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NEW GENERATING FUNCTIONS FOR MULTIVARIATE BIORTHOGONAL POLYNOMIALS ON THE N-SPHERE

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ABSTRACT

Certain multivariate biorthogonal polynomials on the N-sphere arise in connection with quantum chromodynamics. These functions can be expressed in terms of Lauricella functions of the first kind, and multi-dimensional generating functions for them are deduced by means of an extension of Bailey's theorem.

Key words: Generating Functions, Biorthogonal Polynomials.

AMS subject classification: 33C30, 33C45, 81V05.

1. Introduction and Notation

The purpose of this study is to give a number of new generating functions for the biorthogonal polynomials on the N-sphere first presented by Lam and Tratnik [8], and discussed at considerable length by Kalnins, Miller and Tratnik [6]. These polynomials, which arose in certain calculations relating to quantum chromodynamics, may be represented as Lauricella functions $F_A^{(n)}$ in the form

$$F_A^{(n)}(M+G-1, -m_1,\ldots, -m_n; g_1,\ldots,g_n; x_1,\ldots,x_n) \tag{1.1}$$

and

$$(1-X)^M F_A^{(n)}(1-M-g_{n+1}, -m_1,\ldots, -m_n; g_1,\ldots,g_n; x_1/(X-1),\ldots,x_n/(X-1)), \tag{1.2}$$

where $G = g_1 + \ldots + g_{n+1}$.

The basis of the results given is a multi-dimensional generalization of Bailey's theorem as given by Exton [4], p. 139 which extends the discussion of Bailey [2] and Slater [9], p. 58. This very general formulation may be stated as follows:

If $B_{m_1,\ldots,} = \sum_{p_1,\ldots,p_n=0}^{m_1,\ldots,m_n} A_{p_1,\ldots,p_n} U_{m_1-p_1,\ldots,m_n-p_n} V_{m_1+p_1,\ldots,m_n+p_n}$ and

$$C_{m_1,\ldots,m_n} = \sum_{p_1=m_1,\ldots,p_n=m_n}^{\infty} D_{p_1,\ldots,p_n} U_{p_1-m_1,\ldots,p_n-m_n} V_{p_1+m_1,\ldots,p_n+m_n},$$

then, formally,

$$\sum A_{m_1,\ldots,m_n} C_{m_1,\ldots,m_n} = \sum B_{m_1,\ldots,m_n} D_{m_1,\ldots,m_n}. \tag{1.3}$$

It is understood that A, U, D and V are functions of $p_1, \ldots, p_n$ only, and any questions of convergence must be dealt with in each individual case as appropriate. The symbol $\sum$ without further qualification denotes a summation with the indices of summation running over all nonnegative integer values.

As usual, we employ the notation

$$(a,n) = a(a+1)(a+2)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a); \quad (a,0) = 1,$$

and following Tratnik [12] for example, we put

$$x_1 + \ldots + x_n = X \text{ etc.}$$

The generalization hypergeometric function of one variable is given by

$$ {}_AF_B(a_1,\ldots,a_A; b_1,\ldots,b_B; x) = \sum \frac{(a_1,m)\ldots(a_A,m)x^m}{(b_1,m)\ldots(b_B,m)m!}$$

$$= \sum \frac{((a),m)x^m}{((b),m)m!},$$

where, for convenience, the sequence of parameters $a_1, \ldots, a_A$ is denoted by (a) etc. Certain multiple hypergeometric functions also figure in the analysis and are given as follows:

The Lauricella function of the first kind

$$F_A^{(n)}(a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n)$$

$$= \sum \frac{(a,M)(b_1,m_1)\ldots(b_n,m_n)x_1^{m_1}\ldots x_n^{m_n}}{(c_1,m_1)\ldots(c_n,m_n)m_1!\ldots m_n!}$$

the Lauicella function of the third kind

$$F_C^{(n)}(a, b; c_1, \ldots, c_n; x_1, \ldots, x_n)$$

$$= \sum \frac{(a,M)(b,M)x_1^{m_1}\ldots x_n^{m_n}}{(c_1,m_1)\ldots(c_n,m_n)m_1!\ldots m_n!}$$

and Karlsson's [7] generalized Kampé de Fériet function

$$F_{C:D}^{A:B}\left[\begin{matrix}(a):(b_1);\ldots;(b_n);\\(c):(d_1);\ldots;(d_n);\end{matrix} x_1, \ldots, x_n\right]$$

$$= \sum \frac{((a),M)((b_1),m_1)\ldots((b_n),m_n)x_1^{m_1}\ldots x_n^{m_n}}{((c),M)((d_1),m_1)\ldots((d_n),m_n)m_1!\ldots m_n!}.$$

It will be seen that

$$F_A^{(n)}(a, b_1, \ldots, b_n; c_1, \ldots, b_n; x_1, \ldots, x_n)$$

$$= F_{0:1}^{1:1}\left[\begin{matrix}a: b_1; \ldots; b_n;\\ -: c_1; \ldots; c_n;\end{matrix} x_1, \ldots, x_n\right]$$

and

$$F_C^{(n)}(a, b; c_1, \ldots, c_n; x_1, \ldots, x_n)$$

$$= F_{0:1}^{2:0}\left[\begin{matrix}a, b: -; \ldots; -;\\ -: c_1; \ldots; c_n;\end{matrix} x_1, \ldots, x_n\right].$$

Any values of the parameters for which any of the expressions given in this paper do not make sense are tacitly excluded. For any relevant general information, the reader is referred to Appell et Kampé de Fériet [1], Erdélyi [3], Exton [4], Slater [9], Srivastava and Karlsson [10], and Srivastava and Manocha [11], for example.

2. Main Results

A generating relation of quite general character for the polynomial (1.1) may be deduced from (1.3). Put

$$U = 1/(m_1!\ldots m_n!),\ V = (G-1, M)$$
$$A = \frac{(-x_1)^{m_1}\ldots(-x_n)^{-m_n}}{(g_1, m_1)\ldots(g_n, m_n)m_1!\ldots m_n!}$$

and
$$D = \frac{((p), M)((h_1), m_1)\ldots((h_n), m_n)t_1^{m_1}\ldots t_n^{m_n}}{((q), M)((k_1), m_1)\ldots((k_n), m_n)}.$$

After some rather tedious manipulation, we obtain the expression

$$\sum \frac{((p), M)(G-1, M)((h_1), m_1)\ldots((h_n), m_n)t_1^{m_1}\ldots t_n^{m_n}}{((q), M)((k_1), m_1)\ldots((k_n), m_n)m_1!\ldots m_n!}$$

$$\cdot F_A^{(n)}(M+G-1, -m_1, \ldots, -m_n; g_1, \ldots, g_n; x_1, \ldots, x_n)$$

$$= \sum \frac{((p), M)(G-1, 2M)((h_1), m_1)\ldots((h_n), m_n)(-x_1t_1)^{m_1}\ldots(-x_nt_n)^{m_n}}{((q), M)(g_1, m_1)\ldots(g_n, m_n)((k_1), m_1)\ldots((k_n), m_n)m_1!\ldots m_n!}$$

$$\cdot F_{Q:K}^{P+1:H}\left[\begin{matrix}(p)-M, G-1+2M: (h_1)+m_1; \ldots; (h_n)+m_n; \\ (q)+M: (k_1)+m_1; \ldots; (k_n)+m_n;\end{matrix} t_1 k \ldots, t_n\right]. \tag{2.1}$$

This is the required generating function for the polynomial (1.1).

Similarly, on putting

$$U = \frac{1}{(g_{n+1}, M)m_1!\ldots m_n!},\quad V = 1,$$
$$A = \frac{x_1^{m_1}\ldots x_n^{m_n}(X-1)^{-M}}{(g_1, m_1)\ldots(g_n, m_n)m_1!\ldots m_n!}$$

and
$$D = \frac{((p), M)((h_1), m_1)\ldots((h_n), m_n)(X-1)^M t_1^{m_1}\ldots t_n^{m_n}}{((q), M)((k_1), m_1)\ldots((k_n), m_n)}$$

we obtain a general generating function for the polynomial (1.2), namely

$$\sum \frac{((p), M)((h_1), m_1)\ldots((h_n), m_n)t_1^{m_1}\ldots t_n^{m_n}(X-1)^M}{((q), M)((k_1), m_1)\ldots((k_n), m_n)m_1!\ldots m_n!(g_{n+1}, M)}$$

$$\cdot F_A^{(n)}(1-g_{n+1}-M, -m_1, \ldots, -m_n; g_1, \ldots, g_n; x_1/(X-1), \ldots, x_n/(X-1))$$

$$= \sum \frac{((p),M)((h_1),m_1)\ldots((h_n),m_n)(x_1t_1)^{m_1}\ldots(x_nt_n)^{m_n}}{((q),M)(g_1,m_1)\ldots(g_n,m_n)((k_1),m_1)\ldots((k_n),m_n)m_1!\ldots m_n!}$$

$$\cdot F^{P:H}_{Q+1:K}\left[\begin{matrix}(p)+M:(h_1)+m_1;\ldots;(h_n)+m_n;\\(q)+M,g_{n+1}:(k_1)+m_1:\ldots:(k_n)+m_n;\end{matrix} t_1(X-1),\ldots,t_n(X-1)\right]. \qquad (2.1)$$

3. Compact Generating Relations

A number of more compact expressions can be obtained as special cases of (2.1) and (2.2). In the first of these, suppose that $P=Q=H=K=0$, when the inner generalized Kampé de Fériet function on the right becomes

$${}_1F_0(G-1+2M;-;T)=(1-T)^{1-G-2M}$$

as a simple consequence of the binomial theorem.

Hence, we see that

$$\sum \frac{(G-1,M)t_1^{m_1}\ldots t_n^{m_n}}{m_1!\ldots m_n!} F_A^{(n)}(M+G-1,m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1,\ldots,x_n)$$
$$=(1-T)^{1-G}F_C^{(n)}(\tfrac{1}{2}G-\tfrac{1}{2},\tfrac{1}{2}G;g_1,\ldots,g_n;-4t_1x_1(1-T)^{-2},\ldots,t_nx_n(1-T)^{-2}).$$

Next, put $t_1=\ldots=t_n=t$ and take $P=K=0$ and $H=Q=1$. After letting $q=H$, we then have

$$\sum \frac{(G-1,M)(h_1,m_1)\ldots(h_n,m_n)t^M}{(H,M)m_1!\ldots m_n!} F_A^{(n)}(M+G-1,-m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1,\ldots,x_n)$$
$$=(1-t)^{1-G}F^{2:1}_{1:1}\left[\begin{matrix}\tfrac{1}{2}G-\tfrac{1}{2},\tfrac{1}{2}G:h_1;\ldots;h_n;\\H:g_1;\ldots;g_n;\end{matrix} -4x_1t(1-t)^{-2},\ldots,-4x_nt(1-t)^{-2}\right].$$

Further, put $g_1=h_1,\ldots,g_n=h_n$ and obtain

$$\sum \frac{(G-1,M)t^M}{(H,M)m_1!\ldots m_n!}F_A^{(n)}(M+G-1,-m_1,\ldots,-m_n;h_1,\ldots,h_n;x_1,\ldots,x_n)$$
$$=(1-t)^{1-G}{}_2F_1(\tfrac{1}{2}G-\tfrac{1}{2},\tfrac{1}{2}G;H;-4tX(1-t)^{-2}.$$

A general result of quite a different character may be deduced by putting

$$t_n=-t_1-\ldots-t_{n-1},$$

so that, if we let $H=K=0$, the inner generalized Kampé de Fériet on the right of (2.1) reduces to unity by means of the binomial theorem. Compare Exton [5]. Hence,

$$\sum \frac{((p),M)(G-1,M)t_1^{m_1}\ldots t_{n-1}^{m_{n-1}}(-t_1-\ldots-t_{n-1})^{m_n}}{((q),M)m_1!\ldots m_n!}$$

$$\cdot F_A^{(n)}(M+G-1,-m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1,\ldots,x_n)$$

$$=F^{P+2:0}_{Q:1}\left[\begin{matrix}(p),\tfrac{1}{2}G-\tfrac{1}{2},\tfrac{1}{2}G:-;\ldots;-;\\(q):g_1;\ldots;g_n;\end{matrix} -4x_1t_1,\ldots,-rx_{n-1}t_{n-1},4x_n(t_1+\ldots+t_{n-1})\right]. \qquad (3.1)$$

The previous expansion embodies some degree of flexibility, since the p and q parameters can be chosen arbitrarily in order to simplify the result. For example, if $P=0$ and $Q=2$ such that $q_1=\frac{1}{2}G-\frac{1}{2}$ and $q_2=\frac{1}{2}G$, the right-hand member of (3.1) reduces to the product of several ${}_0F_1$ series, which are, effectively, Bessel functions. We then see that

$$\sum\frac{(G-1,M)t_1^{m_1}\ldots t_{n-1}^{m_{n-1}}(-t_1-\ldots-t_{n-1})^{m_n}}{(\frac{1}{2}G-\frac{1}{2},M)(\frac{1}{2}G,M)m_1!\ldots m_n!}$$
$$\cdot F_A^{(n)}(M+G-1,-m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1,\ldots,x_n)$$
$$={}_0F_1(-;g_1;-4x_1t_1)\ldots{}_0F_1(-;g_{n-1};-4x_{n-1}t_{n-1}){}_0F_1(-;g_n;4x_n(t_1+\ldots+t_{n-1})).$$

The second general result, (2.2), may also be used to obtain a number of expansion. Suppose that $P=H=Q=K=0$, when the inner generalized Kampé de Fériet function on the right takes the form

$${}_0F_1(-;g_{n+1};T(X-1))$$

and we have the generating function

$$\sum\frac{t_1^{m_1}\ldots t_n^{m_n}(X-1)^M}{(g_{n+1},M)m_1!\ldots m_n!}$$
$$\cdot F_A^{(n)}(1-M-g_{n+1},-m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1/(X-1),\ldots,x_n/(X-1))$$

Put $H=1$, $K=0$ and $t_1=\ldots=t_n=t$, and obtain the result

$$\sum\frac{((p,M)(h_1,m_1)\ldots(h_n,m_n)t^M(X-1)^M}{((q),M)(g_{n+1},M)m_1!\ldots m_n!}$$
$$\cdot F_A^{(n)}(1-M-g_{n+1},-m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1/(X-1),\ldots,x_n/(X-1))$$
$$=\sum\frac{((p),M)(h_1,m_1)\ldots(h_n,m_n)x_1^{m_1}\ldots x_n^{m_n}t^M}{((q),M)(g_1,m_1)\ldots(g_n,m_n)m_1!\ldots m_n!}$$
$$\cdot{}_{P+1}F_{Q+1}((P)+M,H+M;(q)+M,g_{n+1};t(X-1)).$$

Further, let $P=0$, $Q=1$ and $q=h$, when it is found that

$$\sum\frac{(h_1,m_1)\ldots(h_n,m_n)t^M(X-1)^M}{(H,M)(g_{n+1},M)m_1!\ldots m_n!}$$
$$\cdot F_A^{(n)}(1-M-g_{n+1},m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1/(X-1),\ldots,x_n/(X-1))$$
$$={}_0F_1(-;g_1;x_1t)\ldots{}_0F_1(-;g_n;x_nt){}_0F_1(-;g_{n+1};t(X-1)). \tag{3.2}$$

In (2.2), let $H=K=0$ and $t_n=-t_1-\ldots-t_{n-1}$, when, as in (3.1), the inner generalized Kampé de Fériet function reduces to unity, so that

$$\sum\frac{((p),M)t_1^{m_1}\ldots t_{n-1}^{m_{n-1}}(-t_1-\ldots-t_{n-1})^{m_n}(X-1)^M}{((q),M)(g_{n+1},M)m_1!\ldots m_n!}$$
$$\cdot F_A^{(n)}(1-M-g_{n+1},-m_1,\ldots,-m_n;g_1,\ldots,g_n;x_1/(X-1),\ldots,x_n/(X-1))$$

$$= F^{P:0}_{Q:1}\left[\begin{matrix}(p): -;\ldots; -;\\ (q): g_1;\ldots; g_n;\end{matrix} x_1t_1,\ldots,x_{n-1}t_{n-1}, -x_n(t_1+\ldots+t_{n-1})\right].$$

Put $P = Q = 0$, and obtain the result

$$\sum \frac{t_1^{m_1}\ldots t_{n-1}^{m_{n-1}}(-t_1-\ldots-t_{n-1})^{m_n}(X-1)^M}{((q),M)(t_{n+1},M)m_1!\ldots m_n!}$$

$$\cdot F_A^{(n)}(1-M-g_{n+1}, -m_1,\ldots, -m_n; g_1,\ldots,g_n; x_1/(X-1),\ldots,x_n/(X-1))$$

$$= {}_0F_1(-;g_1;x_1t_1)\ldots {}_0F_1(-;g_{n-1};x_{n-1}t_{n-1}){}_0F_1(-;g_n;-x_n(t_1+\ldots+t_{n-1})). \quad (3.3)$$

The formulas (3.2) and (3.3) again may be expressed in terms of Bessel functions on account of the occurrence of the ${}_0F_1$ series on the right in each case.

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