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Research Article

Mixed Monotone Iterative Technique for Abstract Impulsive Evolution Equations in Banach Spaces

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By constructing a mixed monotone iterative technique under a new concept of upper and lower solutions, some existence theorems of mild ω-periodic (L-quasi) solutions for abstract impulsive evolution equations are obtained in ordered Banach spaces. These results partially generalize and extend the relevant results in ordinary differential equations and partial differential equations.

1. Introduction and Main Result

Impulsive differential equations are a basic tool for studying evolution processes of real life phenomena that are subjected to sudden changes at certain instants. In view of multiple applications of the impulsive differential equations, it is necessary to develop the methods for their solvability. Unfortunately, a comparatively small class of impulsive differential equations can be solved analytically. Therefore, it is necessary to establish approximation methods for finding solutions. The monotone iterative technique of Lakshmikantham et al. (see [1–3]) is such a method which can be applied in practice easily. This technique combines the idea of method of upper and lower solutions with appropriate monotone conditions. Recent results by means of monotone iterative method are obtained in [4–7] and the references therein. In this paper, by using a mixed monotone iterative technique in the presence of coupled lower and upper L-quasisolutions, we consider the existence of mild ω-periodic (L-quasi)solutions for the periodic boundary value problem (PBVP) of impulsive evolution equations

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t), u(t)), \quad \text{a.e. } t \in J, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u(t_k)), \quad k = 1, 2, \ldots, p, \\ u(0) = u(\omega) \end{gathered} \tag{1.1}$$

in an ordered Banach space X, where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)$ $(t \geq 0)$ in X; $f : J \times X \times X \to X$ only satisfies weak Carathéodory condition, $J = [0, \omega]$, $\omega > 0$ is a constant; $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = \omega$; $I_k : X \times X \to X$ is an impulsive function, $k = 1, 2, \ldots, p$; $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, that is, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. Let $\mathrm{PC}(J, X) := \{u : J \to X \mid u(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, p\}$. Evidently, $\mathrm{PC}(J, X)$ is a Banach space with the norm $\|u\|_{\mathrm{PC}} = \sup_{t\in J} \|u(t)\|$. Let $J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, $J'' = J \setminus \{0, t_1, t_2, \ldots, t_p\}$. Denote by X_1 the Banach space generated by $D(A)$ with the norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. An abstract function $u \in \mathrm{PC}(J, X) \cap C^1(J'', X) \cap C(J', X_1)$ is called a solution of the PBVP(1.1) if $u(t)$ satisfies all the equalities of (1.1).

Let X be an ordered Banach space with the norm $\|\cdot\|$ and the partial order "$\leq$", whose positive cone $K := \{u \in X \mid u \geq 0\}$ is normal with a normal constant N. Let $L \geq 0$. If functions $v_0, w_0 \in \mathrm{PC}(J, X) \cap C^1(J'', X) \cap C(J', X_1)$ satisfy

$$\begin{gathered} v_0'(t) + Av_0(t) \leq f(t, v_0(t), w_0(t)) + L(v_0(t) - w_0(t)), \quad t \in J', \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k), w_0(t_k)), \quad k = 1, 2, \ldots, p, \\ v_0(0) \leq v_0(\omega), \end{gathered} \tag{1.2}$$

$$\begin{gathered} w_0'(t) + Aw_0(t) \geq f(t, w_0(t), v_0(t)) + L(w_0(t) - v_0(t)), \quad t \in J', \\ \Delta w_0|_{t=t_k} \geq I_k(w_0(t_k), v_0(t_k)), \quad k = 1, 2, \ldots, p, \\ w_0(0) \geq w_0(\omega), \end{gathered} \tag{1.3}$$

we call v_0, w_0 coupled lower and upper L-quasisolutions of the PBVP(1.1). Only choosing "=" in (1.2) and (1.3), we call (v_0, w_0) coupled ω-periodic L-quasisolution pair of the PBVP(1.1). Furthermore, if $u_0 := v_0 = w_0$, we call u_0 an ω-periodic solution of the PBVP(1.1).

Definition 1.1. Abstract functions $u, v \in \mathrm{PC}(J, X)$ are called a coupled mild ω-periodic L-quasisolution pair of the PBVP(1.1) if $u(t)$ and $v(t)$ satisfy the following integral equations:

$$\begin{aligned} u(t) = {}& T(t)B_1(u, v) + \int_0^t T(t-s)G_1(u, v)(s)\,ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k), v(t_k)), \quad t \in J, \\ v(t) = {}& T(t)B_1(v, u) + \int_0^t T(t-s)G_1(v, u)(s)\,ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(v(t_k), u(t_k)), \quad t \in J, \end{aligned} \tag{1.4}$$

where $B_1(x, y) = (I - T(\omega))^{-1}[\int_0^{\omega} T(\omega - s)G_1(x, y)(s)\,ds + \sum_{k=1}^{p} T(\omega - t_k)I_k(x(t_k), y(t_k))]$ and $G_1(x, y)(s) = f(s, x(s), y(s)) + L(x(s) - y(s))$ for any $x, y \in \mathrm{PC}(J, X)$, I is an identity operator. If $\tilde{u} := u = v$, then $\tilde{u}$ is called a mild ω-periodic solution of the PBVP(1.1).

Without impulse, the PBVP(1.1) has been studied by many authors, see [8–11] and the references therein. In particular, Shen and Li [11] considered the existence of coupled mild ω-periodic quasisolution pair for the following periodic boundary value problem (PBVP) in X:

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t), u(t)), \quad t \in J, \\ u(0) &= u(\omega), \end{aligned} \tag{1.5}$$

where $f : J \times X \times X \to X$ is continuous. Under one of the following situations:

(i) $T(t)$ $(t \geq 0)$ is a compact semigroup,

(ii) K is regular in X and $T(t)$ is continuous in operator norm for $t > 0$,

they built a mixed monotone iterative method for the PBVP(1.5), and they proved that, if the PBVP(1.5) has coupled lower and upper quasisolutions (i.e., $L \equiv 0$ and without impulse in (1.2) and (1.3)) v_0 and w_0 with $v_0 \leq w_0$, nonlinear term f satisfies one of the following conditions:

(F_1) $f : J \times X \times X \to X$ is mixed monotone,

(F_2) There exists a constant $M_1 > 0$ such that

$$f(t, u_2, w) - f(t, u_1, w) \geq -M_1(u_2 - u_1), \quad \forall t \in J, \ v_0(t) \leq u_1 \leq u_2 \leq w_0(t), \ v_0(t) \leq w \leq w_0(t), \tag{1.6}$$

and $f(t, u, v)$ is nonincreasing on v.
Then the PBVP(1.5) has minimal and maximal coupled mild ω-periodic quasisolutions between v_0 and w_0, which can be obtained by monotone iterative sequences from v_0 and w_0. But conditions (i) and (ii) are difficult to satisfy in applications except some special situations.

In this paper, by constructing a mixed monotone iterative technique under a new concept of upper and lower solutions, we will discuss the existence of mild ω-periodic (L-quasi) solutions for the impulsive evolution Equation(1.1) in an ordered Banach space X. In our results, we will delete conditions (i) and (ii) for the operator semigroup $T(t)$ $(t \geq 0)$, and improve conditions (F_1) and (F_2) for nonlinearity f. In addition, we only require that the nonlinear term $f : J \times X \times X \to X$ satisfies weak Carathéodory condition:

(1) for each $u, v \in X$, $f(\cdot, u, v)$ is strongly measurable.

(2) for a.e.$t \in J$, $f(t, \cdot, \cdot)$ is subcontinuous, namely, there exists $e \subset J$ with mes $e = 0$ such that

$$f(t, u_n, v_n) \overset{\text{weak}}{\longrightarrow} f(t, u, v), \quad (n \longrightarrow +\infty), \tag{1.7}$$

for any $t \in J \setminus e$, and $u_n \to u, v_n \to v$ $(n \to +\infty)$.

Our main result is as follows:

Theorem 1.2. *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X. If the PBVP(1.1) has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0 \leq w_0$, nonlinear term f and impulsive functions I_k's satisfy the following conditions*

(H_1) *There exist constants* $M > 0$ *and* $L \geq 0$ *such that*

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -M(u_2 - u_1) + L(v_2 - v_1) \tag{1.8}$$

for any $t \in J$, *and* $v_0(t) \leq u_1 \leq u_2 \leq w_0(t), v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

(H_2) *Impulsive function* $I_k(\cdot, \ \cdot)$ *is continuous, and for any* $u_i, v_i \in X$ $(i = 1, 2)$, *it satisfies*

$$I_k(u_1, v_1) \leq I_k(u_2, v_2), \quad k = 1, 2, \ldots, p \tag{1.9}$$

for any $t \in J$, *and* $v_0(t) \leq u_1 \leq u_2 \leq w_0(t), v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

then the PBVP(1.1) *has minimal and maximal coupled mild* ω*-periodic* L*-quasisolutions between* v_0 *and* w_0, *which can be obtained by monotone iterative sequences starting from* v_0 *and* w_0.

Evidently, condition (H_1) contains conditions (F_1) and (F_2). Hence, even without impulse in PBVP(1.1), Theorem 1.2 still extends the results in [10, 11].

The proof of Theorem 1.2 will be shown in the next section. In Section 2, we also discuss the existence of mild ω-periodic solutions for the PBVP(1.1) between coupled lower and upper L-quasisolutions (see Theorem 2.3). In Section 3, the results obtained will be applied to a class of partial differential equations of parabolic type.

2. Proof of the Main Results

Let X be a Banach space, $A : D(A) \subset X \rightarrow X$ be a closed linear operator, and $-A$ generate a C_0-semigroup $T(t)$ $(t \geq 0)$ in X. Then there exist constants $C > 0$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq Ce^{\delta t}, \quad t \geq 0. \tag{2.1}$$

Definition 2.1. A C_0-semigroup $T(t)$ $(t \geq 0)$ is said to be exponentially stable in X if there exist constants $C \geq 1$ and $\delta > 0$ such that

$$\|T(t)\| \leq Ce^{-\delta t}, \quad t \geq 0. \tag{2.2}$$

Let $I_0 = [t_0, T]$. Denote by $C(I_0, X)$ the Banach space of all continuous X-value functions on interval I_0 with the norm $\|u\|_C = \max_{t \in I_0} \|u(t)\|$. It is well-known ([12, Chapter 4, Theorem 2.9]) that for any $x_0 \in D(A)$ and $h \in C^1(I_0, \mathrm{X})$, the initial value problem(IVP) of linear evolution equation

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t \in I_0, \\ u(t_0) = x_0 \end{gathered} \tag{2.3}$$

has a unique classical solution $u \in C^1(I_0, X) \cap C(I_0, X_1)$ expressed by

$$u(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s)h(s)ds, \quad t \in I_0. \tag{2.4}$$

If $x_0 \in X$ and $h \in C(I_0, X)$, the function u given by (2.4) belongs to $C(I_0, X)$. We call it a mild solution of the IVP(2.3).

To prove Theorem 1.2, for any $h \in \mathrm{PC}(J, X)$, we consider the periodic boundary value problem (PBVP) of linear impulsive evolution equation in X

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \ldots, p, \\ u(0) = u(\omega), \end{gathered} \tag{2.5}$$

where $y_k \in X, \ k = 1, 2, \ldots, p$.

Lemma 2.2. *Let $T(t)$ $(t \geq 0)$ be an exponentially stable C_0-semigroup in X. Then for any $h \in PC(J, X)$ and $y_k \in X, \ k = 1, 2, \ldots, p$, the linear PBVP(2.5) has a unique mild solution $u \in PC(J, X)$ given by*

$$u(t) = T(t)B(h) + \int_0^t T(t-s)h(s)ds + \sum_{0<t_k<t} T(t-t_k)y_k, \quad t \in J, \tag{2.6}$$

where $B(h) = (I - T(\omega))^{-1}[\int_0^{\omega} T(\omega - s)h(s)ds + \sum_{k=1}^{p} T(\omega - t_k)y_k]$.

Proof. For any $h \in \mathrm{PC}(J, X)$, we first show that the initial value problem (IVP) of linear impulsive evolution equation

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \ldots, p, \\ u(0) = x_0 \end{gathered} \tag{2.7}$$

has a unique mild solution $u \in \mathrm{PC}(J, X)$ given by

$$u(t) = T(t)x_0 + \int_0^t T(t-s)h(s)ds + \sum_{0<t_k<t} T(t-t_k)y_k, \quad t \in J, \tag{2.8}$$

where $x_0 \in X$ and $y_k \in X, \ k = 1, 2, \ldots, p$.

Let $J_k = [t_k, t_{k+1}],\ k = 0, 1, 2, \ldots, p$. Let $y_0 = 0$. If $u \in \mathrm{PC}(J, X)$ is a mild solution of the linear IVP(2.7), then the restriction of u on J_k satisfies the initial value problem (IVP) of linear evolution equation without impulse

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t_k < t \le t_{k+1}, \\ u\left(t_k^+\right) = u(t_k) + y_k. \end{gathered} \tag{2.9}$$

Hence, on $(t_k, t_{k+1}]$, $u(t)$ can be expressed by

$$u(t) = T(t - t_k)\left(u(t_k) + y_k\right) + \int_{t_k}^{t} T(t - s)h(s)ds. \tag{2.10}$$

Iterating successively in the above equality with $u(t_j)$ for $j = k, k-1, \ldots, 1, 0$, we see that u satisfies (2.8).

Inversely, we can verify directly that the function $u \in \mathrm{PC}(J, X)$ defined by (2.8) is a solution of the linear IVP(2.7). Hence the linear IVP(2.7) has a unique mild solution $u \in \mathrm{PC}(J, X)$ given by (2.8).

Next, we show that the linear PBVP(2.5) has a unique mild solution $u \in \mathrm{PC}(J, X)$ given by (2.6).

If a function $u \in \mathrm{PC}(J, X)$ defined by (2.8) is a solution of the linear PBVP(2.5), then $x_0 = u(\omega)$, namely,

$$(I - T(\omega))x_0 = \int_0^{\omega} T(\omega - s)h(s)ds + \sum_{k=1}^{p} T(\omega - t_k)y_k. \tag{2.11}$$

Since $T(t)\ (t \ge 0)$ is exponentially stable, we define an equivalent norm in X by

$$|x| = \sup_{t \ge 0} \|e^{\delta t}T(t)x\|. \tag{2.12}$$

Then $\|x\| \le |x| \le C\|x\|$ and $|T(t)| < e^{-\delta t}\ (t \ge 0)$, and especially, $|T(\omega)| < e^{-\delta\omega} < 1$. It follows that $I - T(\omega)$ has a bounded inverse operator $(I - T(\omega))^{-1}$, which is a positive operator when $T(t)(t \ge 0)$ is a positive semigroup. Hence we choose $x_0 = (I - T(\omega))^{-1}[\int_0^{\omega} T(\omega - s)h(s)ds + \sum_{k=1}^{p} T(\omega - t_k)y_k] \triangleq B(h)$. Then x_0 is the unique initial value of the IVP(2.7) in X, which satisfies $u(0) = x_0 = u(\omega)$. Combining this fact with (2.8), it follows that (2.6) is satisfied.

Inversely, we can verify directly that the function $u \in \mathrm{PC}(J, X)$ defined by (2.6) is a solution of the linear PBVP(2.5). Therefore, the conclusion of Lemma 2.2 holds. □

Evidently, $\mathrm{PC}(J, X)$ is also an ordered Banach space with the partial order "$\le$" reduced by positive function cone $K_{\mathrm{PC}} := \{u \in \mathrm{PC}(J, X) \mid u(t) \ge 0, t \in J\}$. K_{PC} is also normal with the same normal constant N. For $v, w \in \mathrm{PC}(J, X)$ with $v \le w$, we use $[v, w]$ to denote the order interval $\{u \in \mathrm{PC}(J, X) \mid v \le u \le w\}$ in $\mathrm{PC}(J, X)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in X \mid v(t) \le u \le w(t)\}$ in X. From Lemma 2.2, if $T(t)\ (t \ge 0)$ is a positive C_0-semigroup, $h \ge 0$ and $y_k \ge 0, k = 1, 2, \ldots, p$, then the mild solution $u \in \mathrm{PC}(J, X)$ of the linear PBVP(2.5) satisfies $u \ge 0$.

Proof of Theorem 1.2. We first show that $f(t,h_1(t),h_2(t)) \in L^1(J,X)$ for any $t \in J$ and $h_1(t),h_2(t) \in [v_0(t),w_0(t)]$. Since $v_0(t) \le h_1(t) \le w_0(t),\ \ v_0(t) \le h_2(t) \le w_0(t)$ for any $t \in J$, from the assumption (H_1), we have

$$\begin{aligned}
&f(t,h_1(t),h_2(t)) + (M+L)h_1(t) - Lh_2(t)\\
&\quad \le f(t,w_0(t),v_0(t)) + L(w_0(t)-v_0(t)) + Mw_0(t)\\
&\quad \le w_0'(t) + (A+MI)w_0(t) \triangleq h_0(t),\\
&f(t,h_1(t),h_2(t)) + (M+L)h_1(t) - Lh_2(t)\\
&\quad \ge f(t,v_0(t),w_0(t)) + L(v_0(t)-w_0(t)) + Mv_0(t)\\
&\quad \ge v_0'(t) + (A+MI)v_0(t) \triangleq g_0(t).
\end{aligned} \tag{2.13}$$

Namely, $g_0(t) \le f(t,h_1(t),h_2(t)) + (M+L)h_1(t) - Lh_2(t) \le h_0(t),\ t \in J$. From the normality of cone K in X, we have

$$\left\| f(t,h_1(t),h_2(t)) + (M+L)h_1(t) - Lh_2(t) \right\| \le N\left\| h_0 - g_0 \right\|_{\mathrm{PC}} + \left\| g_0 \right\|_{\mathrm{PC}} \triangleq M^*. \tag{2.14}$$

Combining this fact with the fact that $f(t,h_1(t),h_2(t))$ is strongly measurable, it follows that $f(t,h_1(t),h_2(t)) \in L^1(J,X)$. Therefore, for any $h_1(t),h_2(t) \in [v_0(t),w_0(t)], t \in J$, we consider the periodic boundary value problem(PBVP) of impulsive evolution equation in X

$$\begin{aligned}
u'(t) + (A+MI)u(t) &= G(h_1,h_2)(t), \quad \text{a.e. } t \in J,\\
\Delta u|_{t=t_k} &= I_k(h_1(t_k),h_2(t_k)), \quad k = 1,2,\dots,p,\\
u(0) &= u(\omega),
\end{aligned} \tag{2.15}$$

where $G(h_1,h_2)(t) = f(t,h_1(t),h_2(t)) + (M+L)h_1(t) - Lh_2(t)$. Let $M > 0$ be large enough such that $M > \delta$ (otherwise, replacing M by $M+\delta$, the assumption (H_1) still holds). Then $-(A+MI)$ generates an exponentially stable C_0-semigroup $S(t) = e^{-Mt}T(t)\ (t \ge 0)$. Obviously, $S(t)\ (t \ge 0)$ is a positive C_0-semigroup and $\|S(t)\| \le Ce^{-(M-\delta)t}$ for $t \ge 0$. From Lemma 2.2, the PBVP(2.15) has a unique mild solution $u \in \mathrm{PC}(J,X)$ given by

$$\begin{aligned}
u(t) &= S(t)B(h_1,h_2) + \int_0^t S(t-s)G(h_1,h_2)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(h_1(t_k),h_2(t_k)), \quad t \in J,\\
B(h_1,h_2) &= (I-S(\omega))^{-1}\left[\int_0^\omega S(\omega-s)G(h_1,h_2)(s)ds + \sum_{k=1}^{p} S(\omega-t_k)I_k(h_1(t_k),h_2(t_k))\right].
\end{aligned} \tag{2.16}$$

Let $D = [v_0, w_0]$. We define an operator $Q : D \times D \to \mathrm{PC}(J, X)$ by

$$\begin{aligned} Q(h_1, h_2)(t) = S(t)B(h_1, h_2) &+ \int_0^t S(t-s)G(h_1, h_2)(s)ds \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(h_1(t_k), h_2(t_k)), \quad t \in J. \end{aligned} \tag{2.17}$$

Then the coupled mild ω-periodic L-quasisolution of the PBVP(1.1) is equivalent to the coupled fixed point of operator Q.

Next, we will prove that the operator Q has coupled fixed points on D. For this purpose, we first show that $Q : D \times D \to \mathrm{PC}(J, X)$ is a mixed monotone operator and $v_0 \le Q(v_0, w_0), Q(w_0, v_0) \le w_0$. In fact, for any $t \in J$, $v_0(t) \le u_1(t) \le u_2(t) \le w_0(t)$, $v_0(t) \le v_2(t) \le v_1(t) \le w_0(t)$, from assumptions (H_1) and (H_2), we have

$$\begin{gathered} G(u_1, v_1)(t) \le G(u_2, v_2)(t), \\ I_k(u_1(t_k), v_1(t_k)) \le I_k(u_2(t_k), v_2(t_k)), \quad k = 1, 2, \ldots, p. \end{gathered} \tag{2.18}$$

Since $S(t)(t \ge 0)$ is a positive C_0-semigroup, it follows that $(I - S(\omega))^{-1} = \sum_{n=0}^{\infty} S(n\omega)$ is a positive operator. Then $B(u_1, v_1) \le B(u_2, v_2)$. Hence from (2.17) we see that $Q(u_1, v_1) \le Q(u_2, v_2)$, which implies that Q is a mixed monotone operator. Since

$$\varphi(t) \triangleq v_0'(t) + (A + MI)v_0(t) \le G(v_0, w_0)(t), \quad t \in J, \tag{2.19}$$

from Lemma 2.2 and (1.2), we have

$$\begin{aligned} v_0(t) &= S(t)v_0(0) + \int_0^t S(t-s)\varphi(s)ds + \sum_{0<t_k<t} S(t-t_k)\Delta v_0|_{t=t_k} \\ &\le S(t)v_0(0) + \int_0^t S(t-s)G(v_0, w_0)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(v_0(t_k), w_0(t_k)) \end{aligned} \tag{2.20}$$

for $t \in J$. Especially, we have

$$v_0(\omega) \le S(\omega)v_0(0) + \int_0^\omega S(\omega - s)G(v_0, w_0)(s)ds + \sum_{k=1}^{p} S(\omega - t_k)I_k(v_0(t_k), w_0(t_k)). \tag{2.21}$$

Combining this inequality with $v_0(0) \le v_0(\omega)$, it follows that

$$\begin{aligned} v_0(0) &\le (I - S(\omega))^{-1}\left[\int_0^\omega S(\omega - s)G(v_0, w_0)(s)ds + \sum_{k=1}^{p} S(\omega - t_k)I_k(v_0(t_k), w_0(t_k))\right] \\ &\triangleq B(v_0, w_0). \end{aligned} \tag{2.22}$$

On the other hand, from (2.17), we have

$$Q(v_0, w_0)(t) = S(t)B(v_0, w_0) + \int_0^t S(t-s)G(v_0, w_0)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(v_0(t_k), w_0(t_k)), \quad t \in J. \tag{2.23}$$

Therefore, $Q(v_0, w_0)(t) - v_0(t) \geq S(t)(B(v_0, w_0) - v_0(0)) \geq 0$ for all $t \in J$. It implies that $v_0 \leq Q(v_0, w_0)$. Similarly, we can prove that $Q(w_0, v_0) \leq w_0$.

Now, we define sequences $\{v_n\}$ and $\{w_n\}$ by the iterative scheme

$$v_n = Q(v_{n-1}, w_{n-1}), \quad w_n = Q(w_{n-1}, v_{n-1}), \quad n = 1, 2, \ldots. \tag{2.24}$$

Then from the mixed monotonicity of operator Q, we have

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{2.25}$$

Therefore, for any $t \in J$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are monotone order-bounded sequences in X. Noticing that X is a weakly sequentially complete Banach space, then $\{v_n(t)\}$ and $\{w_n(t)\}$ are relatively compact in X. Combining this fact with the monotonicity of (2.25) and the normality of cone K in X, it follows that $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly convergent in X. Let

$$v^*(t) = \lim_{n\to\infty} v_n(t), \quad w^*(t) = \lim_{n\to\infty} w_n(t), \quad t \in J. \tag{2.26}$$

Then $v^*, w^* : J \to X$ are strongly measurable, and $v_0(t) \leq v^*(t) \leq w^*(t) \leq w_0(t)$ for any $t \in J$. Hence, $v^*, w^* \in L^1(J, X)$.

At last, we show that v^* and w^* are coupled mild ω-periodic L-quasisolutions of the PBVP(1.1). For any $\phi \in X^*$, from subcontinuity of f and continuity of I_k's, there exists $e \subset J$ with mes $e = 0$ such that

$$\begin{gathered} \phi(G(v_n, w_n)(t)) \longrightarrow \phi(G(v^*, w^*)(t)), \quad n \longrightarrow \infty,\ t \in J \setminus e, \\ I_k(v_n(t_k), w_n(t_k)) \longrightarrow I_k(v^*(t_k), w^*(t_k)), \quad n \longrightarrow \infty,\ k = 1, 2, \ldots, p. \end{gathered} \tag{2.27}$$

Hence, for any $t \in J$ and $s \in [0, t] \setminus e$, denote by $S^*(t-s)$ the adjoint operator of $S(t-s)$, then $S^*(t-s) \in X^*$, and

$$\begin{aligned} &\phi[S(t-s)G(v_n, w_n)(s)] = S^*(t-s)\phi(G(v_n, w_n)(s)) \\ &\quad \longrightarrow S^*(t-s)\phi(G(v^*, w^*)(s)) = \phi[S(t-s)G(v^*, w^*)(s)], \quad n \longrightarrow \infty, \\ &\phi\left(\sum_{0<t_k<t} S(t-t_k)I_k(v_n(t_k)), w_n(t_k)\right) \\ &\quad \longrightarrow \phi\left(\sum_{0<t_k<t} S(t-t_k)I_k(v^*(t_k)), w^*(t_k)\right), \quad n \longrightarrow \infty. \end{aligned} \tag{2.28}$$

On the other hand, we have

$$\left|\phi[S(t-s)G(v_n,w_n)(s)]\right| \le \|\phi\|\cdot S(t-s)\cdot\|G(v_n,w_n)(s)\| \le \|\phi\|CM^* \triangleq M^{**}. \tag{2.29}$$

From Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\phi(B(v_n,w_n)) &= \phi\Bigg((I-S(\omega))^{-1}\bigg[\int_0^{\omega} S(\omega-s)G(v_n,w_n)(s)ds\\&\qquad+\sum_{k=1}^{p} S(\omega-t_k)I_k(v_n(t_k),w_n(t_k))\bigg]\Bigg)\\&\longrightarrow \phi\Bigg((I-S(\omega))^{-1}\bigg[\int_0^{\omega} S(\omega-s)G(v^*,w^*)(s)ds\\&\qquad+\sum_{k=1}^{p} S(\omega-t_k)I_k(v^*(t_k),w^*(t_k))\bigg]\Bigg)\\&= \phi(B(v^*,w^*)),\quad n\longrightarrow\infty.\end{aligned} \tag{2.30}$$

Hence, from (2.17), we have

$$\begin{aligned}\phi(v_{n+1}(t)) &= \phi(Q(v_n,w_n)(t)) = \phi(S(t)B(v_n,w_n)) + \phi\left(\int_0^t S(t-s)G(v_n,w_n)(s)ds\right)\\&\quad+\phi\left(\sum_{0<t_k<t} S(t-t_k)I_k(v_n(t_k),w_n(t_k))\right)\\&\longrightarrow \phi(S(t)B(v^*,w^*)) + \phi\left(\int_0^t S(t-s)G(v^*,w^*)(s)ds\right)\\&\quad+\phi\left(\sum_{0<t_k<t} S(t-t_k)I_k(v^*(t_k),w^*(t_k))\right)\\&= \phi\left(S(t)B(v^*,w^*) + \int_0^t S(t-s)G(v^*,w^*)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(v^*(t_k),w^*(t_k))\right)\\&= \phi(Q(v^*,w^*)(t)),\quad n\longrightarrow\infty.\end{aligned} \tag{2.31}$$

On the other hand, it follows from (2.26) that $\lim_{n\to\infty} v_{n+1}(t) = v^*(t)$, $t\in J$. Hence $\phi(v_{n+1}(t)) \to \phi(v^*(t))$ $(n\to\infty)$. By the uniqueness of limits, we can deduce that

$$\phi(Q(v^*,w^*)(t)) = \phi(v^*(t)),\quad t\in J,\ \phi\in X^*. \tag{2.32}$$

By the arbitrariness of $\phi \in X^*$, we have

$$v^* = Q(v^*, w^*). \tag{2.33}$$

Similarly, we can prove that $w^* = Q(w^*, v^*)$. Therefore, (v^*, w^*) is coupled mild ω-periodic L-quasisolution pair of the PBVP(1.1). □

Now, we discuss the existence of mild ω-periodic solutions for the PBVP(1.1) on $[v_0, w_0]$. We assume that the following assumptions are also satisfied:

(H_3) there exists a constant R with $\max\{2L, M + 2L - 1/\omega NC(CM_0 + 1)\} < R \le M + L$ such that

$$f(t, u, v) - f(t, v, u) \le -R(u - v) \tag{2.34}$$

for any $t \in J$, $v_0(t) \le v \le u \le w_0(t)$, where $M_0 = \|(I - S(\omega))^{-1}\|$,

(H_4) there exist positive constants $\tau_k (k = 1, 2, \ldots, p)$ with $\sum_{k=1}^{p} \tau_k < (1 - \omega NC(M + 2L - R)(CM_0 + 1))/CN(CM_0 + 1)$ such that

$$I_k(u, v) - I_k(v, u) \le \tau_k(u - v), \quad k = 1, 2, \ldots, p \tag{2.35}$$

for any $t \in J$, $v_0(t) \le v \le u \le w_0(t)$.

Then we have the following existence and uniqueness result in general ordered Banach space.

Theorem 2.3. *Let X be an ordered Banach space, whose positive cone K is normal, $A : D(A) \subset X \to X$ be a closed linear operator, and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \ge 0)$ in X. If the PBVP(1.1) has coupled lower and upper L-quasisolution v_0 and w_0 with $v_0 \le w_0$, nonlinear term f and impulsive functions I_k's satisfy the following assumptions:*

$(H_1)^*$ *there exist constants $M > 0$ and $0 \le L < min\{M, 1/\omega NC(CM_0 + 1)\}$ such that*

$$f(t, u_2, v_2) - f(t, u_1, v_1) \ge -M(u_2 - u_1) + L(v_2 - v_1) \tag{2.36}$$

for any $t \in J$, and $v_0(t) \le u_1 \le u_2 \le w_0(t), v_0(t) \le v_2 \le v_1 \le w_0(t)$.

And (H_2)–(H_4), then the PBVP(1.1) has a unique mild ω-periodic solution u^* on $[v_0, w_0]$.

Proof. From the proof of Theorem 1.2, when the conditions $(H_1)^*$ and (H_2) are satisfied, the iterative sequences $\{v_n\}$ and $\{w_n\}$ defined by (2.24) satisfy (2.25). We show that there exists a unique $u^* \in \mathrm{PC}(J, X)$ such that $u^* = Q(u^*, u^*)$. For any $t \in J$, from (H_3), (H_4), (2.17), (2.24) and (2.25), we have

$$\begin{aligned}
0 \leq w_n(t) - v_n(t) &= Q(w_{n-1}, v_{n-1})(t) - Q(v_{n-1}, w_{n-1})(t) \\
&= S(t)[B(w_{n-1}, v_{n-1}) - B(v_{n-1}, w_{n-1})] \\
&\quad + \int_0^t S(t-s)[G(w_{n-1}, v_{n-1})(s) - G(v_{n-1}, w_{n-1})(s)]ds \\
&\quad + \sum_{0<t_k<t} S(t-t_k)[I_k(w_{n-1}(t_k), v_{n-1}(t_k)) - I_k(v_{n-1}(t_k), w_{n-1}(t_k))] \\
&\leq S(t)[B(w_{n-1}, v_{n-1}) - B(v_{n-1}, w_{n-1})] \\
&\quad + (M + 2L - R)\int_0^t S(t-s)(w_{n-1}(s) - v_{n-1}(s))ds \\
&\quad + \sum_{0<t_k<t} S(t-t_k)\tau_k(w_{n-1}(t_k) - v_{n-1}(t_k)).
\end{aligned} \tag{2.37}$$

By means of the normality of cone K in X, we have

$$\begin{aligned}
\|w_n(t) - v_n(t)\| &\leq N\Bigg\| S(t)[B(w_{n-1}, v_{n-1}) - B(v_{n-1}, w_{n-1})] \\
&\qquad + (M + 2L - R)\int_0^t S(t-s)(w_{n-1}(s) - v_{n-1}(s))ds \\
&\qquad + \sum_{0<t_k<t} S(t-t_k)\tau_k(w_{n-1}(t_k) - v_{n-1}(t_k))\Bigg\| \\
&\leq NC\|B(w_{n-1}, v_{n-1}) - B(v_{n-1}, w_{n-1})\| \\
&\quad + NC(M + 2L - R)\omega\|w_{n-1} - v_{n-1}\|_{\mathrm{PC}} + NC\sum_{k=1}^{p}\tau_k\|w_{n-1} - v_{n-1}\|_{\mathrm{PC}} \\
&\leq \left[NC\omega(M + 2L - R)(M_0C + 1) + NC\sum_{k=1}^{p}\tau_k(M_0C + 1)\right]\|w_{n-1} - v_{n-1}\|_{\mathrm{PC}}.
\end{aligned} \tag{2.38}$$

Therefore

$$\|w_n - v_n\|_{\mathrm{PC}} \leq \left[NC(M_0C + 1)\left(\omega(M + 2L - R) + \sum_{k=1}^{p}\tau_k\right)\right]\|w_{n-1} - v_{n-1}\|_{\mathrm{PC}}. \tag{2.39}$$

by Repeating the using of the above inequality, we can obtain that

$$\|w_n - v_n\|_{\mathrm{PC}} \leq \left[NC(M_0C+1)\left(\omega(M+2L-R) + \sum_{k=1}^{p}\tau_k\right)\right]^n \|w_0 - v_0\|_{\mathrm{PC}} \longrightarrow 0 \tag{2.40}$$

as $n \to \infty$. Then there exists a unique $u^* \in \mathrm{PC}(J, X)$ such that $\lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n = u^*$. Therefore, let $n \to \infty$ in (2.24), from the continuity of operator Q, we have $u^* = Q(u^*, u^*)$, which means that u^* is a unique mild ω-periodic solution of the PBVP(1.1). □

3. An Example

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Let $J = [0, 2\pi]$, $f_i : \overline{\Omega} \times J \times \mathbb{R} \to \mathbb{R}$, and $I_{k,i} \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2$. Consider the existence of mild solutions for the boundary value problem of parabolic type:

$$\begin{gathered}
\frac{\partial}{\partial t}u - \nabla^2 u = f_1(x,t,u) + f_2(x,t,u), \quad \forall x \in \Omega, \text{ a.e. } t \in J, \\
\Delta u|_{t=t_k} = I_{k,1}(u(x,t_k)) + I_{k,2}(u(x,t_k)), \quad \forall x \in \Omega, \ k = 1,2,\ldots,p, \\
u|_{\partial\Omega} = 0, \\
u(x,0) = u(x,2\pi), \quad x \in \Omega,
\end{gathered} \tag{3.1}$$

where ∇^2 is the Laplace operator, $0 < t_1 < t_2 < \cdots < t_p < 2\pi$. Let $X := L^2(\Omega, \mathbb{R})$ equipped with the L^2-norm $\|\cdot\|_2$, $K := \{u \in X u(x) \geq 0, \text{ a.e. } x \in \Omega\}$. Then K is a generating normal cone in X. Consider the operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \left\{u \in X \mid \nabla^2 u \in X, \ u|_{\partial\Omega} = 0\right\}, \quad Au = -\nabla^2 u. \tag{3.2}$$

Then $-A$ generates an analytic semigroup $T(t)$ $(t \geq 0)$ in X. By the maximum principle of the equations of parabolic type, it is easy to prove that $T(t)$ $(t \geq 0)$ is a positive C_0-semigroup in X. Let λ_1 be the first eigenvalue of operator A and e_1 be a corresponding positive eigenvector. For solving the problem (3.1), the following assumptions are needed.

(i) There exists a constant $L \geq 0$ such that

(a) $f_1(x,t,0) + f_2(x,t,e_1(x)) \geq Le_1(x)$, $x \in \Omega$, $t \in J'$, $I_{k,1}(0) + I_{k,2}(e_1(x)) = 0$, $x \in \Omega$.

(b) $f_1(x,t,e_1(x)) + f_2(x,t,0) \leq (\lambda_1 - L)e_1(x)$, $x \in \Omega$, $t \in J'$, $I_{k,1}(e_1(x)) + I_{k,2}(0) = 0$, $x \in \Omega$.

(ii) (a) The partial derivative of $f_1(x,t,u)$ on u is continuous on any bounded domain.

(b) The partial derivative of $f_2(x,t,u)$ on u has upper bound, and $\sup((\partial/\partial u)f_2(x,t,u)) \leq L$.

(iii) For any $u_1, u_2 \in [0, e_1]$ with $u_1 \leq u_2$, we have

$$I_{k,1}(u_1(x,t_k)) \leq I_{k,1}(u_2(x,t_k)), \quad I_{k,2}(u_2(x,t_k)) \leq I_{k,2}(u_1(x,t_k)), \quad x \in \Omega, \ k = 1,2,\ldots,p. \tag{3.3}$$

Let $f : J \times X \times X \to X$ and $I_k : X \times X \to X$ be defined by $f(t,u,u) = f_1(\cdot,t,u(\cdot)) + f_2(\cdot,t,u(\cdot))$ and by $I_k(u,u) = I_{k,1}(u(\cdot)) + I_{k,2}(u(\cdot))$. Then the problem (3.1) can be transformed into the PBVP(1.1). Assumption (i) implies that $v_0 \equiv 0$ and $w_0 \equiv e_1$ are coupled lower and upper L-quasisolutions of the PBVP(1.1). From assumption (ii)(a), there exists a constant $M > 0$ such that, for any $(x,t) \in \overline{\Omega} \times J$, we have

$$\left|\frac{\partial}{\partial u} f_1(x,t,u)\right| \le M. \tag{3.4}$$

This implies that

$$\left|f_1(x,t,u_2) - f_1(x,t,u_1)\right| = \left|\frac{\partial}{\partial u} f_1(x,t,\xi)(u_2 - u_1)\right| \le M(u_2 - u_1) \tag{3.5}$$

for any $0 \le u_1 \le u_2 \le e_1$ and $\xi \in (u_1,u_2)$. Hence for any $0 \le u_1 \le u_2 \le e_1$ and $\xi \in (u_1,u_2)$, we have

$$f_1(x,t,u_2) - f_1(x,t,u_1) \ge -M(u_2 - u_1). \tag{3.6}$$

Therefore, for any $u_i, v_i \in X$ with $0 \le u_1 \le u_2 \le e_1,\ 0 \le v_2 \le v_1 \le e_1$, from the assumption (ii), we have

$$\begin{aligned}
f(t,u_2,v_2) - f(t,u_1,v_1) &= f_1(\cdot,t,u_2(\cdot)) + f_2(\cdot,t,v_2(\cdot)) - f_1(\cdot,t,u_1(\cdot)) - f_2(\cdot,t,v_1(\cdot)) \\
&= \left[f_1(\cdot,t,u_2(\cdot)) - f_1(\cdot,t,u_1(\cdot))\right] + \left[f_2(\cdot,t,v_2(\cdot)) - f_2(\cdot,t,v_1(\cdot))\right) \\
&\ge -M(u_2(\cdot) - u_1(\cdot)) + \sup \frac{\partial}{\partial u} f_2(\cdot,t,\xi)(v_2(\cdot) - v_1(\cdot)) \\
&\ge -M(u_2 - u_1) + L(v_2 - v_1),
\end{aligned} \tag{3.7}$$

That is, assumption (H_1) is satisfied. From (iii), it is easy to see that assumption (H_2) is satisfied. Therefore, the following result is deduced from Theorem 1.2.

Theorem 3.1. *If the assumptions* (*i*)–(*iii*) *are satisfied, then the problem* (3.1) *has coupled mild ω-periodic L-quasisolution pair on* $[0,e_1]$.

Remark 3.2. In applications of partial differential equations, we often choose Banach space $L^p(1 \le p < \infty)$ as working space, which is weakly sequentially complete. Hence the result in Theorem 1.2 is more valuable in applications. In particular, we obtain a unique mild ω-periodic solution of the PBVP(1.1) in general ordered Banach space in Theorem 2.3.

Remark 3.3. If $L \equiv 0$, then the coupled lower and upper L-quasisolutions are equivalent to coupled lower and upper quasisolutions of the PBVP(1.1). Since condition (H_1) contains conditions (F_1) and (F_2), even without impulse in PBVP(1.1), the results in this paper still extend the results in [10, 11].

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