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Research Article

The Best Approximation of the Sinc Function by a Polynomial of Degree n with the Square Norm

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The polynomial of degree n which is the best approximation of the sinc function on the interval $(0, \pi/2]$ with the square norm is considered. By using Lagrange's method of multipliers, we construct the polynomial explicitly. This method is also generalized to the continuous function on the closed interval $[a, b]$. Numerical examples are given to show the effectiveness.

1. Introduction

Let $\sin c(x) = (\sin x)/x$ be the sinc function; the following result is known as Jordan inequality [1]:

$$\frac{2}{\pi} \leq \sin c(x) < 1, 0 < x \leq \frac{\pi}{2}, \tag{1.1}$$

where the left-handed equality holds if and only if $x = \pi/2$. This inequality has been further refined by many scholars in the past few years [2–30]. Özban [12] presented a new lower bound for the sinc function and obtained the following inequality:

$$\frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \frac{4(\pi - 3)}{\pi^3}\left(x - \frac{\pi}{2}\right)^2 \leq \sin \mathrm{c}(x). \tag{1.2}$$

The above inequality was generalized to an upper bound by Zhu [26]:

$$\sin \mathrm{c}(x) \leq \frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \frac{12 - \pi^2}{\pi^3}\left(x - \frac{\pi}{2}\right)^2. \tag{1.3}$$

Later, Agarwal and his collaborators [2] proposed a more refined two-sided inequality:

$$1-\frac{4\left(-66+43\pi-7\pi^2\right)}{\pi^2}x-\frac{4\left(124-83\pi+14\pi^2\right)}{\pi^3}x^2-\frac{4(12-4\pi)}{\pi^4}x^3$$
$$\leq \sin\mathrm{c}(x)\leq 1-\frac{4\left(-75+49\pi-8\pi^2\right)}{\pi^2}x+\frac{4\left(-142+95\pi-16\pi^2\right)}{\pi^3}x^2-\frac{4(12-4\pi)}{\pi^4}x^3, \tag{1.4}$$

where the two-sided equalities hold if x tends to zero or $x=\pi/2$.

Note that the bounds of the sinc function $\mathrm{sinc}(x)$ listed above are estimated by the given polynomials with the boundary constraints; the smaller the residual between the polynomial and the sinc function is, the more refined the estimation will be. Hence, our aim is to seek a polynomial of degree n, $p_n(x)$, which is the best approximation of the sinc function with the square norm. In view of that, the sinc function is defined on $(0,\pi/2]$ and two boundary constrained conditions are imposed. So we want to solve the following minimum problem:

$$\min_{p_n(x)\in\mathcal{P}_n}\left(\int_0^{\pi/2}\left(\sin\mathrm{c}(x)-p_n(x)\right)^2\mathrm{d}x\right)^{1/2}$$
$$\text{s.t.}\quad \lim_{x\to 0^+}p_n(x)=\lim_{x\to 0^+}\sin\mathrm{c}(x),\qquad \lim_{x\to\pi/2}p_n(x)=\lim_{x\to\pi/2}\sin c(x), \tag{1.5}$$

where $\mathcal{P}_n$ is the set of the polynomial of degree n and it is denoted by

$$P_n=\left\{p_n\mid p_n(x)=a_0+a_1x+\cdots+a_nx^n,\, a_i\in R, i=1,2,\ldots,n\right\} \tag{1.6}$$

In this paper, an explicit representation for the approximating polynomial of $\mathrm{sinc}(x)$ is presented by using Lagrange's method of multipliers, and numerical examples are given to show the effectiveness. Moreover, this method can be generalized to the continuous function $g(x)$ on the closed interval $[a,b]$. However, the residual error between the approximating polynomial $p_n(x)$ and $g(x)$ is concussive, that is, it cannot keep positive or negative always.

The rest of paper is organized as follows. In Section 2, we solve the problem (5) by Lagrange's method of multipliers and this method is generalized to a continuous function on $[a,b]$ in Section 3. Numerical examples are given in Section 4 to display the effectiveness of our estimations.

2. The Best Approximation of the Sinc Function by a Polynomial of Degree n on $(0,\pi/2]$

Obviously, the constraints of (1.5) imply

$$a_0=1,\qquad p_n\left(\frac{\pi}{2}\right)=\frac{2}{\pi}. \tag{2.1}$$

Note that

$$\int_0^{\pi/2}(\sin c(x)-p_n(x))^2 dx=\int_0^{\pi/2}\Bigg(\sin^2 c(x)+1-2\sin c(x)-2\sum_{i=1}^n a_i x^{i-1}\sin x+2\sum_{i=1}^n a_i x^i$$
$$+2\sum_{1\le i<j\le n}^n a_i a_j x^{i+j}+\sum_{i=1}^n a_i^2 x^{2i}\Bigg)dx$$
$$=\int_0^{\pi/2}\left(\sin^2 c(x)+1-2\sin c(x)-2\sum_{i=1}^n a_i x^{i-1}\sin x\right)dx$$
$$+\sum_{i=1}^n\frac{2a_i}{i+1}\left(\frac{\pi}{2}\right)^{i+1}+\sum_{1\le i<j\le n}\frac{2a_i a_j}{i+j+1}\left(\frac{\pi}{2}\right)^{i+j+1}$$
$$+\sum_{i=1}^n\frac{a_i^2}{2i+1}\left(\frac{\pi}{2}\right)^{2i+1}. \tag{2.2}$$

Denote

$$G(a_1,\ldots,a_n)=h+\sum_{1\le i<j\le n}\frac{2a_i a_j}{i+j+1}\left(\frac{\pi}{2}\right)^{i+j+1}+\sum_{i=1}^n\frac{a_i^2}{2i+1}\left(\frac{\pi}{2}\right)^{2i+1} \tag{2.3}$$

with

$$h=\int_0^{\pi/2}\left(-2\sum_{i=1}^n a_i x^{i-1}\sin x\right)dx+\sum_{i=1}^n\frac{2a_i}{i+1}\left(\frac{\pi}{2}\right)^{i+1}, \tag{2.4}$$

where $a_i\in\mathcal{R}, i=1,2,\ldots,n$. So (1.5) is equivalent to solving the following minimum problem:

$$\min_{a_i\in\mathcal{R}} G(a_1,\ldots,a_n)$$
$$\text{s.t.}\quad a_1\frac{\pi}{2}+\cdots+a_n\left(\frac{\pi}{2}\right)^n=\frac{2}{\pi}-1. \tag{2.5}$$

This can be solved by using Lagrange's method of multipliers. We construct the Lagrange function by

$$L(a_1,a_2,\ldots,a_n,\lambda)=G(a_1,\ldots,a_n)+\lambda\left(a_1\frac{\pi}{2}+\cdots+a_n\left(\frac{\pi}{2}\right)^n-\frac{2}{\pi}+1\right) \tag{2.6}$$

with Lagragian multiplier λ. Thus we need to equate to zero the partial derivatives of L with respect to each $a_j (j = 1, 2, \dots, n)$ and λ, that is,

$$\frac{\partial L}{\partial a_j} = 0, \quad j = 1, \dots, n,$$
$$a_1 \frac{\pi}{2} + \cdots + a_n \left(\frac{\pi}{2}\right)^n - \frac{2}{\pi} + 1 = 0. \tag{2.7}$$

It gives a system of linear equations

$$Au = f, \tag{2.8}$$

where

$$A = \begin{pmatrix} \frac{2}{3}\left(\frac{\pi}{2}\right)^3 & \frac{2}{4}\left(\frac{\pi}{2}\right)^4 & \cdots & \frac{2}{n+2}\left(\frac{\pi}{2}\right)^{n+2} & \frac{\pi}{2} \\ \frac{2}{4}\left(\frac{\pi}{2}\right)^4 & \frac{2}{5}\left(\frac{\pi}{2}\right)^5 & \cdots & \frac{2}{n+3}\left(\frac{\pi}{2}\right)^{n+3} & \left(\frac{\pi}{2}\right)^2 \\ \frac{2}{5}\left(\frac{\pi}{2}\right)^5 & \frac{2}{6}\left(\frac{\pi}{2}\right)^6 & \cdots & \frac{2}{n+4}\left(\frac{\pi}{2}\right)^{n+4} & \left(\frac{\pi}{2}\right)^3 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{2}{n+2}\left(\frac{\pi}{2}\right)^{n+2} & \frac{2}{n+3}\left(\frac{\pi}{2}\right)^{n+3} & \cdots & \frac{2}{2n+1}\left(\frac{\pi}{2}\right)^{2n+1} & \left(\frac{\pi}{2}\right)^n \\ \frac{\pi}{2} & \left(\frac{\pi}{2}\right)^2 & \cdots & \left(\frac{\pi}{2}\right)^n & 0 \end{pmatrix}, \tag{2.9}$$

$$u = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \lambda \end{pmatrix}, \qquad f = -\begin{pmatrix} \frac{\partial h}{\partial a_1} \\ \frac{\partial h}{\partial a_2} \\ \vdots \\ \frac{\partial h}{\partial a_n} \\ 1 - \frac{2}{\pi} \end{pmatrix}. \tag{2.10}$$

To consider the consistence of the equations (2.8), we introduce the following lemma for the square matrix A of order $n+1$.

Lemma 2.1. *The square matrix A of order $n+1$ defined by* (2.9) *is nonsingular.*

Proof. We want to prove that $\det(A) \neq 0$. Note that

$$
\begin{aligned}
\det(A) &= \left(\frac{\pi}{2}\right)^{3+4+\cdots+(n+2)+1} \det\begin{pmatrix}
\frac{2}{3} & \frac{2}{4}\left(\frac{\pi}{2}\right)^{1} & \cdots & \frac{2}{n+2}\left(\frac{\pi}{2}\right)^{n-1} & \left(\frac{\pi}{2}\right)^{-2} \\
\frac{2}{4} & \frac{2}{5}\left(\frac{\pi}{2}\right)^{1} & \cdots & \frac{2}{n+3}\left(\frac{\pi}{2}\right)^{n-1} & \left(\frac{\pi}{2}\right)^{-2} \\
\frac{2}{5} & \frac{2}{6}\left(\frac{\pi}{2}\right)^{1} & \cdots & \frac{2}{n+4}\left(\frac{\pi}{2}\right)^{n-1} & \left(\frac{\pi}{2}\right)^{-2} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\frac{2}{n+2} & \frac{2}{n+3}\left(\frac{\pi}{2}\right)^{1} & \cdots & \frac{2}{2n+1}\left(\frac{\pi}{2}\right)^{n-1} & \left(\frac{\pi}{2}\right)^{-2} \\
1 & \left(\frac{\pi}{2}\right)^{1} & \cdots & \left(\frac{\pi}{2}\right)^{n-1} & 0
\end{pmatrix} \\
&= \left(\frac{\pi}{2}\right)^{(n+5)n/2+(n-1)n/2-1} \det\begin{pmatrix}
\frac{2}{3} & \frac{2}{4} & \cdots & \frac{2}{n+2} & 1 \\
\frac{2}{4} & \frac{2}{5} & \cdots & \frac{2}{n+3} & 1 \\
\frac{2}{5} & \frac{2}{6} & \cdots & \frac{2}{n+4} & 1 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\frac{2}{n+2} & \frac{2}{n+3} & \cdots & \frac{2}{2n+1} & 1 \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix} \\
&= \left(\frac{\pi}{2}\right)^{n(n+2)-1} \det\begin{pmatrix} 2H_n & e \\ e^T & 0 \end{pmatrix},
\end{aligned}
\tag{2.11}
$$

where

$$H_n = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\ \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{n+4} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{pmatrix}, \qquad e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.12}$$

Since H_n is one n-order principal square submatrix of $(n+2)$-order Hilbert matrix, together with Hilbert matrix being positive definite [31, volume 1, page 401], then H_n is also positive definite. Hence, H_n^{-1} exists and it is positive definite, which implies $e^T H_n^{-1} e \neq 0$. Moreover,

$$\det\begin{pmatrix} 2H_n & e \\ e^T & 0 \end{pmatrix} = \det\begin{pmatrix} 2H_n & e \\ 0 & -\frac{1}{2}e^T H_n^{-1} e \end{pmatrix}. \tag{2.13}$$

So, $\det\begin{pmatrix} 2H_n & e \\ e^T & 0 \end{pmatrix} \neq 0$, that is, A is nonsingular. □

Because A is nonsingular, the solution of the equations (2.8) exists and is unique, as well as the best approximation of $\sin c(x)$ by a polynomial of degree n. Therefore, we obtain the following theorem.

Theorem 2.2. *Let $0 < x \leq \pi/2$; suppose the matrix A and vector f are denoted by* (2.9). *Then the best approximation of* $\sin c(x)$ *by a polynomial of degree n on interval* $(0, \pi/2]$ *with the square norm is given by*

$$p_n(x) = 1 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n, \tag{2.14}$$

where $a_1, \ldots, a_n$ is the 1,2, $\ldots$ n-th components of the vector $A^{-1}f$.

3. The Best Approximation of the Continuous Function $g(x)$ by a Polynomial of Degree n on $[a, b]$

In this section, we generalize the above conclusion to the continuous function $g(x)$ on interval $[a, b]$, that is, we want to consider the following minimum problem:

$$\min_{p_n(x)\in\mathcal{P}_n} \left(\int_a^b (g(x) - p_n(x))^2 \mathrm{d}x \right)^{1/2} \tag{3.1}$$

with the constraints

$$p_n(a) = g(a), \qquad p_n(b) = g(b), \tag{3.2}$$

where the polynomial $p_n(x)$ is rewritten as

$$p_n(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n \tag{3.3}$$

and $\mathcal{P}_n$ is defined by (1.6). If we set $t = x - a$, problem (3.1) is equivalent to

$$\min_{\tilde{p}_n(t)\in\mathcal{P}_n} \left(\int_0^{b-a} \left(g(t+a) - \tilde{p}_n(t)\right)^2 \mathrm{d}t \right)^{1/2} \tag{3.4}$$

with

$$a_0 = g(a), \qquad \tilde{p}_n(b-a) = g(b), \tag{3.5}$$

where

$$\tilde{p}_n(t) = a_0 + a_1 t + \cdots + a_n t^n. \tag{3.6}$$

If we replace $a_0 = 1$, $\pi/2$, $\sin c(x)$, $p_n(x)$ in Section 2 by $a_0 = g(a)$, $b - a$, $g(x)$, and $\tilde{p}_n(t)$, respectively, then (2.4) is rewritten as

$$h = \int_a^b \left(-2\sum_{i=1}^{n} a_i g(x)(x-a)^i \right) \mathrm{d}x + \sum_{i=1}^{n} \frac{2a_i g(a)}{i+1}(b-a)^{i+1}, \tag{3.7}$$

$$A = \begin{pmatrix} \frac{2(b-a)^3}{3} & \frac{2(b-a)^4}{4} & \cdots & \frac{2(b-a)^{n+2}}{n+2} & (b-a) \\ \frac{2(b-a)^4}{4} & \frac{2(b-a)^5}{5} & \cdots & \frac{2(b-a)^{n+3}}{n+3} & (b-a)^2 \\ \frac{2(b-a)^5}{5} & \frac{2(b-a)^6}{6} & \cdots & \frac{2(b-a)^{n+4}}{n+4} & (b-a)^3 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{2(b-a)^{n+2}}{n+2} & \frac{2(b-a)^{n+3}}{n+3} & \cdots & \frac{2(b-a)^{2n+1}}{2n+1} & (b-a)^n \\ (b-a) & (b-a)^2 & \cdots & (b-a)^n & 0 \end{pmatrix}, \qquad f = -\begin{pmatrix} \frac{\partial h}{\partial a_1} \\ \frac{\partial h}{\partial a_2} \\ \vdots \\ \frac{\partial h}{\partial a_n} \\ g(a) - g(b) \end{pmatrix}. \tag{3.8}$$

So we have the following theorem.

Theorem 3.1. *Let $g(x)$ be continuous on $[a,b]$, and we denote the matrix A and f by* (3.8)*. Then the best approximation of $g(x)$ by the polynomial of degree n on $[a,b]$ with the square norm is given by*

$$p_n(x) = g(a) + a_1(x-a) + \cdots + a_{n-1}(x-a)^{n-1} + a_n(x-a)^n, \tag{3.9}$$

where $a_1,\ldots,a_n$ is the 1,2, ... n-th components of the vector $A^{-1}f$.

Remark 3.2. The interval $[a,b]$ in Theorem 3.1 can be generalized to (a,b), where

$$\lim_{x\to a^+} g(x), \quad \lim_{x\to b^-} g(x) \quad \text{both exist.} \tag{3.10}$$

4. Numerical Examples

In this section, we present some numerical examples to illustrate the effectiveness of our methods based on Theorems 2.2 and 3.1. For any function $g(x)$, two kinds of errors are used as measures of accuracy. One is the residual error

$$\epsilon_{g(x)-p_n} = g(x) - p_n(x). \tag{4.1}$$

The other is the integration error

$$\epsilon^{\text{int}}_{g(x)-p_n} = \left(\int_a^b (g(x)-p_n(x))^2 \mathrm{d}x\right)^{1/2}. \tag{4.2}$$

Example 4.1. Let $a = 0$, $b = \pi/2$, and $g(x) = \sin c(x)$; we compare the approximation effectiveness between the approximating polynomial of degree 3 and $\sin c(x)$ by Theorem 2.2 and that in [2]. Denote the left-handed polynomial in inequality (1.4) by $p_3^l(x)$, and the right-handed one by $p_3^r(x)$, that is,

$$\begin{aligned} p_3^l(x) &= 1 - \frac{4(-66+43\pi-7\pi^2)}{\pi^2}x - \frac{4(124-83\pi+14\pi^2)}{\pi^3}x^2 - \frac{4(12-4\pi)}{\pi^4}x^3, \\ p_3^r(x) &= 1 - \frac{4(-75+49\pi-8\pi^2)}{\pi^2}x + \frac{4(-142+95\pi-16\pi^2)}{\pi^3}x^2 - \frac{48-16\pi}{\pi^4}x^3. \end{aligned} \tag{4.3}$$

With Theorem 2.2, it is easy to compute that

$$\begin{aligned} p_3(x) = 1 &- \frac{2(13440-1440\pi-960\pi^2-4\pi^3+7\pi^4)}{\pi^5}x \\ &+ \frac{4(40320-4800\pi-2640\pi^2-16\pi^3+13\pi^4)}{\pi^6}x^2 \\ &- \frac{56(3840-480\pi-240\pi^2-2\pi^3+\pi^4)}{\pi^7}x^3. \end{aligned} \tag{4.4}$$

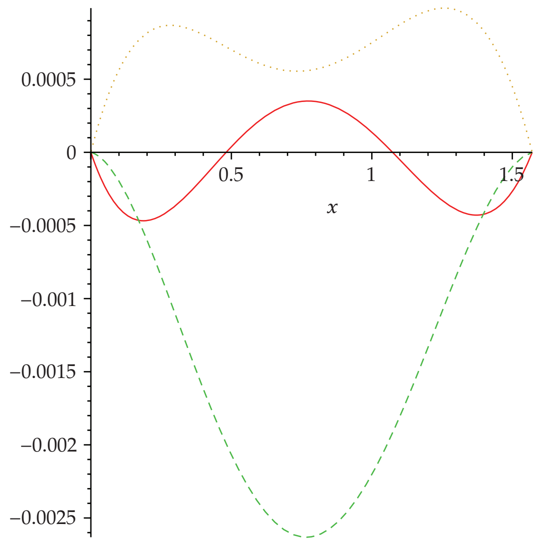

Figure 1: The residual errors between the approximating polynomial of degree 3 and $\sin c(x)$ with the square norm, where we denote the yellow dotted line by $\epsilon_{\sin c(x)-p_3^l}$, green dash line by $\epsilon_{\sin c(x)-p_3^r(x)}$, and red line by $\epsilon_{\sin c(x)-p_3}$.

Table 1: The residual error $\epsilon_{\sin\, c(x)-p_n}$ and integration error $\epsilon^{int}_{\sin\, c(x)-p_n}$ between the approximating polynomial of degree n and $\sin c(x)$ with the square norm on interval $(0, \pi/2]$, where $n = 2, 3, 4, 5$.

n	maximal $\epsilon_{\sin c(x)-p_n}$	minimal $\epsilon_{\sin c(x)-p_n}$	$\epsilon^{\text{int}}_{\sin c(x)-p_n}$
2	$4.12 * 10^{-3}$	$-4.73 * 10^{-3}$	$3.97 * 10^{-3}$
3	$3.51 * 10^{-4}$	$-4.68 * 10^{-4}$	$4.59 * 10^{-4}$
4	$3.28 * 10^{-5}$	$-2.16 * 10^{-5}$	$5.08 * 10^{-4}$
5	$1.72 * 10^{-6}$	$-1.16 * 10^{-6}$	$5.12 * 10^{-4}$

We plot the residual error for $p_3^l(x)$, $p_3^r(x)$, and $p_3(x)$, respectively. In Figure 1, we will find that the total error of $p_3(x)$ is smaller than that of $p_3^l(x)$ and $p_3^r(x)$. However, the curve of $\epsilon_{\sin\, c(x)-p_3}$ is concussive at $y = 0$.

Example 4.2. In this example, we consider the residual error $\epsilon_{g(x)-p_n}$ and integration error $\epsilon^{\text{int}}_{g(x)-p_n}$ for $n = 2, 3, 4, 5$ with $g(x) = \sin c(x)$ and interval $(0, \pi/2]$. In Table 1, we will find that the order of the residual errors $\epsilon_{\sin\, c(x)-p_n}$ will decrease with increasing n. However, the precision of integration error $\epsilon^{\text{int}}_{\sin\, c(x)-p_n}$ can remain 10^{-4} when $n = 3, 4, 5$.

Example 4.3. In this example, let $g(x) = \cos x$ and the interval be $[0, \pi]$; we consider its approximating polynomial of degree 3: $p_3(x)$. By Theorem 3.1, we have

$$p_3(x) = 1 - \frac{3\left(140\pi^2 + 3\pi^4 - 1680\right)}{\pi^5}x - \frac{21\left(60\pi^2 + \pi^4 - 720\right)}{\pi^6}x^2 + \frac{14\left(60\pi^2 + \pi^4 - 720\right)}{\pi^7}x^3, \tag{4.5}$$

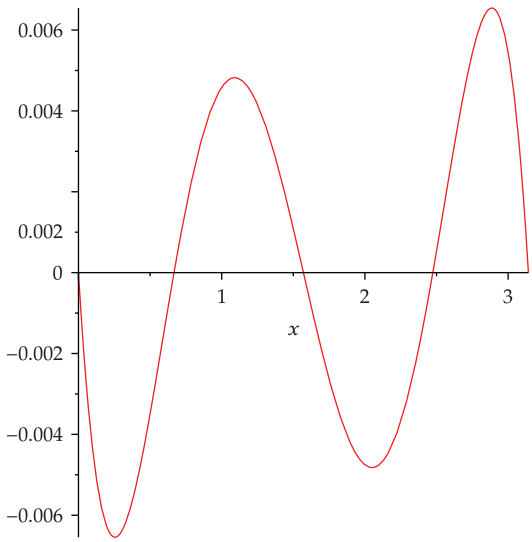

Figure 2: The residual error $\epsilon_{\cos x - p_3(x)}$ between $\cos x$ and $p_3(x)$ on $[0, \pi]$.

and the residual error $\epsilon_{\cos x - p_3}$ can be represented by Figure 2 . Obviously, the curve is concussive; however, the residual error can reach 10^{-3}.

Example 4.4. Let $g(x) = \sin x$ and the interval be $[\pi/2, \pi]$; we consider its approximating polynomial of degree 4 ($p_4(x)$) by Theorem 3.1. It is easy to verify

$$\begin{aligned} p_4(x) = 1 &- \frac{23\pi^5 + 8400\pi^3 - 127680\pi^2 - 1532160\pi + 5806080}{\pi^6}\left(x - \frac{\pi}{2}\right) \\ &+ \frac{14(11\pi^5 + 6000\pi^3 - 110400\pi^2 - 1209600\pi + 4700160)}{\pi^7}\left(x - \frac{\pi}{2}\right)^2 \\ &- \frac{56(7\pi^5 + 4560\pi^3 - 95040\pi^2 - 979200\pi + 3870720)}{\pi^8}\left(x - \frac{\pi}{2}\right)^3 \\ &+ \frac{336(\pi^5 + 720\pi^3 - 16320\pi^2 - 161280\pi + 645120)}{\pi^9}\left(x - \frac{\pi}{2}\right)^4. \end{aligned} \tag{4.6}$$

We plot the residual error $\epsilon_{\sin x - p_4(x)}$ in Figure 3, where we find it can reach 10^{-4}.

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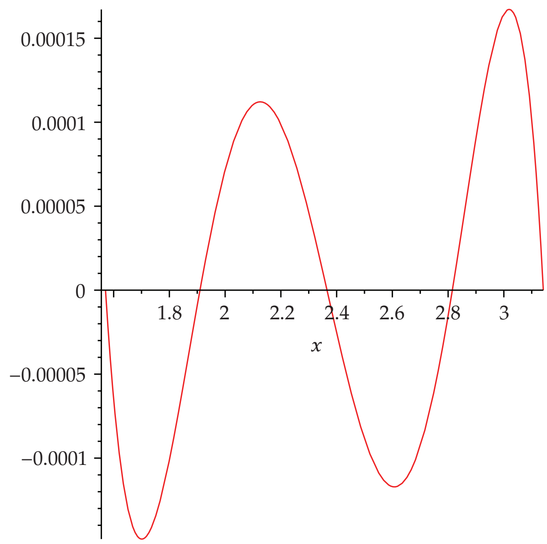

Figure 3: The residual error $\epsilon_{\sin x - p_4(x)}$ between $\sin x$ and $p_4(x)$ on $[\pi/2, \pi]$.

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