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NOTE ON STRONG SOLUTIONS OF A STOCHASTIC INCLUSION

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ABSTRACT

Two different definitions of strong solutions of a stochastic integral set-valued equation are discussed. A selection property of a set-valued stochastic integral is given.

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1. Introduction

In the theory of stochastic equations the definition of their solutions is quite natural. A process x is a *solution* of the equation

$$x_t = \int_0^t f_\tau(x) dM_\tau, \quad 0 \le t < \infty \tag{1}$$

if the above is satisfied for all t.

In the set-valued approach there are two possibilities for defining a solution of a stochastic inclusion.

Let F be a set-valued predictable process and let the following stochastic inclusion be given:

$$x_t \in \int_0^t F_\tau(x) dM_\tau, \quad 0 \le t < \infty \tag{2}$$

(for required definitions see the next section).

Definition A: A process x is a *solution of problem* (2) if it satisfies

$$x_t - x_s \in \int_s^t F_\tau(x) dM_\tau \tag{3}$$

for all $0 \le s < t < \infty$.

Definition B: A process x is a *solution of problem* (2) if there exists an M-integrable selector f of $F(x)$ such that

$$x_t = \int_0^t f_\tau dM_\tau, \tag{4}$$

for all $t, 0 \leq t < \infty$.

Definition **A** is more natural because of its similarity to a single-valued case. In stochastic set-valued investigations the two definitions have been used. In [1,10,11] the solutions were investigated in the sense of **B** , while in [7,8] they were investigated in the sense of **A**. Avgerinos and Papageorgiou in [3] used a combination of these definitions. They investigated a random inclusion of the type

$$\dot{x}(\omega,t) \in A(\omega)x(\omega,t) + F(\omega,t,x(\omega,t))$$

and as a solution they meant a process satisfying the inclusion

$$\dot{x}(\omega,t) \in A(\omega)x(\omega,t) + f(\omega)(t)$$

for $f(\omega)$ being a selection of $F(\omega,\cdot,x(\omega,\cdot))$.

It is well known that, in the ordinary differential inclusion case, these two concepts of solutions coincide only for convex-valued set-valued functions (see e.g., Integral Representation Property in [2, p. 99]). The same is true for a stochastic inclusion with a Wiener process ([7, Th. 4.1]), but it is an open problem for the semimartingale case. It is clear that if x is a solution of problem (2) in the sense of definition **B**, it is also a solution in the sense of **A**. The purpose of this paper is to prove the converse, and this requires some selection-type theorem.

2. Preliminaries

Throughout the paper $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ denotes a complete filtered probability space satisfying the usual hypothesis: (i) $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, (ii) $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$, for all t, $0 \leq t < \infty$; This means that a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is right continuous. By a *stochastic process* x on $(\Omega, \mathcal{F}, P)$ we mean a collection $(x_t)_{t \geq 0}$ of n-dimensional random variables $x_t: \Omega \to \mathbb{R}^n$, $t \geq 0$. The process x is said to be *adapted* if x_t belongs to $\mathcal{F}_t$ (which means it is $\mathcal{F}_t$-measurable) for each $t \geq 0$. A stochastic process x is called *cádlág* if it a.s. has sample paths which are right continuous, with left limits. Similarly, a stochastic process x is said to be *cáglád* if it a.s. has sample paths which are left continuous, with right limits. The family of all adapted cádlág (cáglád) processes is denoted by D $[L]$.

Let $\mathcal{P}(\mathcal{F}_t)$ denote the smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ with respect to which every cáglád adapted process is measurable in (t,ω), i.e. $\mathcal{P}(\mathcal{F}_t) = \sigma(L)$. A stochastic process x is said to be *predictable* if x is $\mathcal{P}(\mathcal{F}_t)$-measurable. The family of all such processes is denoted by $\mathcal{P}$. One has $\mathcal{P}(\mathcal{F}_t) \subset \beta_+ \otimes \mathcal{F}$, where β_+ denotes the Borel σ-algebra on $\mathbb{R}_+$.

Denote $X^2 = \{x \in \mathcal{P}: \|x\|_{S^2} < \infty\}$, where $\|x\|_{S^2} = \|\sup_{t \geq 0} |x_t|\|_{L^2}$. It can be verified that $(X^2, \|\cdot\|_{S^2})$ is a Banach space (see e.g., [12, 13]).

Let $\mathcal{M}$ [or $\mathcal{M}_0$] denote the set of all one-dimensional semimartingales [or vanishing at $t = 0$ respectively]. Given $M \in \mathcal{M}$, let $M = N + A$ be a decomposition of M, where N is a local martingale, A denotes a process with path of finite variation on compacts and $[N,N]$ denotes the quadratic variation process of N. Define

$$j_2(N,A) = \| [N,N]_\infty^{\frac{1}{2}} + \int_{0^-}^{\infty} |dA_s| \|_{L^2}$$

and

$$\| M \|_{\mathcal{H}^2} = \inf_{M = N + A} j_2(N, A),$$

where $\int_{0^-}^{t} | dA_s | = \int_{[0,t]} | dA_s |$ and the infimum is taken over all possible decompositions of M. Define $\mathcal{H}^2 = \{M \in \mathcal{M}: \| M \|_{\mathcal{H}^2} < \infty\}$. We also let $L(M) = \{H \in \mathcal{P}: H$ is integrable with respect to $M\}$ with a norm $\| H \|_M = \| H \cdot M \|_{\mathcal{H}^2}$. Moreover, by $H \cdot M$ we denote $\int H_\tau dM_\tau$.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $Cl(\mathbb{R}^n)$, $\text{Comp}(\mathbb{R}^n)$ and $\text{Conv}(\mathbb{R}^n)$ denote spaces of all nonempty closed, compact, compact and convex, respectively, subsets of $\mathbb{R}^n$. Denote by $\text{dist}(a, A)$ the distance between $a \in \mathbb{R}^n$ and $A \in Cl(\mathbb{R}^n)$. We put $\bar{h}(A, B) = \sup_{a \in B} \text{dist}(a, A)$, and $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$ for all $A, B \in Cl(\mathbb{R}^n)$.

Consider a set-valued stochastic process $\mathcal{R} = (\mathcal{R}_t)_{t \geq 0}$ with values in $Cl(\mathbb{R}^n)$, i.e. a family of $\mathcal{F}$-measurable set-valued mappings $\mathcal{R}_t: \Omega \to Cl(\mathbb{R}^n)$ for each $t \geq 0$. We call $\mathcal{R}$ *predictable* if $\mathcal{R}$ is $\mathcal{P}(\mathcal{F}_t)$-measurable in the sense of set-valued functions.

Given a predictable set-valued process $\mathcal{R} = (\mathcal{R}_t)_{t \geq 0}$ and $M \in \mathcal{M}_0$, let

$$\mathcal{S}_M(\mathcal{R}) := \{H \in L(M): H_t \in \mathcal{R}_t \text{ for all } t\}.$$

A set $\mathcal{S}_M(\mathcal{R})$ is called a *subtrajectory integral* of $\mathcal{R}$.

A predictable set-valued process $\mathcal{R}$ is said to be *integrable with respect to a semimartingale* M or, simply, M*-integrable*, if $\mathcal{S}_M(\mathcal{R})$ is a nonempty set. It follows immediately from the properties of stochastic integrals with respect to semimartingales (see Th. 3.2 of [6]) and Kuratowski and Ryll-Nardzewski measurable selection theorem (see e.g. [9]), that every M-integrably bounded and predictable set-valued stochastic process $\mathcal{R}$ is M-integrable. Recall a set-valued stochastic process $\mathcal{R} = (\mathcal{R}_t)_{t \geq 0}$ is M*-integrably bounded* if there exists $m \in L(M) \cap X^2$ such that $h(\mathcal{R}_t, \{0\}) \leq m_t$ a.s., for each $t \geq 0$.

3. Selection Properties of Integrals

Convention: In this section we employ a notation $\int_a^b H dM$ instead of $\int_a^b H_s dM_s$ for clarity of formulas.

Lemma 1: *Let M be a semimartingale in $\mathcal{H}^2$, let $x = (x_t)_{t \geq 0}$ be a cádlág process and let a predictable set-valued process $\mathcal{G}$ be integrably bounded by a process $m = (m_t)_{t \geq 0}$, $m \in L(M)$. If $x_t - x_s \in cl_{L^2} \int_s^t \mathcal{G} dM$ for every $0 \leq s < t < \infty$, then for all stopping times α, β, $0 \leq \alpha < \beta < \infty$, there exists a sequence $(g^n) \subset cl_{L(M)} \mathcal{S}_M(\mathcal{G})$ such that*

$$\lim_{n \to \infty} \| (x_\beta - x_\alpha) - \int_\alpha^\beta g^n dM \|_{L^2} = 0.$$

Proof: Let $\alpha_n = k \cdot 2^{-n}$ for ω such that $(k-1)2^{-n} \leq \alpha(\omega) < k2^{-n}$ and $\beta_n = k \cdot 2^{-n}$ for ω such that $(k-1)2^{-n} \leq \beta(\omega) < k2^{-n}$. Let $A_k^n = \{\omega: \alpha \geq k \cdot 2^{-n}\}$ and $B_k^n = \{\omega: \beta \geq k \cdot 2^{-n}\}$. Then we have

$$[0, \alpha_n] = (\{0\} \times \Omega) \cup (\bigcup_{k=0}^{\infty} (k \cdot 2^{-n}, (k+1)2^{-n}] \times A_k^n),$$

$$[0,\beta_n] = (\{0\}\times\Omega)\cup(\bigcup_{k=0}^{\infty}(k\cdot 2^{-n},(k+1)2^{-n}]\times B_k^n).$$

Now, for each $n=1,2,\dots$ we obtain

$$x_{\alpha_n} = x_0 + \sum_{k=0}^{\infty} I_{A_k^n}(x_{(k+1)2^{-n}} - x_{k2^{-n}})$$

and

$$x_{\beta_n} = x_0 + \sum_{k=0}^{\infty} I_{B_k^n}(x_{(k+1)2^{-n}} - x_{k2^{-n}}).$$

Since $A_k^n \subset B_k^n$ then

$$x_{\beta_n} - x_{\alpha_n} = \sum_{k=0}^{\infty} I_{B_k^n\setminus A_k^n}(x_{(k+1)2^{-n}} - x_{k2^{-n}}).$$

For every $k=0,1,\dots$ and $n=1,2,\dots$ we can select $g^{n,k}\in\mathcal{S}_M(\mathcal{G})$ such that

$$\| x_{(k+1)2^{-n}} - x_{k2^{-n}} - \int_{k2^{-n}}^{(k+1)2^{-n}} g^{n,k}dM \|_{L^2} < \epsilon/(3\cdot 2^k)$$

and put

$$g^n = I_{[0,\alpha_n]}\overline{g} + \sum_{k=0}^{\infty} I_{(k2^{-n},(k+1)2^{-n}]\times B_k^n\setminus A_k^n}\cdot g^{n,k} + I_{(\beta_n,\infty)}\overline{g},$$

where $\overline{g}\in\mathcal{S}_M(\mathcal{G})$ is an arbitrary selector.

It is easy to see that g^n belongs to $cl_{L(M)}\mathcal{S}_M(\mathcal{G})$ because of decomposability of $\mathcal{S}_M(\mathcal{G})$ and the Lebesgue Dominated Convergence Theorem. Moreover,

$$\int_{\alpha_n}^{\beta_n} g^n dM = \sum_{k=0}^{\infty} I_{B_k^n\setminus A_k^n}\int_{k2^{-n}}^{(k+1)2^{-n}} g^{n,k}dM.$$

Since $|g_t^n(\omega)| \le m_t(\omega)$ for every (t,ω), we obtain

$$\| \int_{\alpha}^{\beta} g^n dM - \int_{\alpha_n}^{\beta_n} g^n dM \|_{L^2} \le \sqrt{8}\,\| \int_0^{\infty} I_{(\alpha,\beta]\Delta(\alpha_n,\beta_n]} m dM \|_{\mathcal{H}^2},$$

where $A\Delta B$ denotes the set $(A\setminus B)\cup(B\setminus A)$. Therefore,

$$\| x_\beta - x_\alpha - \int_{\alpha}^{\beta} g^n dM \|_{L^2}$$

$$\le \| x_\beta - x_\alpha - (x_{\beta_n} - x_{\alpha_n}) \|_{L^2} + \| x_{\beta_n} - x_{\alpha_n} - \int_{\alpha_n}^{\beta_n} g^n dM \|_{L^2}$$

$$+ \sqrt{8}\,\| \int_0^{\infty} I_{(\alpha,\beta]\Delta(\alpha_n,\beta_n]} m dM \|_{\mathcal{H}^2}.$$

Since $(x_t)_{t\geq 0}$ and the stochastic integral are cádlág processes, $\alpha_n \to \alpha$, $\beta_n \to \beta$ as $n\to\infty$, we can select n_0 so great that first and third components are less than $\epsilon/3$ for $n > n_0$.

Next we have

$$\| x_{\beta_n} - x_{\alpha_n} - \int_{\alpha_n}^{\beta_n} g^n dM \|_{L^2}$$

$$= \| \sum_{k=0}^{\infty} I_{B_k^n \setminus A_k^n} [x_{(k+1)2^{-n}} - x_{k2^{-n}} - \int_{k2^{-n}}^{(k+1)2^{-n}} g^{n,k} dM] \|_{L^2}$$

$$\leq \sum_{k=0}^{\infty} \epsilon/(3\cdot 2^k) = \epsilon/3 \text{ for } n = 1,2,\ldots.$$

Since $\epsilon > 0$ is arbitrary and fixed, we obtain

$$\lim_{n\to\infty} \| (x_\beta - x_\alpha) - \int_{\alpha}^{\beta} g^n dM \|_{L^2} = 0. \qquad \square$$

Theorem 1: *Let M be a semimartingale in $\mathcal{H}^2$ and let m be a process in $L(M)\cap X^2$. Suppose $\mathcal{R}$ is a predictable set-valued process integrably bounded by m. If $x = (x_t)_{t\geq 0}$ is a cádlág process such that $x_t - x_s \in cl_{L^2}\int_s^t \mathcal{R} dM$ a.s. for every stopping time T and $s,t,T\leq s\leq t\leq\infty$, then for every $\epsilon > 0$ there exists a process $H\in cl_{L(M)}\mathcal{S}_M(\mathcal{R})$ such that*

$$\sup_{t\geq T} \| x_t - x_T - \int_T^t H dM \|_{L^2} < \epsilon.$$

Proof: Let $\epsilon > 0$ be fixed. By the Fundamental Theorem of Local Martingales and the Bichteler-Dellacherie Theorem M has a decomposition $M = N + A$ such that the jumps of the local martingale N are bounded by $\epsilon(3C_2 \| m \|_{S^2})^{-1}$. Define recursively

$$T_0 = T$$

$$T_{k+1} = \inf\{t\geq T_k : (\int_{T_{k-1}}^t d[N,N])^{1/2} + \int_{T_{k-1}}^t |dA| \geq \epsilon(3C_2\| m \|_{S^2})^{-1}$$
$$\text{or } \; |x_t - x_{T_{k-1}}| > \epsilon/3\}.$$

Then (T_k) increase to infinity a.s. [12, p. 192].

By Lemma 1, for every $k = 1,2,\ldots$, there exists a selector $H_k\in\mathcal{S}_M(\mathcal{R})$ such that

$$\| x_{T_k} - x_{T_{k-1}} - \int_{T_{k-1}}^{T_k} H_k dM \|_{L^2} < \frac{1}{2^k}\cdot\epsilon/3.$$

Next, take any $H_0\in\mathcal{S}_M(\mathcal{R})$ and define $H = H_0 I_{[0,T)} + \sum_{k=1}^{\infty} H_k I_{[T_{k-1},T_k)}$. Let us claim that $H\in cl_{L(M)}\mathcal{S}_M(\mathcal{R})$. Indeed, the set $cl_{L(M)}\mathcal{S}_M(\mathcal{R})$ is closed in $L(M)$ and decomposable.

Then $H_n = H_0 I_{[0,T)} + \sum_{k=1}^n H_k I_{[T_{k-1},T_k)}$ belongs to $\mathcal{S}_M(\mathcal{R})$. Since H_n tends to H for all (t,ω) and $|H_n| < m \in L(M)$ for each $n = 1,2,\ldots$, then by the Lebesgue Dominated Convergence Theorem H_n tends to H in $L(M)$ [12].

Now we have

$$\sup_{t \geq T} \| x_t - x_T - \int_T^t HdM \|_{L^2} = \sup_{k \geq 1} \sup_{T_{k-1} \leq t < T_k} \| x_t - x_T - \int_T^t HdM \|_{L^2}$$

$$\leq \sup_{k \geq 1} \sup_{T_{k-1} \leq t < T_k} \| x_t - x_{T_{k-1}} \|_{L^2} + \sup_{k \geq 2} \| \sum_{i=1}^{k-1} (x_{T_i} - x_{T_{i-1}} - \int_{T_{i-1}}^{T_i} HdM) \|_{L^2}$$

$$+ \sup_{k \geq 1} \sup_{T_{k-1} \leq t < T_k} \| \int_{T_{k-1}}^t HdM \|_{L^2} = I_1 + I_2 + I_3.$$

By the definition of T_k we obtain $\sup_{T_{k-1} \leq t < T_k} | x_t - x_{T_{k-1}} | < \epsilon/3$ for $k = 1,2,\ldots$, and a.e. $\omega \in \Omega$. Therefore, $I_1 < \epsilon/3$.

$$I_2 \leq \sup_{k \geq 2} \sum_{i=1}^{k-1} \| x_{T_i} - x_{T_{i-1}} - \int_{T_{i-1}}^{T_i} H_i dM \|_{L^2}$$

$$\leq \sum_{i=1}^{\infty} \| x_{T_i} - x_{T_{i-1}} - \int_{T_{i-1}}^{T_i} HdM \|_{L^2} \leq \epsilon/3 \sum_{i=1}^{\infty} \frac{1}{2^i} = \epsilon/3.$$

Now let us observe that

$$\| \int_{T_{k-1}}^t HdM \|_{L^2} \leq \| H \cdot I_{(T_{k-1},t]} \cdot M \|_{S^2} \leq C_2 \| H \cdot I_{(T_{k-1},t]} \cdot M \|_{\mathcal{H}^2}$$

$$\leq C_2 \| m \|_{S^2} \| (\int_{T_{k-1}}^t d[N,N])^{1/2} + \int_{T_{k-1}}^t | dA | \|_{L^2}.$$

Therefore, by the definition of (T_k) we get $I_3 \leq \epsilon/3$ and we are done. □

Theorem 2: *Let all assumptions of Theorem 1 be satisfied. If, moreover, $\mathcal{R}$ takes on convex values, then there exists a process $H \in cl_{L(M)} \mathcal{S}_M(\mathcal{R})$ such that*

$$x_t = x_T + \int_T^t HdM \quad a.s. \text{ for each } t \geq T.$$

Proof: By virtue of Theorem 1, there exists a sequence (H^n) in $cl_{L(M)} \mathcal{S}_M(\mathcal{R})$ such that

$$\sup_{t \geq T} \| x_t - x_T - \int_T^t H^n dM \|_{L^2} \to 0 \text{ as } n \to \infty.$$

We show that the set (H^n) is weakly compact in $L(M)$. Since

$$\| H \|_M \leq \Big(\int_\Omega \int H^2 d[N,N] dP \Big)^{1/2} + \Big(\int_\Omega \Big(\int | H | \, | dA | \Big)^2 \Big)^{1/2},$$

then this norm is weaker from the norm defined by the sum of norms in $\mathcal{L}^2(\Omega, \mathcal{L}^2(\mathbb{R}_+, \mu))$ and $\mathcal{L}^2(\Omega, \mathcal{L}^1(\mathbb{R}_+, \nu))$, where μ and ν denote measures generated by $[N,N]$ and $|A|$ respectively. The set (H^n) is integrably bounded, so it is weakly compact in the first space mentioned above by [4, Th. II.9]. It is also weakly compact in the second space, because the weak compactness of bounded sets in $\mathcal{L}^2(\Omega, E)$ and $\mathcal{L}^1(\Omega, E)$ is equivalent ([4]) and it follows by [9, Th. 2.1] that the set of selectors of integrable bounded set-valued functions is weakly compact in $\mathcal{L}^1(\Omega, \mathcal{L}^1(\nu)) = \mathcal{L}^1(\Omega \times \mathbb{R}_+, P \times \nu)$. Therefore, we deduce that (H^n) has a weak cluster point H in $cl_{L(M)} S_M(\mathcal{R})$. On the other hand, $x_t - x_T$ and $\int_T^t H dM$ are weak cluster points of a weak convergent sequence $\int_T^t H^{n_k} dM$ in $L^2(\mathcal{F}_t)$ for each $t \geq T$. Therefore $x_t - x_T$ is a modification of $(\int_T^t H dM)_{t \geq T}$. Then, by [12, I. Th. 2],

$$x_t = x_T + \int_T^t H dM \text{ a.s. for each } t \geq T. \qquad \square$$

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