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Research Article

A Summability Factor Theorem for Quasi-Power-Increasing Sequences

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We establish a summability factor theorem for summability $|A,\delta|_k$, where A is lower triangular matrix with nonnegative entries satisfying certain conditions. This paper is an extension of the main result of the work by Rhoades and Savaş (2006) by using quasi f-increasing sequences.

1. Introduction

Recently, Rhoades and Savaş [1] obtained sufficient conditions for $\sum a_n\lambda_n$ to be summable $|A,\delta|_k$, $k \geq 1$ by using almost increasing sequence. The purpose of this paper is to obtain the corresponding result for quasi f-increasing sequence.

A sequence $\{\lambda_n\}$ is said to be of bounded variation (bv) if $\sum_n |\Delta\lambda_n| < \infty$. Let $\mathrm{bv}_0 = \mathrm{bv} \cap c_0$, where c_0 denotes the set of all null sequences.

Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$A_n := \sum_{\nu=0}^{n} a_{n\nu}s_\nu. \tag{1.1}$$

A series $\sum a_n$, with partial sums (s_n), is said to be summable $|A|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|A_n - A_{n-1}|^k < \infty, \tag{1.2}$$

and it is said to be summable $|A,\delta|_k$, $k \geq 1$ and $\delta \geq 0$ if (see, [2])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|A_n - A_{n-1}|^k < \infty. \tag{1.3}$$

A positive sequence $\{b_n\}$ is said to be an almost increasing sequence if there exist an increasing sequence $\{c_n\}$ and positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see, [3]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = e^{(-1)^n} n$.

A positive sequence $\gamma := \{\gamma_n\}$ is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$K n^{\beta} \gamma_n \geq m^{\beta} \gamma_m \tag{1.4}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking an example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$ (see, [4]). A sequence satisfying (1.4) for $\beta = 0$ is called a quasi-increasing sequence. It is clear that if $\{\gamma_n\}$ is quasi β-power increasing then $\{n^{\beta}\gamma_n\}$ is quasi-increasing.

A positive sequence $\gamma = \{\gamma_n\}$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(\gamma, f) \geq 1$ such that $K f_n \gamma_n \geq f_m \gamma_m$ holds for all $n \geq m \geq 1$, where $f := \{f_n\} = \{n^{\beta}(\log n)^{\mu}\}, \mu > 0, 0 < \beta < 1$, (see, [5]).

We may associate with A two lower triangular matrices $\overline{A}$ and $\widehat{A}$ as follows:

$$\begin{aligned} \overline{a}_{nv} &= \sum_{r=v}^{n} a_{nr}, \quad n, v = 0, 1, \ldots, \\ \widehat{a}_{nv} &= \overline{a}_{nv} - \overline{a}_{n-1,v}, \quad n = 1, 2, \ldots, \end{aligned} \tag{1.5}$$

where

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}. \tag{1.6}$$

Given any sequence $\{x_n\}$, the notation $x_n \asymp O(1)$ means that $x_n = O(1)$ and $1/x_n = O(1)$. For any matrix entry a_{nv}, $\Delta_v a_{nv} := a_{nv} - a_{n,v+1}$.

Rhoades and Savaş [1] proved the following theorem for $|A, \delta|_k$ summability factors of infinite series.

Theorem 1.1. *Let $\{X_n\}$ be an almost increasing sequence and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that*

(i) $|\Delta \lambda_n| \leq \beta_n$,

(ii) $\lim \beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty$,

(iv) $|\lambda_n| X_n = O(1)$.

Let A be a lower triangular matrix with nonnegative entries satisfying

(v) $n a_{nn} \asymp O(1)$,

(vi) $a_{n-1,\nu} \geq a_{n\nu}$ *for* $n \geq \nu + 1$,

(vii) $\overline{a}_{n0} = 1$ *for all* n,

(viii) $\sum_{\nu=1}^{n-1} a_{\nu\nu}\widehat{a}_{n\nu+1} = O(a_{nn})$,

(ix) $\sum_{n=\nu+1}^{m+1} n^{\delta k}|\Delta_\nu \widehat{a}_{n\nu}| = O(\nu^{\delta k} a_{\nu\nu})$ *and*

(x) $\sum_{n=\nu+1}^{m+1} n^{\delta k}\widehat{a}_{n\nu+1} = O(\nu^{\delta k})$.

If

(xi) $\sum_{n=1}^{m} n^{\delta k-1}|t_n|^k = O(X_m)$, *where* $t_n := (1/(n+1))\sum_{k=1}^{n} k a_k$,

then the series $\sum a_n\lambda_n$ *is summable* $|A,\delta|_k, k \geq 1$.

It should be noted that, if $\{X_n\}$ is an almost increasing sequence, then condition (iv) implies that the sequence $\{\lambda_n\}$ is bounded. However, if $\{X_n\}$ is a quasi β-power increasing sequence or a quasi f-increasing sequence, (iv) does not imply that λ is bounded. For example, the sequence $\{X_m\}$ defined by $X_m = m^{-\beta}$ is trivially a quasi β-power increasing sequence for each $\beta > 0$. If $\lambda = \{m^\delta\}$, for any $0 < \delta < \beta$, then $\lambda_m X_m = m^{\delta-\beta} = O(1)$, but λ is not bounded, (see, [6, 7]).

The purpose of this paper is to prove a theorem by using quasi f-increasing sequences. We show that the crucial condition of our proof, $\{\lambda_n\} \in \mathrm{bv}_0$, can be deduced from another condition of the theorem.

2. The Main Results

We now will prove the following theorems.

Theorem 2.1. *Let* A *satisfy conditions (v)–(x) and let* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions (i) and (ii) of Theorem 1.1 and*

$$\sum_{n=1}^{m}\lambda_n = o(m), \quad m \longrightarrow \infty. \tag{2.1}$$

If $\{X_n\}$ *is a quasi* f*-increasing sequence and condition (xi) and*

$$\sum_{n=1}^{\infty} nX_n(\beta,\mu)\left|\Delta\beta_n\right| < \infty \tag{2.2}$$

are satisfied then the series $\sum a_n\lambda_n$ *is summable* $|A,\delta|_k$, $k \geq 1$, *where* $\{f_n\} := \{n^\beta(\log n)^\mu\}, \mu \geq 0, 0 \leq \beta < 1$, *and* $X_n(\beta,\mu) := (n^\beta(\log n)^\mu X_n)$.

The following theorem is the special case of Theorem 2.1 for $\mu = 0$.

Theorem 2.2. *Let* A *satisfy conditions (v)–(x) and let* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions (i), (ii), and* (2.1). *If* $\{X_n\}$ *is a quasi* β*-power increasing sequence for some* $0 \leq \beta < 1$ *and conditions (xi) and*

$$\sum_{n=1}^{\infty} nX_n(\beta)\left|\Delta\beta_n\right| < \infty \tag{2.3}$$

are satisfied, where $X_n(\beta) := (n^\beta X_n)$, *then the series* $\sum a_n\lambda_n$ *is summable* $|A,\delta|_k$, $k \geq 1$.

Remark 2.3. The conditions $\{\lambda_n\} \in \mathrm{bv}_0$, and (iv) do not appear among the conditions of Theorems 2.1 and 2.2. By Lemma 3.3, under the conditions on $\{X_n\}, \{\beta_n\}$, and $\{\lambda_n\}$ as taken in the statement of the Theorem 2.1, also in the statement of Theorem 2.2 with the special case $\mu = 0$, conditions $\{\lambda_n\} \in \mathrm{bv}_0$ and (iv) hold.

3. Lemmas

We will need the following lemmas for the proof of our main Theorem 2.1.

Lemma 3.1 (see [8]). *Let $\{\varphi_n\}$ be a sequence of real numbers and denote*

$$\Phi_n := \sum_{k=1}^{n} \varphi_k, \qquad \Psi_n := \sum_{k=n}^{\infty} |\Delta\varphi_k|. \tag{3.1}$$

If $\Phi_n = o(n)$ then there exists a natural number $\mathbb{N}$ such that

$$|\varphi_n| \leq 2\Psi_n \tag{3.2}$$

for all $n \geq \mathbb{N}$.

Lemma 3.2 (see [9]). *If $\{X_n\}$ is a quasi f-increasing sequence, where $\{f_n\} = \{n^{\beta}(\log n)^{\mu}\}, \mu \geq 0, 0 \leq \beta < 1$, then conditions* (2.1) *of Theorem 2.1,*

$$\sum_{n=1}^{m} |\Delta\lambda_n| = o(m), \quad m \longrightarrow \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} nX_n(\beta,\mu)|\Delta|\Delta\lambda_n|| < \infty, \tag{3.4}$$

where $X_n(\beta,\mu) = (n^{\beta}(\log n)^{\mu}X_n)$, imply conditions (iv) and

$$\lambda_n \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.5}$$

Lemma 3.3 (see [7]). *If $\{X_n\}$ is a quasi f-increasing sequence, where $\{f_n\} = \{n^{\beta}(\log n)^{\mu}\}, \mu \geq 0, 0 \leq \beta < 1$, then under conditions (i), (ii),* (2.1)*, and* (2.2)*, conditions (iv) and* (3.5) *are satisfied.*

Lemma 3.4 (see [7]). *Let $\{X_n\}$ be a quasi f-increasing sequence, where $\{f_n\} = \{n^{\beta}(\log n)^{\mu}\}$, $\mu \geq 0, 0 \leq \beta < 1$. If conditions (i), (ii), and* (2.2) *are satisfied, then*

$$n\beta_n X_n = O(1), \tag{3.6}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.7}$$

4. Proof of Theorem 2.1

Proof. Let (y_n) be the nth term of the A transform of the partial sums of $\sum_{i=0}^{n} \lambda_i a_i$. Then we have

$$\begin{aligned} y_n &:= \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu \\ &= \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} a_{ni} = \sum_{\nu=0}^{n} \overline{a}_{n\nu} \lambda_\nu a_\nu, \end{aligned} \tag{4.1}$$

and, for $n \geq 1$, we have

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} \widehat{a}_{n\nu} \lambda_\nu a_\nu. \tag{4.2}$$

We may write (noting that (vii) implies that $\widehat{a}_{n0} = 0$),

$$\begin{aligned} Y_n &= \sum_{\nu=1}^{n} \left(\frac{\widehat{a}_{n\nu} \lambda_\nu}{\nu} \right) \nu a_\nu \\ &= \sum_{\nu=1}^{n} \left(\frac{\widehat{a}_{n\nu} \lambda_\nu}{\nu} \right) \left[\sum_{r=1}^{\nu} r a_r - \sum_{r=1}^{\nu-1} r a_r \right] \\ &= \sum_{\nu=1}^{n-1} \Delta_\nu \left(\frac{\widehat{a}_{n\nu} \lambda_\nu}{\nu} \right) \sum_{r=1}^{\nu} r a_r + \frac{\widehat{a}_{nn} \lambda_n}{n} \sum_{r=1}^{n} r a_r \\ &= \sum_{\nu=1}^{n-1} (\Delta_\nu \widehat{a}_{n\nu}) \lambda_\nu \frac{\nu+1}{\nu} t_\nu + \sum_{\nu=1}^{n-1} \widehat{a}_{n,\nu+1} (\Delta \lambda_\nu) \frac{\nu+1}{\nu} t_\nu \\ &\quad + \sum_{\nu=1}^{n-1} \widehat{a}_{n,\nu+1} \lambda_{\nu+1} \frac{1}{\nu} t_\nu + \frac{(n+1) a_{nn} \lambda_n t_n}{n} \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say}. \end{aligned} \tag{4.3}$$

To complete the proof it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |T_{nr}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \tag{4.4}$$

From the definition of $\widehat{A}$ and using (vi) and (vii) it follows that

$$\widehat{a}_{n,\nu+1} \geq 0. \tag{4.5}$$

Using Hölder's inequality

$$
\begin{aligned}
I_1 &:= \sum_{n=1}^{m} n^{\delta k+k-1}|T_{n1}|^k = \sum_{n=1}^{m} n^{\delta k+k-1}\left|\sum_{\nu=1}^{n-1}\Delta_\nu \widehat{a}_{n\nu}\lambda_\nu \frac{\nu+1}{\nu}t_\nu\right|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\left(\sum_{\nu=1}^{n-1}|\Delta_\nu \widehat{a}_{n\nu}||\lambda_\nu||t_\nu|\right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\left(\sum_{\nu=1}^{n-1}|\Delta_\nu \widehat{a}_{n\nu}||\lambda_\nu|^k|t_\nu|^k\right)\left(\sum_{\nu=1}^{n-1}|\Delta_\nu \widehat{a}_{n\nu}|\right)^{k-1}, \\
\Delta_\nu \widehat{a}_{n\nu} &= \widehat{a}_{n\nu} - \widehat{a}_{n,\nu+1} \\
&= \overline{a}_{n\nu} - \overline{a}_{n-1,\nu} - \overline{a}_{n,\nu+1} + \overline{a}_{n-1,\nu+1} \\
&= a_{n\nu} - a_{n-1,\nu} \le 0.
\end{aligned}
\tag{4.6}
$$

Thus, using (vii),

$$
\sum_{\nu=0}^{n-1}|\Delta_\nu \widehat{a}_{n\nu}| = \sum_{\nu=0}^{n-1}(a_{n-1,\nu} - a_{n\nu}) = 1 - 1 + a_{nn} = a_{nn}. \tag{4.7}
$$

Since (λ_n) is bounded by Lemma 3.3, using (v), (ix), (xi), (i), and condition (3.7) of Lemma 3.4

$$
\begin{aligned}
I_1 &= O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_\nu|^k|t_\nu|^k|\Delta_\nu \widehat{a}_{n\nu}| \\
&= O(1)\sum_{n=1}^{m+1} n^{\delta k}\left(\sum_{\nu=1}^{n-1}|\lambda_\nu|^{k-1}|\lambda_\nu||\Delta_\nu \widehat{a}_{n\nu}||t_\nu|^k\right) \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu||t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}|\Delta_\nu \widehat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\delta k}|\lambda_\nu|a_{\nu\nu}|t_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\delta k-1}|\lambda_\nu||t_\nu|^k \\
&= O(1)\left[\sum_{\nu=1}^{m-1}\Delta(|\lambda_\nu|)\sum_{r=1}^{\nu} r^{\delta k-1}|t_r|^k + |\lambda_m|\sum_{r=1}^{m} r^{\delta k-1}|t_r|^k\right] \\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta\lambda_\nu|X_\nu + O(1)|\lambda_m|X_m \\
&= O(1)\sum_{\nu=1}^{m}\beta_\nu X_\nu + O(1)|\lambda_m|X_m \\
&= O(1).
\end{aligned}
\tag{4.8}
$$

Using Hölder's inequality,

$$
\begin{aligned}
I_2 &:= \sum_{n=2}^{m+1} n^{\delta k+k-1} |T_{n2}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1} \left| \sum_{\nu=1}^{n-1} \widehat{a}_{n,\nu+1} (\Delta \lambda_\nu) \frac{\nu+1}{\nu} t_\nu \right|^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left[\sum_{\nu=1}^{n-1} \widehat{a}_{n,\nu+1} |\Delta \lambda_\nu| |t_\nu| \right]^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left[\sum_{\nu=1}^{n-1} |\Delta \lambda_\nu| |t_\nu|^k \widehat{a}_{n,\nu+1} \right] \left[\sum_{\nu=1}^{n-1} \widehat{a}_{n,\nu+1} |\Delta \lambda_\nu| \right]^{k-1}.
\end{aligned}
\tag{4.9}
$$

By Lemma 3.1, condition (3.3), in view of Lemma 3.3 implies that

$$
\sum_{n=1}^{\infty} |\Delta \lambda_n| \le 2 \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} |\Delta |\Delta \lambda_k|| = 2 \sum_{k=1}^{\infty} |\Delta |\Delta \lambda_k|| \tag{4.10}
$$

holds. Thus by Lemma 3.3, (3.4) implies that $\sum_{n=1}^{\infty} |\Delta \lambda_n|$ converges. Therefore, there exists a positive constant M such that $\sum_{n=1}^{\infty} |\Delta \lambda_n| \le M$ and from the properties of matrix A, we obtain

$$
\sum_{\nu=1}^{n-1} \widehat{a}_{n,\nu+1} |\Delta \lambda_k| \le M a_{nn}. \tag{4.11}
$$

We have, using (v) and (x),

$$
\begin{aligned}
I_2 &= O(1) \sum_{n=2}^{m+1} n^{\delta k} (n a_{nn})^{k-1} \sum_{\nu=1}^{n-1} \widehat{a}_{n,\nu+1} \beta_\nu |t_\nu|^k \\
&= O(1) \sum_{\nu=1}^{m} \beta_\nu |t_\nu|^k \sum_{n=\nu+1}^{m+1} n^{\delta k} \widehat{a}_{n,\nu+1}.
\end{aligned}
\tag{4.12}
$$

Therefore,

$$
\begin{aligned}
I_2 &= O(1) \sum_{\nu=1}^{m} \nu^{\delta k} \beta_\nu |t_\nu|^k \\
&= O(1) \sum_{\nu=1}^{m} \nu \beta_\nu \frac{|t_\nu|^k}{\nu} \nu^{\delta k}.
\end{aligned}
\tag{4.13}
$$

Using summation by parts, (2.2), (xi), and condition (3.6) and (3.7) of Lemma 3.4

$$
\begin{aligned}
I_2 &:= O(1)\sum_{\nu=1}^{m-1}\Delta(\nu\beta_\nu)\sum_{r=1}^{\nu} r^{\delta k-1}|t_r|^k + O(1)m\beta_m\sum_{r=1}^{m} r^{\delta k-1}|t_r|^k \\
&= O(1)\sum_{\nu=1}^{m-1}\nu\left|\Delta(\beta_\nu)\right|X_\nu + O(1)\sum_{\nu=1}^{m-1}\beta_{\nu+1}X_{\nu+1} + O(1) \\
&= O(1).
\end{aligned}
\tag{4.14}
$$

Using Hölder's inequality and (viii),

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{k-1}|T_{n3}|^k &= \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\sum_{\nu=1}^{n-1}\widehat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu}t_\nu\right|^k \\
&\le \sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|\frac{\widehat{a}_{n,\nu+1}}{\nu}|t_\nu|\right]^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|\widehat{a}_{n,\nu+1}|t_\nu|a_{\nu\nu}\right]^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k a_{\nu\nu}|t_\nu|^k\widehat{a}_{n,\nu+1}\right]\left[\sum_{\nu=1}^{n-1}a_{\nu\nu}|\widehat{a}_{n,\nu+1}|\right]^{k-1}.
\end{aligned}
\tag{4.15}
$$

Using boundedness of $\{\lambda_n\}$, (v), (x), (xi), Lemmas 3.3 and 3.4

$$
\begin{aligned}
I_3 &= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k a_{\nu\nu}|t_\nu|^k\widehat{a}_{n,\nu+1} \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|a_{\nu\nu}|t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}\widehat{a}_{n,\nu+1} \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|\nu^{\delta k}a_{\nu\nu}|t_\nu|^k \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|(va_{vv})v^{\delta k-1}|t_v|^k \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|v^{\delta k-1}|t_v|^k.
\end{aligned}
\tag{4.16}
$$

Using summation by parts

$$
\begin{aligned}
I_3 &= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v} r^{\delta k-1}|t_r|^k + O(1)|\lambda_{m+1}|\sum_{v=1}^{m} v^{\delta k-1}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v+1} r^{\delta k-1}|t_r|^k + O(1)|\lambda_{m+1}|\sum_{v=1}^{m+1} v^{\delta k-1}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1} + O(1)|\lambda_{m+1}|X_{m+1} \\
&= O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)|\lambda_{m+1}|X_{m+1} \\
&= O(1).
\end{aligned}
\tag{4.17}
$$

Finally, using boundedness of $\{\lambda_n\}$, and (v) we have

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\delta k+k-1}|T_{n4}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1}\left|\frac{(n+1)a_{nn}\lambda_n t_n}{n}\right|^k \\
&= O(1)\sum_{n=1}^{m} n^{\delta k}a_{nn}|\lambda_n||t_n|^k \\
&= O(1),
\end{aligned}
\tag{4.18}
$$

as in the proof of I_1. □

5. Corollaries and Applications to Weighted Means

Setting $\delta = 0$ in Theorem 2.1 and Theorem 2.2 yields the following two corollaries, respectively.

Corollary 5.1. *Let A satisfy conditions (v)–(viii) and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions (i), (ii), and* (2.1)*. If $\{X_n\}$ is a quasi f-increasing sequence, where $\{f_n\} := \{n^\beta(\log n)^\mu\}$, $\mu \ge 0, 0 \le \beta < 1$, and conditions* (2.2) *and*

$$\sum_{n=1}^{m}\frac{1}{n}|t_n|^k = O(X_m), \quad m \longrightarrow \infty, \tag{5.1}$$

are satisfied then the series $\sum a_n\lambda_n$ is summable $|A|_k, k \ge 1$.

Proof. If we take $\delta = 0$ in Theorem 2.1 then condition (xi) reduces condition (5.1). □

Corollary 5.2. *Let A satisfy conditions (v)–(viii) and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions (i), (ii), and* (2.1)*. If $\{X_n\}$ is a quasi β-power increasing sequence for some $0 \le \beta < 1$ and conditions* (2.3) *and* (5.1) *are satisfied then the series $\sum a_n\lambda_n$ is summable $|A|_k$, $k \ge 1$.*

Corollary 5.3. *Let $\{p_n\}$ be a positive sequence such that $P_n := \sum_{i=0}^{n} p_i \to \infty$, as $n \to \infty$ satisfies*

$$np_n \asymp O(P_n), \quad \text{as } n \longrightarrow \infty, \tag{5.2}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k} \frac{p_n}{P_n P_{n-1}} = O\left(\frac{v^{\delta k}}{P_v}\right) \tag{5.3}$$

and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions (i), (ii), and (2.1). *If $\{X_n\}$ is a quasi f-increasing sequence, where $\{f_n\} := \{n^{\beta}(\log n)^{\mu}\}$, $\mu \geq 0, 0 \leq \beta < 1$, and conditions (xi) and* (2.2) *are satisfied then the series, $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n, \delta|_k$ for $k \geq 1$.*

Proof. In Theorem 2.1, set $A = (\overline{N}, p_n)$. Conditions (i) and (ii) of Corollary 5.3 are, respectively, conditions (i) and (ii) of Theorem 2.1. Condition (v) becomes condition (5.2) and conditions (ix) and (x) become condition (5.3) for weighted mean method. Conditions (vi), (vii), and (viii) of Theorem 2.1 are automatically satisfied for any weighted mean method. □

The following Corollary is the special case of Corollary 5.3 for $\mu = 0$.

Corollary 5.4. *Let $\{p_n\}$ be a positive sequence satisfying* (5.2), (5.3) *and let $\{X_n\}$ be a quasi β-power increasing sequence for some $0 \leq \beta < 1$. Then under conditions (i), (ii), (xi),* (2.1), *and* (2.3), *$\sum a_n \lambda_n$ is summable $|\overline{N}, p_n, \delta|_k$, $k \geq 1$.*

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