: Nonlinear Volterra equations, Banach contraction principle.

AMS (MOS) subject classification: 45D05.

Consider the integral equation:

$$x(t) = f(t) + \int_0^t F(t,s,x(s))ds, \tag{1}$$

where $\int_0^t$ is understood as iterated integral $\int_0^{t_1} \ldots \int_0^{t_n}$ and:

(*i*) $f \in L^p(J, \mathbb{R}^k)$ i.e. is vector-valued L^p function on n-dimensional cube $J: = \{t \in \mathbb{R}^n: 0 \leq t \leq a\}, a \leq +\infty, p > 1$, $F: T \times \mathbb{R}^k \to \mathbb{R}^k$ is measurable, $T: = \{(t,s) \in J \times J: t \leq s\}$.

Assume that:

(*ii*) the function $t \to \int_0^t F(t,s,f(s))ds$ belongs to $L^p(J, \mathbb{R}^k)$;

(*iii*) $|F(t,s,x) - F(t,s,y)| \leq L(t,s)|x-y|, x,y \in \mathbb{R}^n, (t,s) \in T$, where L is measurable function for which:

$$M(t): = \left(\int_0^t L(t,s)^q ds\right)^{p/q},\ t \in J,\ 1/p + 1/q = 1,$$

exists and is integrable over J.

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By the successive approximation, it can be proved that under these assumptions there is a unique solution $x^* \in L^p(J, \mathbb{R}^k)$ of (1). The question whether the result can be obtained by direct application of the Banach contraction principle for the case $p = 2$ was discussed by Beesack [1]. To obtain the result in this way he tried to employ in $L^p(J, \mathbb{R}^k)$ a weighted norm of the form:

$$\| x \| : = (\int_0^a | x(t) |^p G(t) dt)^{1/p}, \tag{2}$$

with some positive weight G determined by L. Unfortunately, his approach was not successful because the resulting inequality for $H = 1/G$:

$$\int_0^s L(t,s)^2 H(t) dt \leq \rho H(s), \text{ with some } \rho \in (0,1), \tag{3}$$

in general has no positive, bounded away from zero solution. For this reason Beesack [2] claimed: "... this technique fails in the L^2-case ...". On the other hand a positive answer to our question was given in [3] by the use of the norm:

$$\| x \| : = sup\{\omega^{-1}(t) \int_0^t | x(s) |^p ds : t \in J\}, \tag{4}$$

where ω is found as a solution of the inequality:

$$\int_0^t M(s)\omega(s) ds \leq \rho\omega(t), \rho \in (0,1). \tag{5}$$

The aim of the present note is to show, in contrary to Bessack's conclusion, that there always exists a lot of weight functions G positive and continuous on J such that the operator ϕ defined by right-hand side of (1) is a contraction with respect to the norm (2). Obviously for such G this norm is equivalent to the standard norm in $L^p(J, \mathbb{R}^k)$.

We have:

Theorem: *Suppose that conditions (i), (ii), (iii) are satisfied. Then for any $\rho \in (0,1)$ there exists positive continuous function $G: J \to \mathbb{R}$ such that ϕ is a contraction with respect to the norm $\| \cdot \|$ defined by (2) i.e.:*

$$\| \phi x - \phi y \| \leq \rho \| x - y \| \text{ for every } x, y \in L^p(J, \mathbb{R}^k). \tag{6}$$

Proof. Take a positive continuous function $G: J \to \mathbb{R}$ and $\rho \in (0,1)$. For $x, y \in L^p(J, \mathbb{R}^k)$ by the Lipschitz condition, the Hölder inequality and the Fubini theorem one obtains:

$$\| \phi x - \phi y \|^p =$$

$$= \int_0^a | \int_0^t [F(t,s,x(s)) - F(t,s,y(s))] ds |^p G(t) dt \leq$$

$$\leq \int_0^a [\int_0^t L(t,s) | x(s) - y(s) | ds]^p G(t) dt \leq$$

$$\leq \int_0^a [\int_0^t | x(s) - y(s) |^p ds]^{p\ 1/p} [\int_0^t L(t,u)^q du]^{p/q} G(t) dt \leq$$

$$\leq \int_0^a [\int_0^t | x(s) - y(s) |^p M(t) G(t) ds] dt =$$

$$= \int_0^a | x(s) - y(s) |^p [\int_s^a M(t) G(t) dt] ds.$$

It is clear that so as to fulfill inequality (6), one should require that:

$$\int_s^a M(t) G(t) dt \leq \rho^p G(s), s \in J. \tag{7}$$

One can rewrite this condition in terms of the function $G_1(u) := G(a-u)$ as:

$$\int_0^s N(u) G_1(u) du \leq G_1(s), s \in J, \tag{8}$$

where $N(u) := \rho^{-p} M(a-u)$ is positive integrable function on J. To conclude the proof, one has to show that inequality (8) has positive and continuous solutions. It is easy to check that function

$$G_1(s) := exp(\int_0^s N(u) du) \tag{9}$$

is a solution of (8). (This fact is also a consequence of Lemma 1 [1], [2]).

Observe that for any positive continuous $c: J \to \mathbb{R}$, a solution of equation:

$$G_1(t) = c(t) + \int_0^t N(u)G_1(u)du, t \in J \tag{10}$$

satisfies (8). The unique solution of (10) can be found by Neumann series. The right-hand side operator of (10) is a contraction in $C(J, \mathbb{R})$ with respect to the weighted norm:

$$\| v \|_c = sup\{ | v(t) | exp(\lambda \int_0^t N(u)du) : t \in J\}, \ \lambda > 1.$$

So one can obtain plenty of weight functions for which the assertion of the theorem holds.

Remark. The case $p = 1$ can be treated in almost the same way (replace Hölder inequality with its analog in L^1). The resulting inequality for the corresponding function G_1 has the form of (8) with:

$$M(t) := esssup\{L(t,u) : u \in J\}.$$

There is also no reason for restricting ourselves to $\mathbb{R}^k$ -valued functions. Replacement of $\mathbb{R}^k$ with any Banach space would affect no harm to our considerations.

Remark. Our approach works fairly well also for the case when equation (1) and the condition (iii) are replaced by the general Volterra operator equation:

$$x(t) = [\Phi x](t), \tag*{(1)$'$}$$

$$| [\Phi x](t) - [\Phi y](t) | \leq \int_0^t L(t,s) | x(s) - y(s) | ds, \tag*{$(iii)'$}$$

respectively.

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