

RANDOM APPROXIMATIONS AND RANDOM FIXED POINT THEOREMS

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ABSTRACT

Some results regarding random best approximation are proved. As applications, several random fixed point theorems are obtained.

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Random approximations and random fixed point theorems are stochastic generalizations of (classical) approximations and fixed point theorems. Random fixed point theory has received much attention for the last two decades, since the publication of the paper by Bharucha-Reid [3]. On the other hand, random approximation has recently received further attention after the papers by Sehgal and Waters [11], Sehgal and Singh [10], Papageorgiou [9], Lin [8] and Beg and Shahzad [1]. The aim of this paper is to prove some theorems on random best approximations. As applications, some stochastic fixed point theorems are also derived.

Throughout this paper, (Ω, \mathcal{A}) denotes a measurable space and X be a Banach space. Let 2^X be the family of all subsets of X , $CD(X)$ all nonempty closed subsets of X , $CB(X)$ all nonempty bounded closed subsets of X and $K(X)$ all nonempty compact subsets of X . A mapping $F: \Omega \rightarrow 2^X$ is called *measurable* if for any open subset C of X , $F^{-1}(C) = \{\omega \in \Omega: F(\omega) \cap C \neq \emptyset\} \in \mathcal{A}$. This type of measurability is usually called a *weak measurability* (cf. Himmelberg [6]). However, in this paper we always use this type of measurability, thus we omit the term "weak" for simplicity. Let S be a nonempty subset of X , a map $f: \Omega \times S \rightarrow X$ is called a *random operator* if for any $x \in S$, $f(\cdot, x)$ is measurable. Similarly, a mapping $F: \Omega \times S \rightarrow CD(X)$ is a *random multivalued operator* if for every $x \in S$, $F(\cdot, x)$ is measurable. A measurable mapping $\zeta: \Omega \rightarrow S$ is called a *measurable selector* of the measurable mapping $F: \Omega \rightarrow CD(X)$ if

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$\zeta(\omega) \in F(\omega)$ for each $\omega \in \Omega$. A measurable mapping $\zeta: \Omega \rightarrow S$ is called a *random fixed point* of a random operator $f: \Omega \times S \rightarrow X$ (or $F: \Omega \times S \rightarrow CD(X)$) if for every $\omega \in \Omega$, $\zeta(\omega) = f(\omega, \zeta(\omega))$ (or $\zeta(\omega) \in F(\omega, \zeta(\omega))$). A map $f: S \rightarrow X$ is called *nonexpansive* if $\|f(x) - f(y)\| \leq \|x - y\|$ for $x, y \in S$; f is a *Banach operator* of type k on S , if there exists a constant k , $0 \leq k < 1$, such that for $x \in S$, $\|f(x) - f^2(x)\| \leq k \|x - f(x)\|$; f is *compact*, if for any bounded subset S of X , $\overline{f(S)}$ (that is closure of $f(S)$) is compact. A random operator $f: \Omega \times S \rightarrow X$ is *continuous* (*nonexpansive*, *Banach operator*, *compact* etc.) if for each $\omega \in \Omega$, $f(\omega, \cdot)$ is *continuous* (*nonexpansive*, *Banach operator*, *compact* etc.). For $x \in X$, define

$$d(x, S) = \inf_{y \in S} \|x - y\|$$

and

$$P_S(x) = \{y \in S: \|x - y\| = d(x, S)\}.$$

Here $P_S(x)$ denotes the set of all *best approximations* of y out of S and $P_S(x)$, is called the *metric projection* on S . In the case when S is a *Chebyshev set*, $P_S(x)$ is a point map $X \rightarrow S$, is called a *proximity map*. The set $P_S(x)$ is closed and convex for any subspace (or convex subset) S of X . If a mapping $f: X \rightarrow X$ leaves a subset M of X invariant, then a restriction of f to M will be denoted by the symbol f/M .

The following lemma is a consequence of [2, Theorem 2.1].

Lemma 1: *A continuous Banach random operator $f: \Omega \times S \rightarrow S$, where S is a separable closed subset of a Banach space, has a random fixed point.*

Remark 2: Lemma 1 remains true if S is a separable, closed subset of a normed space and $\overline{f(\omega, S)}$ is compact, for any $\omega \in \Omega$.

The following is a random fixed point theorem for a nonexpansive random operator. For a corresponding fixed point theorem, we refer to Habinaik [5].

Theorem 3: *If $f: \Omega \times S \rightarrow S$ is nonexpansive random operator, where S is a separable closed and starshaped subset of a normed space E , and $\overline{f(\omega, S)}$ is compact, for any $\omega \in \Omega$, then f has a random fixed point.*

Proof: Let p be the starcenter of S and let $\{k_n\}$ be a sequence of positive numbers less than 1 and converging to 1. For each n , define the random operator

$$f_n(\omega, x) = (1 - k_n)p + k_n f(\omega, x).$$

For each n and $\omega \in \Omega$, $f_n(\omega, \cdot)$ maps S into itself because $f(\omega, \cdot): S \rightarrow S$ and S is starshaped. Moreover, each f_n is a continuous Banach operator:

$$\begin{aligned} & \|f_n(\omega, x) - f_n^2(\omega, x)\| \\ &= \|(1 - k_n)p + k_n f(\omega, x) - (1 - k_n)p - k_n f(\omega, (1 - k_n)p + k_n f(\omega, x))\| \\ &= k_n \|f(\omega, x) - f(\omega, (1 - k_n)p + k_n f(\omega, x))\| \\ &\leq k_n \|x - f_n(\omega, x)\|, \text{ for each } \omega \in \Omega. \end{aligned}$$

Since $\overline{f(\omega, S)}$ compact, $\overline{f_n(\omega, S)}$ is compact too. By Remark 2, f_n has a random fixed point ξ_n . For each n , define $G_n: \Omega \rightarrow K(S)$ as $G_n(\omega) = \overline{\{\xi_i(\omega): i \geq n\}}$. Define $G: \Omega \rightarrow K(S)$ by $G(\omega)$

$= \bigcap_{n=1}^{\infty} G_n(\omega)$. Then, G is measurable [5] and has a measurable selector ξ . Thus, ξ is a desired random fixed point of f . Fix any $\omega \in \Omega$. Since $\overline{f(\omega, S)}$ is compact, $\{f(\omega, \xi_n(\omega))\}$ has a subsequence $\{f(\omega, \xi_{n_m}(\omega))\}$ convergent to $\xi(\omega)$; since $k_{n_m} \rightarrow 1$, the subsequence $\xi_{n_m}(\omega) = (1 - k_{n_m})p + k_{n_m} f(\omega, \xi_{n_m}(\omega))$ converges to $\xi(\omega)$. By continuity of f , $f(\omega, \xi_{n_m}(\omega))$ converges to $f(\omega, \xi(\omega))$. Thus $f(\omega, \xi(\omega)) = \xi(\omega)$.

Theorem 4: *Let E be a normed space. If $f: \Omega \times E \rightarrow E$ is a nonexpansive random operator with deterministic fixed point x , leaving a subspace M of E invariant, $f(\omega, \cdot)/M$ is compact and $P_M(x)$ is separable, then the point x has a best random approximation $\xi: \Omega \rightarrow M$ which is also a random fixed point of f .*

Proof: The set $P_M(x)$ is invariant with respect to the operator $f(\omega, \cdot)$ for each $\omega \in \Omega$. Indeed, from [12] it follows that there exists $b \in P_M(x)$, that is, $d(x, M) = \|x - b\|$. Since f is nonexpansive and $x = f(\omega, x)$, we have

$$d(x, M) = \|x - b\| \geq \|x - f(\omega, b)\|.$$

So, $\|x - f(\omega, b)\| \leq d(x, M)$ for each $b \in P_M(x)$. Therefore, $P_M(x)$ is $f(\omega, \cdot)$ -invariant, that is, $f(\omega, P_M(x)) \subset P_M(x)$ for each $\omega \in \Omega$. The set $P_M(x)$ is closed and starshaped since it is convex; since $P_M(x)$ is a bounded subset of M and $f(\omega, \cdot)/M$ is compact, the set $\overline{f(\omega, P_M(x))}$ is compact, for any $\omega \in \Omega$.

Theorem 3 further implies that f has a random fixed point in subset $P_M(x)$. Hence the random operator f has a random fixed point in the set of best approximation $P_M(x)$.

Theorem 5: *Let $f: \Omega \times X \rightarrow X$ be a nonexpansive random operator with deterministic fixed point x , leaving convex subset S of X invariant and $f(\omega, \cdot)/S$ compact. If $P_S(x)$ is nonempty and separable, then point x has a best random approximation $\xi: \Omega \rightarrow S$ which is also a random fixed point of f .*

Proof: Proof is exactly same as that of Theorem 4; instead of Theorem 3, we use Itoh [7, Corollary 2.2] to show that f has a random fixed point.

Theorem 6: *Let S be a nonempty separable closed convex subset of a Hilbert space H and $f: \Omega \times S \rightarrow H$ a nonexpansive random operator with $f(\omega, S)$ compact for any $\omega \in \Omega$. Then there exists a measurable map $\xi: \Omega \rightarrow S$ such that*

$$\|\xi(\omega) - f(\omega, \xi(\omega))\| = d(f(\omega, \xi(\omega)), S)$$

for each $\omega \in \Omega$.

Proof: Let $P: H \rightarrow S$ be a proximity map. From [3] we have that, P is nonexpansive in the Hilbert space. Then $P \circ f: \Omega \times S \rightarrow S$ is also a nonexpansive random operator. Now $\overline{P \circ f(\omega, S)}$ is compact since $P \circ f(\omega, S) \subset P \circ \overline{f(\omega, S)}$ and $P \circ \overline{f(\omega, S)}$ is compact, for any $\omega \in \Omega$. By Theorem 3, there exists a measurable map $\xi: \Omega \rightarrow S$ such that $P \circ f(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. Hence,

$$\begin{aligned} \|\xi(\omega) - f(\omega, \xi(\omega))\| &= \|P \circ f(\omega, \xi(\omega)) - f(\omega, \xi(\omega))\| \\ &= d(f(\omega, \xi(\omega)), S), \end{aligned}$$

for each $\omega \in \Omega$.

Theorem 7: *Let S be a nonempty separable closed convex subset of a Hilbert space H , and $f: \Omega \times S \rightarrow H$ be a nonexpansive random operator with $\overline{f(\omega, S)}$ compact for any $\omega \in \Omega$, such that*

for each x on the boundary of S , $\|f(\omega, x) - \zeta_1(\omega)\| \leq \|x - \zeta_1(\omega)\|$ for some measurable map $\zeta_1: \Omega \rightarrow S$. Then f has a random fixed point.

Proof: Theorem 6 implies that there exists a measurable map $\zeta: \Omega \rightarrow S$ such that

$$\|\zeta(\omega) - f(\omega, \zeta(\omega))\| = d(f(\omega, \zeta(\omega)), S).$$

If for any $\omega \in \Omega$, $f(\omega, \zeta(\omega)) \in S$ then $\zeta(\omega) = f(\omega, \zeta(\omega))$. Suppose $f(\omega, \zeta(\omega)) \notin S$. Let P be the proximity map on S , then P is nonexpansive. Therefore $P \circ f: \Omega \times S \rightarrow S$ is nonexpansive random operator. By definition

$$\|f(\omega, \zeta(\omega)) - P \circ f(\omega, \zeta(\omega))\| = d(f(\omega, \zeta(\omega)), S).$$

It implies that $P \circ f(\omega, \zeta(\omega)) = \zeta(\omega)$ for each $\omega \in \Omega$. By hypothesis, there is a measurable map $\zeta_1: \Omega \rightarrow S$ such that $\|f(\omega, \zeta(\omega)) - \zeta_1(\omega)\| \leq \|\zeta(\omega) - \zeta_1(\omega)\|$. Now it suffices to show that $\zeta(\omega) = \zeta_1(\omega)$ for each $\omega \in \Omega$. Otherwise, we have

$$\|\zeta(\omega) - f(\omega, \zeta(\omega))\| < \|f(\omega, \zeta(\omega)) - \zeta_1(\omega)\| \leq \|\zeta(\omega) - \zeta_1(\omega)\|.$$

It follows that there is a measurable map $\gamma: \Omega \rightarrow S$ defined by

$$\gamma(\omega) = \alpha \zeta(\omega) + (1 - \alpha) \zeta_1(\omega),$$

with $0 < |\alpha| < 1$ for which

$$\|\gamma(\omega) - f(\omega, \zeta(\omega))\| < \|\zeta(\omega) - f(\omega, \zeta(\omega))\|.$$

This is a contradiction. Hence $\zeta(\omega) = \zeta_1(\omega)$ for each $\omega \in \Omega$.

Theorem 8: Let S be a nonempty separable closed convex subset of a Hilbert space H . Let $f: \Omega \times S \rightarrow H$ be nonexpansive random operator with $\overline{f(\omega, S)}$ compact for any $\omega \in \Omega$, and assume that for any measurable map $\zeta_1: \Omega \rightarrow \partial S$ (where ∂S denotes the boundary of S), with $\zeta_1(\omega) = P \circ f(\omega, \zeta_1(\omega))$ that is a random fixed point of $P \circ f$, then f has a random fixed point.

Proof: From Theorem 6, there exists measurable map $\zeta: \Omega \rightarrow S$ such that

$$\|\zeta(\omega) - f(\omega, \zeta(\omega))\| = d(f(\omega, \zeta(\omega)), S).$$

If for any $\omega \in \Omega$, $f(\omega, \zeta(\omega)) \in S$ then $f(\omega, \zeta(\omega)) = \zeta(\omega)$. Suppose $f(\omega, \zeta(\omega)) \notin S$. As in Theorem 7 we have $P \circ f(\omega, \zeta(\omega)) = \zeta(\omega)$ for each $\omega \in \Omega$. If $\zeta: \Omega \rightarrow \partial S$, then by the above assumption, $f(\omega, \zeta(\omega)) = \zeta(\omega)$ for each $\omega \in \Omega$. Otherwise, $f(\omega, \zeta(\omega)) \in S$. Thus for $\omega \in \Omega$, $\zeta(\omega) = P \circ f(\omega, \zeta(\omega)) = f(\omega, \zeta(\omega))$.

A random operator $f: \Omega \times S \rightarrow H$ is said to satisfy property (I), if $f(\omega, x) = \alpha(\omega)x$ for some x in ∂S , then $\alpha(\omega) \leq 1$ for each $\omega \in \Omega$. [Here $\alpha: \Omega \rightarrow \mathbb{R}$ is a measurable map.]

Theorem 9: Let S be a separable closed ball centered at the origin and with radius r in a Hilbert space H . Let $f: \Omega \times S \rightarrow H$ be a nonexpansive random operator with compact $\overline{f(\omega, S)}$ for any $\omega \in \Omega$ and satisfying property (I). Then, f has a random fixed point.

Proof: There exists a measurable map $\zeta: \Omega \rightarrow S$ such that

$$\|\zeta(\omega) - f(\omega, \zeta(\omega))\| = d(f(\omega, \zeta(\omega)), S).$$

If for any $\omega \in \Omega$, $f(\omega, \zeta(\omega)) \in S$ then $f(\omega, \zeta(\omega)) = \zeta(\omega)$. Suppose $\|f(\omega, \zeta(\omega))\| > r$. By con-

vexity, $\zeta(\omega)$ is on the boundary of S and $f(\omega, \zeta(\omega)) = \alpha(\omega)\zeta(\omega)$ for some measurable map $\alpha: \Omega \rightarrow (0, \infty)$. It gives for each $\omega \in \Omega$,

$$\alpha(\omega) = \frac{\|\alpha(\omega)\zeta(\omega)\|}{\|\zeta(\omega)\|} = \frac{\|f(\omega, \zeta(\omega))\|}{\|\zeta(\omega)\|} > \frac{r}{r} = 1.$$

Hence, this is a contradiction to property (I).

Theorem 10: *Let S be a nonempty separable closed convex subset of a Hilbert space H and $f: \Omega \times S \rightarrow H$ be a nonexpansive random operator with compact $\overline{f(\omega, S)}$, for any $\omega \in \Omega$. Moreover, f satisfies one of the following conditions:*

- (i) *For each $\omega \in \Omega$, each $x \in S$ with $x \neq f(\omega, x)$, there exists y , dependent on ω and x , in $I_S(x) = \{x + c(z - x) : \text{some } z \in S, c > 0\}$ such that $\|y - f(\omega, x)\| < \|x - f(\omega, x)\|$.*
- (ii) *f is weakly inward (that is, for each $\omega \in \Omega$, $f(\omega, x) \in \overline{I_S(x)}$ for any $x \in S$).*

Then, f has a random fixed point.

For the proof, see Theorem 6 and Lin [8, Proof of Theorem 4].

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