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OSCILLATIONS IN LINEAR DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS[1]

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ABSTRACT

A class of linear difference equations with variable coefficients is considered. Sufficient conditions and necessary conditions for the oscillation of the solutions are established. In the special cases where the coefficients are constant or periodic the conditions become both necessary and sufficient.

Key words: Linear difference equation, oscillation.

AMS (MOS) subject classification: Primary 39A10.

1. INTRODUCTION

Recently there has been a great deal of work on the oscillation of solutions of difference equations (see, for example, [2]-[5], [7]-[10], [14]-[16], [18], [19], and [21]).

Within the past two decades, the study of difference equations has acquired a new significance. This comes about, in large part, from the fact that difference equations appear as natural descriptions of observed evolution phenomena as well as in the study of discretization methods for differential equations. Furthermore, the theory of difference equations is rapidly gaining attention because of its use in such fields as numerical analysis, control theory, finite mathematics and computer science; in particular, because of the successful use in recent years of computers to solve difficult problems arising in applications, the connection between the theory of difference equations and computer science has become more important.

For a systematic treatment of the theory of difference equations and its applications to numerical analysis, the reader is referred to the recent book by Lakshmikantham and Trigiante [12]. Chapter 7 of this book is devoted to some applications of difference equations to many fields such as economics, chemistry, population dynamics and queueing theory. Some discrete models in

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population dynamics have appeared in [6].

Consider the linear difference equation

$$A_{n+1} - A_n + \sum_{k=0}^{m} p_k(n) A_{n-\ell_k} = 0, \tag{E}$$

where m is a positive integer, $(p_k(n))_{n \geq 0}$ $(k = 0, 1, \ldots, m)$ are sequences of nonnegative numbers, and $\ell_k (k = 0, 1, \ldots, m)$ are integers such that $0 = \ell_0 < \ell_1 < \ldots < \ell_m$.The sequences $(p_k(n))_{n \geq 0} (k = 1, \ldots, m)$ are (supposed to be) not identically zero.

Let n_0 be a nonnegative integer and set

$$\mathsf{N}_{n_0} = \{n_0, n_0 + 1, \ldots\}.$$

By a solution on N_{n_0} of the difference equation (E) we mean a sequence $(A_n)_{n \geq n_0 - \ell_m}$ which satisfies (E) for all $n \geq n_0$. As usual, a solution of (E) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory. A solution $(A_n)_{n \geq n_0 - \ell_m}$ on N_{n_0} of (E) is called positive if $A_n > 0$ for all $n \geq n_0 - \ell_m$.

Let us consider the linear difference equation with constant coefficients

$$A_{n+1} - A_n + \sum_{k=0}^{m} P_k A_{n-\ell_k} = 0, \tag{E_0}$$

where $P_k (k = 0, 1, \ldots, m)$ are real numbers with $P_0 \geq 0, P_1 > 0, \ldots, P_m > 0$. This equation is a special case of the difference equation (E). For the autonomous difference equation (E_0) the following "if and only if" oscillation criterion is known (see [4], [9], [14]), which is the discrete analogue of a result due to Tramov [22] (see also [1], [11]) concerning the oscillation of first order linear delay differential equations with constant coefficients.

Theorem A: *A necessary and sufficient condition for the oscillation of all solutions of the difference equation* (E_0) *is that its characteristic equation*

$$F_0(\lambda) \equiv \lambda - 1 + \sum_{k=0}^{m} P_k \lambda^{-\ell_k} = 0 \tag{$*$}_0$$

has no roots in $(0, 1)$.

Next, let us consider another special case (which includes the previous one), i.e., the case where the coefficients in (E) are periodic sequences with a common period and the integers $\ell_k (k = 1, \ldots, m)$ are multiples of this period. In this case, the oscillation of the solutions of the difference equation (E) is described by the following result due to Philos [18], which is the discrete analogue of an oscillation criterion of the same author [17] for first order linear delay differential equations with periodic coefficients.

Theorem B: *Assume that $(p_k(n))_{n \geq 0}(k = 0,1,\ldots,m)$ are periodic sequences with a common period L (where L is a positive integer) and that there exist positive integers $v_1,\ldots,v_m$ such that*

$$\ell_1 = v_1 L,\ldots,\ell_m = v_m L.$$

Introduce the equation

$$F(\lambda) \equiv \lambda^L - \prod_{r=0}^{L-1}\Big(1 - \sum_{k=0}^{m} p_k(r)\lambda^{-\ell_k}\Big) = 0, \tag{$*$}$$

which is associated with (E). Then we have:

(*i*) *A necessary condition for the oscillation of all solutions of (E) is that there is no root λ_0 in $(0,1)$ of $(*)$ with the property: If $L > 1$, then*

$$\sum_{k=0}^{m} p_k(r)\lambda_0^{-\ell_k} < 1 \quad (r = 1,\ldots,L-1).$$

(*ii*) *A sufficient condition for the oscillation of all solutions of (E) is that $(*)$ has no roots in $(0,1)$.*

It is easy to see that Theorem A can be obtained from Theorem B for $L = 1$.

Our purpose in this paper is to examine the oscillation of the solutions of the general difference equation (E) in which the coefficients are variable (and not necessarily periodic). More precisely, our aim is to establish sufficient conditions for the oscillation of all solutions of (E) and also conditions under which (E) has at least one nonoscillatory solution. In the special case of the difference equation (E_0) in which the coefficients are constant, our results lead to Theorem A. Also, Theorem B can be obtained from the results of this paper by applying them to the special case where the coefficients are periodic with a common period and $\ell_k(k = 1,\ldots,m)$ are multiples of this period. The application of our results to the above special cases of constant or periodic coefficients will be presented in Section 5. The results of the present paper are motivated by Theorems A and B as well as by the recent results of Philos [20] concerning the oscillation of first order delay differential equations with variable coefficients (see also [13] for similar results).

2. STATEMENT OF THE MAIN RESULTS

Our main results are Theorems 1 and 2 below. Theorem 1 provides sufficient conditions for the oscillation of all solutions of the difference equation (E). Conditions under which (E) has at least one nonoscillatory solution are established by Theorem 2.

Theorem 1: *Let L and $v_k(k=1,\ldots,m)$ be positive integers such that*

$$\ell_1 = v_1 L, \ldots, \ell_m = v_m L.$$

Suppose that

$$\lim_{n\to\infty} \inf \sum_{i=0}^{\ell_1} \sum_{k=1}^{m} p_k(i+n-\ell_1) > 0. \tag{H_1}$$

Suppose also that, for some integer $\nu_0 \geq \ell_m$,

$$\sup_{n \geq \nu_0} \max_{1 \leq k \leq m} \max_{1 \leq s \leq v_k} \prod_{r=(s-1)L}^{sL-1} \Big(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\Big) < 1. \tag{C_0}$$

Moreover, assume that, for every $\lambda \in (0,1)$, there exists an integer $\nu \geq \ell_m$ such that

$$\lambda^L - \sup_{n \geq \nu_0} \max_{1 \leq k \leq m} \max_{1 \leq s \leq v_k} \prod_{r=(s-1)L}^{sL-1} \Big(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\lambda^{-\ell_j}\Big) > 0. \tag{C_1}$$

Then all solutions of the difference equation (E) are oscillatory.

Theorem 2: *Assume that there exists a $\lambda \in (0,1)$ and an integer $n_0 \geq 1+\ell_m$ such that*

$$\sum_{k=0}^{m} p_k(n)\lambda^{-\ell_k} < 1 \text{ for every } n \geq 0, \tag{H}$$

$$\sum_{i=0}^{\ell_1 - 1} \sum_{k=1}^{m} p_k(n+i) > 0 \text{ for all } n \geq n_0 \tag{H_2}$$

and

$$\sup_{n \geq n_0} \sum_{k=0}^{m} p_k(n) \Big\{ -\lambda^{-\ell_k} + \Big[\prod_{r=0}^{\ell_k - 1} \Big(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\lambda^{-\ell_j}\Big) \Big]^{-1} \Big\} \leq 0. \tag{C_2}$$

Then the difference equation (E) has a positive solution $(A_n)_{n \geq n_0 - \ell_m}$ on $\mathbb{N}_{n_0}$ with $\lim_{n\to\infty} A_n = 0$ and such that

$$A_n \leq \prod_{r=0}^{n-1} \Big(1 - \sum_{k=0}^{m} p_k(r)\lambda^{-\ell_k}\Big) \text{ for every } n \geq n_0 - \ell_m.$$

<u>**Note:**</u> In condition (C_2), we have used the convention $\prod_{\rho}^{\sigma} = 1$ when $\sigma < \rho$. This convention will also be used in the proof of Theorem 2.

3. PROOF OF THEOREM 1

Assume, for the sake of contradiction, that the difference equation (E) has a nonoscillatory solution $(A_n)_{n \geq n_0 - \ell_m}$ on $\mathbb{N}_{n_0}$, where n_0 is a nonnegative integer. As the negative of a solution of (E) is also a solution of the same equation, we may (and do) assume that $(A_n)_{n \geq n_0 - \ell_m}$ is eventually positive. Furthermore, without loss of generality, we can suppose that $A_n > 0$ for all $n \geq n_0 - \ell_m$. Then from (E) it follows that $A_{n+1} - A_n \leq 0$ for $n \geq n_0$ and so the sequence $(A_n)_{n \geq n_0}$ is decreasing.

For any $\lambda \in (0,1)$, we define

$$c_\lambda(n) = 1 - \sum_{k=0}^{m} p_k(n)\lambda^{-\ell_k} \text{ for } n \geq 0.$$

Furthermore, we consider the set

$$\Lambda = \{\lambda \in (0,1): A_{n+1} - c_\lambda(n)A_n \leq 0 \text{ for all large } n\}.$$

The set Λ is nonempty. In fact, by taking into account the decreasing character of the sequence $(A_n)_{n \geq n_0}$, from (E) we obtain for $n \geq n_0 + \ell_m$

$$0 = A_{n+1} - A_n + \sum_{j=0}^{m} p_j(n)A_{n-\ell_j} \geq A_{n+1} - A_n + \sum_{j=0}^{m} p_j(n)A_n$$

and consequently

$$A_{n+1} - \Big(1 - \sum_{j=0}^{m} p_j(n)\Big)A_n \leq 0 \text{ for every } n \geq n_0 + \ell_m. \tag{1}$$

This in particular implies that

$$1 - \sum_{j=0}^{m} p_j(n) > 0 \text{ for every } n \geq n_0 + \ell_m. \tag{2}$$

Set

$$\alpha = [\, \sup_{n \geq \nu_0} \ \max_{1 \leq k \leq m} \ \max_{1 \leq s \leq v_k} \ \prod_{r=(s-1)L}^{sL-1} \Big(1 - \sum_{j=0}^{m} p_j(r + n - \ell_k)\Big)]^{1/L}.$$

Because of (2) and assumption (C_0), we have $0 < \alpha < 1$. If $i \in \{1, 2, \ldots, m\}$, then from (1) we obtain for every integer n with $n \geq \max\{\nu_0, n_0 + 2\ell_m\}$

$$\frac{A_{n-\ell_i}}{A_n} \geq [\prod_{r=n-\ell_i}^{n-1} \Big(1 - \sum_{j=0}^{m} p_j(r)\Big)]^{-1} = [\prod_{r=0}^{\ell_i - 1}\Big(1 - \sum_{j=0}^{m} p_j(r + n - \ell_i)\Big)]^{-1}$$

$$= [\prod_{s=1}^{v_i} \ \prod_{r=(s-1)L}^{sL-1} \Big(1 - \sum_{j=0}^{m} p_j(r + n - \ell_i)\Big)]^{-1}$$

$$\geq [\ \max_{1 \leq s \leq v_i} \prod_{r=(s-1)L}^{sL-1} \Big(1 - \sum_{j=0}^{m} p_j(r+n-\ell_i)\Big)\ ]^{-v_i}$$

$$\geq [\ \sup_{n \geq v_0} \max_{1 \leq k \leq m} \max_{1 \leq s \leq v_k} \prod_{r=(s-1)L}^{sL-1} \Big(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\Big)\ ]^{-v_i}$$

$$= (\alpha^L)^{-v_i} = \alpha^{-v_i L} = \alpha^{-\ell_i}.$$

Therefore,

$$A_{n-\ell_i} \geq \alpha^{-\ell_i} A_n \text{ for } n \geq \max\{\nu_0, n_0 + 2\ell_m\} \quad (i = 0, 1, \ldots, m).$$

(The last inequality is obvious when $i = 0$.) Thus, (E) gives for every $n \geq \max\{\nu_0, n_0 + 2\ell_m\}$

$$0 = A_{n+1} - A_n + \sum_{i=0}^{m} p_i(n) A_{n-\ell_i} \geq A_{n+1} - \Big(1 - \sum_{i=0}^{m} p_i(n)\alpha^{-\ell_i}\Big) A_n$$

$$= A_{n+1} - c_\alpha(n) A_n.$$

This means that $\alpha \in \Lambda$ and so $\Lambda \neq \emptyset$. We can immediately see that, if $\lambda_1 \in \Lambda$, then every number λ_2 with $\lambda_1 < \lambda_2 < 1$ also belongs to Λ. So, Λ is a subinterval of the interval $(0,1)$ and $\sup\Lambda = 1$.

Next, we will establish that $\inf\Lambda > 0$. By assumption (H_1), there exist a constant $\beta > 0$ and an integer $n_1 \geq \ell_1$ so that

$$\sum_{i=n-\ell_1}^{n} \sum_{k=1}^{m} p_k(i) = \sum_{i=0}^{\ell_1} \sum_{k=1}^{m} p_k(i+n-\ell_1) \geq \beta \text{ for } n \geq n_1.$$

Hence, for any $n \geq n_1$, we can choose an integer $n^* \equiv n^*(n)$ with $n - \ell_1 \leq n^* \leq n$ such that

$$\sum_{i=n-\ell_1}^{n^*} \sum_{k=1}^{m} p_k(i) \geq \frac{\beta}{2} \text{ and } \sum_{i=n^*}^{n} \sum_{k=1}^{m} p_k(i) \geq \frac{\beta}{2}. \tag{3}$$

Since the sequence $(A_n)_{n \geq n_0}$ is decreasing, from (E) it follows that

$$A_n - A_{n+1} \geq [\ \sum_{k=1}^{m} p_k(n)\ ] A_{n-\ell_1} \text{ for all } n \geq n_0 + \ell_m.$$

Thus, by using again the fact that $(A_n)_{n \geq n_0}$ is a decreasing sequence, we obtain for $n \geq n_0 + \ell_m + \ell_1$

$$A_{n^*} > A_{n^*} - A_{n+1} = \sum_{i=n^*}^{n} (A_i - A_{i+1})$$

$$\geq \sum_{i=n^*}^{n} [\sum_{k=1}^{m} p_k(i)]A_{i-\ell_1} \geq [\sum_{i=n^*}^{n} \sum_{k=1}^{m} p_k(i)]A_{n-\ell_1}.$$

Therefore, in view of (3), we derive

$$A_{n^*} > \frac{\beta}{2} A_{n-\ell_1} \text{ for every } n \geq max\{n_1, n_0 + \ell_m + \ell_1\}. \tag{4}$$

Furthermore, consider an arbitrary element λ of Λ. Then there exists an integer $n_\lambda \geq n_0$ such that

$$A_{n+1} - \left(1 - \sum_{k=0}^{m} p_k(n)\lambda^{-\ell_k}\right)A_n \leq 0 \text{ for } n \geq n_\lambda.$$

This gives

$$A_n - A_{n+1} \geq [\sum_{k=1}^{m} p_k(n)]\lambda^{-\ell_1} A_n \text{ for all } n \geq n_\lambda.$$

Hence, by the decreasing nature of the sequence $(A_n)_{n \geq n_0}$, we obtain for $n \geq n_\lambda + \ell_1$

$$A_{n-\ell_1} > A_{n-\ell_1} - A_{n^*+1} = \sum_{i=n-\ell_1}^{n^*} \left(A_i - A_{i+1}\right)$$

$$\geq \left\{ \sum_{i=n-\ell_1}^{n^*} [\sum_{k=1}^{m} p_k(i)]A_i \right\} \lambda^{-\ell_1} \geq [\sum_{i=n-\ell_1}^{n^*} \sum_{k=1}^{m} p_k(i)]A_{n^*}\lambda^{-\ell_1}.$$

So, by (3), we have

$$A_{n-\ell_1} > \frac{\beta}{2}\lambda^{-\ell_1} A_{n^*} \text{ for all } n \geq max\{n_1, n_\lambda + \ell_1\}. \tag{5}$$

Combining (4) and (5), we conclude that

$$1 > \left(\frac{\beta}{2}\right)^2 \lambda^{-\ell_1} \text{ or } \lambda > \left(\frac{\beta}{2}\right)^{2/\ell_1}.$$

Thus, as λ is an arbitrary element of Λ, we see that the positive number $\left(\frac{\beta}{2}\right)^{2/\ell_1}$ is a lower bound of Λ and so $inf\Lambda > 0$.

Now, we set $inf\Lambda_0 = \lambda_0 \in (0,1)$ and we consider an arbitrary integer $\nu \geq \ell_m$. Moreover, we consider an arbitrary number θ in the interval $(\lambda_0, 1)$ and we put $\gamma = \lambda_0/\theta$. Then $\lambda_0 < \gamma < 1$ and consequently $\gamma \in \Lambda$. Hence, there exists an integer $n_\gamma \geq n_0$ so that

$$A_{n+1} - c_\gamma(n)A_n \leq 0 \text{ for all } n \geq n_\gamma,$$

i.e.

$$A_{n+1} - \left(1 - \sum_{j=0}^{m} p_j(n)\gamma^{-\ell_j}\right)A_n \leq 0 \text{ for } n \geq n_\gamma. \tag{6}$$

This in particular implies that

$$1-\sum_{j=0}^{m} p_j(n)\gamma^{-\ell_j} > 0 \text{ for every } n \geq n_\gamma. \tag{7}$$

Define

$$q = [\ \sup_{n \geq \nu}\ \max_{1 \leq k \leq m}\ \max_{1 \leq s \leq v_k} \prod_{r=(s-1)L}^{sL-1} \Big(1-\sum_{j=0}^{m} p_j(r+n-\ell_k)\gamma^{-\ell_j}\Big)\ ]^{1/L}.$$

By taking into account (7), we can see that q is a positive number. Furthermore, if $i \in \{1,2,\ldots,m\}$, then from (6) it follows that for every $n \geq max\{\nu, n_\gamma + \ell_m\}$

$$\frac{A_{n-\ell_i}}{A_n} \geq [\ \prod_{r=n-\ell_i}^{n-1}\Big(1-\sum_{j=0}^{m} p_j(r)\gamma^{-\ell_j}\Big)\ ]^{-1} = [\ \prod_{r=0}^{\ell_i-1}\Big(1-\sum_{j=0}^{m} p_j(r+n-\ell_i)\gamma^{-\ell_i}\Big)\ ]^{-1}$$

$$= [\ \prod_{s=1}^{v_i}\ \prod_{r=(s-1)L}^{sL-1}\Big(1-\sum_{j=0}^{m} p_j(r+n-\ell_i)\gamma^{-\ell_j}\Big)\ ]^{-1}$$

$$\geq [\ \max_{1 \leq s \leq v_i}\ \prod_{r=(s-1)L}^{sL-1}\Big(1-\sum_{j=0}^{m} p_j(r+n-\ell_i)\gamma^{-\ell_j}\Big)\ ]^{-v_i}$$

$$\geq [\ \sup_{n \geq \nu}\ \max_{1 \leq k \leq m}\ \max_{1 \leq s \leq v_k}\ \prod_{r=(s-1)L}^{sL-1}\Big(1-\sum_{j=0}^{m} p_j(r+n-\ell_k)\gamma^{-\ell_j}\Big)\ ]^{-v_i}$$

$$= (q^L)^{-v_i} = q^{-v_iL} = q^{-\ell_i}.$$

So, we have

$$A_{n-\ell_i} \geq q^{-\ell_i}A_n \text{ for } n \geq max\{\nu, n_\gamma + \ell_m\}\ \ (i = 0,1,\ldots,m). \tag{8}$$

(For $i = 0$ the last inequality is obvious.) We now claim that

$$q \geq \lambda_0. \tag{9}$$

If $q \geq 1$, then (9) is obvious. So, let us assume that $0 < q < 1$. By using (8), from (E) we obtain for $n \geq max\{\nu, n_\gamma + \ell_m\}$

$$0 = A_{n+1} - A_n + \sum_{i=0}^{m} p_i(n)A_{n-\ell_i} \geq A_{n+1} - \Big(1-\sum_{i=0}^{m} p_i(n)q^{-\ell_i}\Big)A_n$$

$$A_{n+1} - (1-\sum_{i=0}^{m} p_i(n)q^{\ell_i})A_n$$

$$= A_{n+1} - c_q(n)A_n,$$

which means that $q \in \Lambda$ and consequently $q \geq \lambda_0$. Thus, (9) is true. Finally, (9) gives

$$\sup_{n \geq \nu} \max_{1 \leq k \leq m} \max_{1 \leq s \leq v_k} \prod_{r=(s-1)L}^{sL-1} \left(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\gamma^{-\ell_j}\right) \geq \lambda_0^L,$$

namely

$$\sup_{n \geq \nu} \max_{1 \leq k \leq m} \max_{1 \leq s \leq v_k} \prod_{r=(s-1)L}^{sL-1} \left(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\lambda_0^{-\ell_j \theta^{\ell_j}}\right) \geq \lambda_0^L.$$

Therefore, as $\theta \to 1-0$, we obtain

$$\sup_{n \geq \nu} \max_{1 \leq k \leq m} \max_{1 \leq s \leq v_k} \prod_{r=(s-1)L}^{sL-1} \left(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\lambda_0^{-\ell_j}\right) \geq \lambda_0^L. \tag{10}$$

We have thus proved that there exists a number $\lambda_0 \in (0,1)$ such that, for every integer $\nu \geq \ell_m$, (10) holds. This contradicts the assumption that, for every $\lambda \in (0,1)$, there exists an integer $\nu \geq \ell_m$ so that (C_1) is satisfied. The proof of Theorem 1 is complete.

4. PROOF OF THEOREM 2

To prove Theorem 2 we will make use of the following result, which is interesting in its own right.

Lemma: *Let $(B_n)_{n \geq n_0 - \ell_m}$ be a positive solution on $\mathbb{N}_{n_0}$, where n_0 is a nonnegative integer, of the difference inequality*

$$B_{n+1} - B_n + \sum_{k=0}^{m} p_k(n) B_{n-\ell_k} \leq 0. \tag{I}$$

Moreover, assume that condition (H_2) is satisfied.

Then there exists a positive solution $(A_n)_{n \geq n_0 - \ell_m}$ on $\mathbb{N}_{n_0}$ of the difference equation (E) with $\lim_{n\to\infty} A_n = 0$ and such that

$$A_n \leq B_n \text{ for every } n \geq n_0 - \ell_m.$$

Note: If n_0 is a nonnegative integer, by a solution on $\mathbb{N}_{n_0}$ of the difference inequality (I) we mean a sequence $(B_n)_{n \geq n_0 - \ell_m}$ which satisfies (I) for all $n \geq n_0$. A solution $(B_n)_{n \geq n_0 - \ell_m}$ on $\mathbb{N}_{n_0}$ of (I) is said to be positive if $B_n > 0$ for every $n \geq n_0 - \ell_m$.

Proof of the Lemma: The method of proof is similar to that of Theorem 3 in [10] concerning a special case (see also Lemma in Section 3 of [16]).

From (I) we obtain for $\nu \geq n \geq n_0$

$$B_n > -(B_{\nu+1} - B_n) = -\sum_{j=n}^{\nu}(B_{j+1} - B_j) \geq \sum_{j=n}^{\nu} \sum_{k=0}^{m} p_k(j) B_{j-\ell_k}$$

and consequently

$$B_n \geq \sum_{j=n}^{\infty} \sum_{k=0}^{m} p_k(j) B_{j-\ell_k} \text{ for all } n \geq n_0. \tag{11}$$

Consider the space $\mathcal{A}$ of all sequences $(A_n)_{n \geq n_0 - \ell_m}$ satisfying

$$A_n = B_n \text{ for } n_0 - \ell_m \leq n < n_0, \text{ and } 0 \leq A_n \leq B_n \text{ for } n \geq n_0.$$

For any sequence $(A_n)_{n \geq n_0 - \ell_m}$ in $\mathcal{A}$, we define

$$SA_n = \begin{cases} B_n, \text{ if } n_0 - \ell_m \leq n < n_0 \\ \sum_{j=n}^{\infty} \sum_{k=0}^{m} p_k(j) A_{j-\ell_k}, \text{ if } n \geq n_0 \end{cases}$$

and, by using (11), we see that $(SA_n)_{n \geq n_0 - \ell_m}$ is a sequence in the space $\mathcal{A}$. Thus, the above formula defines an operator $S: \mathcal{A} \to \mathcal{A}$. We can easily see that this operator is monotonic in the following sense: If $(A_n^1)_{n \geq n_0 - \ell_m}$ and $(A_n^2)_{n \geq n_0 - \ell_m}$ are two sequences in $\mathcal{A}$ such that

$$A_n^1 \leq A_n^2 \text{ for every } n \geq n_0 - \ell_m,$$

then we also have

$$SA_n^1 \leq SA_n^2 \text{ for all } n \geq n_0 - \ell_m.$$

Now, we introduce the sequence $\left((A_n^r)_{n \geq n_0 - \ell_m}\right)_{r \geq 0}$ of points in $\mathcal{A}$ which is defined by

$$A_n^0 = B_n \text{ for } n \geq n_0 - \ell_m$$

and

$$A_n^r = SA_n^{r-1} \text{ for } n \geq n_0 - \ell_m \quad (r = 1, 2, \ldots).$$

It follows easily that

$$A_n^0 \geq A_n^1 \geq A_n^2 \geq \ldots \text{ for } n \geq n_0 - \ell_m$$

and so we can define

$$A_n = \lim_{r\to\infty} A_n^r \text{ for } n \geq n_0 - \ell_m.$$

Then we observe that

$$0 \leq A_n \leq B_n \text{ for all } n \geq n_0 - \ell_m.$$

Moreover, we can obtain

$$A_n = \sum_{j=n}^{\infty} \sum_{k=0}^{m} p_k(j) A_{j-\ell_k} \text{ for } n \geq n_0.$$

This gives

$$\lim_{n\to\infty} A_n = 0$$

and

$$A_{n+1} - A_n = -\sum_{k=0}^{m} p_k(n) A_{n-\ell_k} \text{ for all } n \geq n_0.$$

The last equation means that the sequence $(A_n)_{n \geq n_0 - \ell_m}$ is a solution on $\mathbb{N}_{n_0}$ of the difference equation (E). We see that $A_n = B_n > 0$ for $n_0 - \ell_m \leq n < n_0$. It remains to show that A_n is also positive for $n \geq n_0$. Assume, for the sake of contradiction, that $N \geq n_0$ is the first zero of $(A_n)_{n \geq n_0 - \ell_m}$. That is,

$$A_n > 0 \text{ for } n_0 - \ell_m \leq n < N, \text{ and } A_N = 0.$$

Then, by using the hypothesis (H_2), we obtain

$$0 \leq A_{N+\ell_1} = A_{N+\ell_1} - A_N = \sum_{\mu=N}^{N+\ell_1-1} (A_{\mu+1} - A_\mu)$$

$$= -\sum_{\mu=N}^{N+\ell_1-1} \sum_{k=0}^{m} p_k(\mu) A_{\mu-\ell_k} \leq -\sum_{\mu=N}^{N+\ell_1-1} \sum_{k=1}^{m} p_k(\mu) A_{\mu-\ell_k}$$

$$\leq -\Big(\min_{N \leq \mu \leq N+\ell_1-1} \min_{1 \leq k \leq m} A_{\mu-\ell_k} \Big) \sum_{\mu=N}^{N+\ell_1-1} \sum_{k=1}^{m} p_k(\mu)$$

$$\leq -\Big(\min_{N-\ell_m \leq \rho \leq N-1} A_\rho \Big) \sum_{i=0}^{\ell_1-1} \sum_{k=1}^{m} p_k(N+i) < 0.$$

This is a contradiction and hence the proof of our lemma is complete.

We are now ready to give the proof of Theorem 2. Define

$$B_n = \prod_{r=0}^{n-1} c_\lambda(r) \text{ for } n \geq 1,$$

where

$$c_\lambda(n) = 1 - \sum_{k=0}^{m} p_k(n)\lambda^{-\ell_k} \text{ for } n \geq 0.$$

By assumption (H), we can see that

$$B_n > 0 \text{ for every } n \geq 1.$$

Next, by using condition (C_2), we obtain for every $n \geq n_0$

$$\begin{aligned}
&B_{n+1} - B_n + \sum_{k=0}^{m} p_k(n) B_{n-\ell_k} = \\
&= [c_\lambda(n) - 1] \prod_{r=0}^{n-1} c_\lambda(r) + \sum_{k=0}^{m} p_k(n) \prod_{r=0}^{n-\ell_k-1} c_\lambda(r) \\
&= \left\{ c_\lambda(n) - 1 + \sum_{k=0}^{m} p_k(n) \Big[\prod_{r=n-\ell_k}^{n-1} c_\lambda(r) \Big]^{-1} \right\} \prod_{r=0}^{n-1} c_\lambda(r) \\
&= \left\{ - \sum_{k=0}^{m} p_k(n)\lambda^{-\ell_k} + \sum_{k=0}^{m} p_k(n) \Big[\prod_{r=n-\ell_k}^{n-1} \Big(1 - \sum_{j=0}^{m} p_j(r)\lambda^{-\ell_j}\Big) \Big]^{-1} \right\} B_n \\
&= B_n \sum_{k=0}^{m} p_k(n) \left\{ -\lambda^{-\ell_k} + \Big[\prod_{r=n-\ell_k}^{n-1} \Big(1 - \sum_{j=0}^{m} p_j(r)\lambda^{-\ell_j}\Big) \Big]^{-1} \right\} \\
&= B_n \sum_{k=0}^{m} p_k(n) \left\{ -\lambda^{-\ell_k} + \Big[\prod_{r=0}^{\ell_k-1} \Big(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\lambda^{-\ell_j}\Big) \Big]^{-1} \right\} \\
&\leq B_n \sup_{n \geq n_0} \sum_{k=0}^{m} p_k(n) \left\{ -\lambda^{-\ell_k} + \Big[\prod_{r=0}^{\ell_k-1} \Big(1 - \sum_{j=0}^{m} p_j(r+n-\ell_k)\lambda^{-\ell_j}\Big) \Big]^{-1} \right\} \\
&\leq 0.
\end{aligned}$$

Thus, the sequence $(B_n)_{n \geq n_0 - \ell_m}$ is a positive solution on $\mathbb{N}_{n_0}$ of the difference inequality (I). So, it suffices to apply our lemma to complete the proof of Theorem 2.

5. APPLICATION OF THE MAIN RESULTS TO THE SPECIAL CASES OF CONSTANT OR PERIODIC COEFFICIENTS

In this section, we shall apply our results to the special cases where the coefficients in the difference equation (E) are constant or periodic. More precisely, we will show that Theorems A and

B can be obtained from Theorems 1 and 2.

The case of constant coefficients: Consider the special case where the coefficients are constant, i.e., the case of the autonomous difference equation (E_0).

Assume first that the characteristic equation $(*)_0$ of (E_0) has no roots in $(0,1)$. Since $F_0(1) = \sum_{k=0}^{m} P_k > 0$, we must have

$$\lambda - 1 + \sum_{k=0}^{m} P_k \lambda^{-\ell_k} > 0 \text{ for all } \lambda \in (0,1). \tag{12}$$

In order to apply here Theorem 1, we choose $L = 1, v_k = \ell_k (k = 1,\ldots,m)$, and $\nu_0 = \nu = \ell_m$. Then we can immediately see that conditions (H_1) and (C_0) hold by themselves. Moreover, one can verify that, for every $\lambda \in (0,1)$, condition (C_1) is satisfied because of (12). Thus, Theorem 1 can be applied to guarantee the oscillation of all solutions of (E_0).

Suppose, conversely, that $(*)_0$ has a root $\lambda \in (0,1)$. Then $\sum_{k=0}^{m} P_k \lambda^{-\ell_k} = 1 - \lambda < 1$ and so (H) is true. Choose $n_0 = 1 + \ell_m$. We can see that (H_2) holds by itself. Furthermore, assumption (H_2) takes the form

$$\sum_{k=0}^{m} P_k[-\lambda^{-\ell_k} + \Big(1 - \sum_{j=0}^{m} P_j \lambda^{-\ell_j}\Big)^{-\ell_k}] \leq 0.$$

This inequality is satisfied, since $\lambda \in (0,1)$ is a root of $(*)_0$. So, Theorem 2 ensures that (E_0) has a nonoscillatory solution.

The case of periodic coefficients: Assume that the sequences $(p_k(n))_{n \geq 0}$ $(k = 0,1,\ldots,m)$ are periodic with a common period L (where L is a positive integer) and that there exist positive integers v_k $(k = 1,\ldots,m)$ such that

$$\ell_1 = v_1 L, \ldots, \ell_m = v_m L.$$

Let there exist a root $\lambda \in (0,1)$ of the equation $(*)$ with the property: If $L > 1$, then

$$\sum_{k=0}^{m} p_k(r) \lambda^{-\ell_k} < 1 \quad (r = 1,\ldots,L-1).$$

Then from $(*)$ it follows that

$$\sum_{k=0}^{m} p_k(r) \lambda^{-\ell_k} < 1 \text{ for all } r \in \{0,1,\ldots,L-1\}.$$

This means that (H) holds, since the sequences $(p_k(n))_{n \geq 0}$ $(k = 0,1,\ldots,m)$ are L-periodic. We now choose $n_0 = 1 + \ell_m$. Since the sequences $(p_k(n))_{n \geq 0}$ $(k = 1,\ldots,m)$ are L-periodic and not identically zero and $\ell_1 = v_1 L$, we can easily verify that condition (H_2) is also satisfied.

Furthermore, by taking into account the fact that $(p_j(n))_{n\geq 0}$ $(j=0,1,\ldots,m)$ are L-periodic sequences and that $\ell_k = v_k L$ $(k=1,\ldots,m)$, we obtain for any $k \in \{1,\ldots,m\}$ and for every $n \geq n_0$

$$\prod_{r=0}^{\ell_k-1}\left(1-\sum_{j=0}^{m} p_j(r+n-\ell_k)\lambda^{-\ell_j}\right)=\prod_{r=n-\ell_k}^{n-1}\left(1-\sum_{j=0}^{m} p_j(r)\lambda^{-\ell_j}\right)$$

$$=\prod_{r=0}^{\ell_k-1}\left(1-\sum_{j=0}^{m} p_j(r)\lambda^{-\ell_j}\right)=\left[\prod_{r=0}^{L-1}\left(1-\sum_{j=0}^{m} p_j(r)\lambda^{-\ell_j}\right)\right]^{v_k}.$$

Thus, condition (C_2) becomes

$$\sup_{n\geq n_0}\sum_{k=0}^{m} p_k(n)\left\{-(\lambda^L)^{-v_k}+\left[\prod_{r=0}^{L-1}\left(1-\sum_{j=0}^{m} p_j(r)\lambda^{-\ell_j}\right)\right]^{-v_k}\right\}\leq 0,$$

which is true since $\lambda \in (0,1)$ is a root of $(*)$. So, by Theorem 2, the difference equation (E) admits a nonoscillatory solution.

Now, let us suppose that the equation $(*)$ has no roots in the interval $(0,1)$. Moreover, assume for the sake of contradiction that (E) has at least one nonoscillatory solution. Then, as in the proof of Theorem 1 (cf. inequality (2)), we can verify that

$$1-\sum_{k=0}^{m} p_k(n) > 0 \text{ for all large } n.$$

This means that

$$1-\sum_{k=0}^{m} p_k(r) > 0 \text{ for } r = 0,1,\ldots,L-1.$$

Furthermore, since the sequences $(p_k(n))_{n\geq 0}$ $(k=1,\ldots,m)$ are L-periodic and not identically zero, we can see that

$$\prod_{r=0}^{L-1}\left(1-\sum_{k=0}^{m} p_k(r)\right)<1. \tag{13}$$

This gives $F(1)>0$ and so, as $F(\lambda)=0$ has no roots in $(0,1)$, we always have

$$\lambda^L-\prod_{r=0}^{L-1}\left(1-\sum_{k=0}^{m} p_k(r)\lambda^{-\ell_k}\right)>0 \text{ for all } \lambda\in(0,1). \tag{14}$$

We will apply Theorem 1. Since the sequences $(p_k(n))_{n\geq 0}$ $(k=1,\ldots,m)$ are L-periodic and not identically zero, we have for every $n \geq \ell_1$

$$\sum_{i=0}^{\ell_1}\sum_{k=1}^{m} p_k(i+n-\ell_1)\geq\sum_{i=0}^{\ell_1-1}\sum_{k=1}^{m} p_k(i+n-\ell_1)=\sum_{\mu=0}^{\ell_1-1}\sum_{k=1}^{m} p_k(\mu)>0$$

and hence (H_1) is fulfilled. Next, we choose $\nu_0 = \nu = \ell_m$. By using again the L-periodicity of the sequences $(p_j(n))_{n \geq 0}$ $(j = 0, 1, \ldots, m)$ and the fact that $\ell_k = v_k L$, we obtain for all $n \geq \nu_0$, $k \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, v_k\}$

$$\prod_{r=(s-1)L}^{sL-1}\Big(1 - \sum_{j=0}^{m} p_j(r + n - \ell_k)\Big) = \prod_{r=0}^{L-1}\Big(1 - \sum_{j=0}^{m} p_j(r + n - \ell_k)\Big)$$

$$= \prod_{r=n-\ell_k}^{n-\ell_k+L-1}\Big(1 - \sum_{j=0}^{m} p_j(r)\Big) = \prod_{r=0}^{L-1}\Big(1 - \sum_{j=0}^{m} p_j(r)\Big).$$

So, assumption (C_0) holds because of (13). In a similar way, we can see that for every $\lambda \in (0,1)$

$$\prod_{r=(s-1)L}^{sL-1}\Big(1 - \sum_{j=0}^{m} p_j(r + n - \ell_k)\lambda^{-\ell_j}\Big) = \prod_{r=0}^{L-1}\Big(1 - \sum_{j=0}^{m} p_j(r)\lambda^{-\ell_j}\Big).$$

for $n \geq \nu$, $k \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, v_k\}$. Thus, in view of (14), condition (C_1) is satisfied for each $\lambda \in (0,1)$. So, we can apply Theorem 1 to conclude that (E) has no nonoscillatory solutions, a contradiction.

6. A BRIEF DISCUSSION

Concerning Theorem 1 it remains an open question to the authors if an analogous oscillation result can be obtained without the restriction that there exist positive integers L and v_k $(k = 1, \ldots, m)$ so that $\ell_1 = v_1 L, \ldots, \ell_m = v_m L$. Furthermore, it is an open problem to extend the results of this paper to the more general case of difference equations of the form

$$A_{n+1} - A_n + \sum_{k=0}^{m} p_k(n) A_{n - \ell_k(n)} = 0,$$

where $\ell_0(n) = 0$ for $n \geq 0$, and $(\ell_k(n))_{n \geq 0}$ $(k = 1, \ldots, m)$ are sequences of positive integers such that

$$\lim_{n \to \infty}(n - \ell_k(n)) = \infty \quad (k = 1, \ldots, m).$$

Equation (E) is also called a delay difference equation. It will be the subject of a future work to present an analogous investigation with that of this paper for the advanced difference equation

$$A_{n+1} - A_n - \sum_{k=0}^{m} p_k(n) A_{n + \ell_k} = 0$$

(see [15] for the case of advanced difference equations with constant coefficients).

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