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Research Article

Volterra Discrete Inequalities of Bernoulli Type

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We obtain the discrete versions of integral inequalities of Bernoulli type obtained in Choi (2007) and give an application to study the boundedness of solutions of nonlinear Volterra difference equations.

1. Introduction

Integral inequalities of Gronwall type have been very useful in the study of ordinary differential equations. Sugiyama [1] proved the discrete analogue of the well-known Gronwall-Bellman inequality [2–5] which find numerous applications in the theory of finite difference equations. See [6–11] for differential inequalities and difference inequalities.

Willett and Wong [12] established some discrete generalizations of the results of Gronwall [5]. The discrete analogue of the result of Bihari [13] was partially given by Hull and Luxemburg [14] and was used by them for the numerical treatment of ordinary differential equations. Pachpatte [15] obtained some general versions of Gronwall-Bellman inequality. Oguntuase [16] established some generalizations of the inequalities obtained in [15]. However, there were some defects in the proofs of Theorems 2.1 and 2.7 in [16]. Choi et al. [17] improved the results of [16] and gave an application to boundedness of the solutions of nonlinear integrodifferential equations.

In this paper, we establish the discrete analogues of integral inequalities of Bernoulli type in [17] and give an application to study the boundedness of solutions of nonlinear Volterra difference equations.

2. Main Results

Pachpatte [11] proved the following useful discrete inequality which can be used in the proof of various discrete inequalities. Let $\mathbb{N}(n_0) = \{n_0, n_0+1, \ldots, n_0+k, \ldots\}$ and $\mathbb{N}(n_0, l) = \{n_0, n_0+1, \ldots, n_0+k, \ldots, l\}$ for fixed nonnegative integers n_0 and l.

Lemma 2.1 (see [11, Theorem 2.3.4]). *Let $u(n)$ be a positive sequence defined on $\mathbb{N}(n_0)$, and let $b(n)$ and $k(n)$ be nonnegative sequences defined on $\mathbb{N}(n_0)$. Suppose that*

$$\Delta u(n) \le b(n)u(n) + k(n)u^p(n), \quad n \in \mathbb{N}(n_0), \tag{2.1}$$

where $p \ge 0$, $p \ne 1$ is a constant. Then, one has

$$u(n) \le \frac{1}{e(n)}\left[u^{1-p}(n_0) + (1-p)\sum_{s=n_0}^{n-1} k(s)e^{1-p}(s+1)\right]^{1/(1-p)}, \quad n \in \mathbb{N}(n_0, \beta), \tag{2.2}$$

where $e(n) = \prod_{s=n_0}^{n-1}[1+b(s)]^{-1}$ and

$$\beta = \sup\left\{n \in \mathbb{N}(n_0) : u^{1-p}(n_0) + (1-p)\sum_{s=n_0}^{n-1} k(s)e^{1-p}(s+1) > 0\right\}. \tag{2.3}$$

If we set $b(n) = 0$ in Lemma 2.1, then we can obtain the following corollary.

Corollary 2.2. *Suppose that*

$$\Delta u(n) \le k(n)u^p(n), \quad n \in \mathbb{N}(n_0),\ p \ge 0,\ p \ne 1. \tag{2.4}$$

Then,

$$u(n) \le \left[u^{1-p}(n_0) + (1-p)\sum_{s=n_0}^{n-1} k(s)\right]^{1/(1-p)}, \quad n \in \mathbb{N}(n_0, \beta), \tag{2.5}$$

where

$$\beta = \sup\left\{n \in \mathbb{N}(n_0) : u^{1-p}(n_0) + (1-p)\sum_{s=n_0}^{n-1} k(s) > 0\right\}. \tag{2.6}$$

Willet and Wong [12, Theorem 4] proved the nonlinear difference inequality by using the mean value theorem. We obtain the following result which is slightly different from Willet and Wong's Theorem 4.

Theorem 2.3. *Let $u(n), b(n), k(n)$ be nonnegative sequences defined on $\mathbb{N}(n_0)$ and $b(n) < 1$ for each $n \in \mathbb{N}(n_0)$. Suppose that*

$$u(n) \le u_0 + \sum_{s=n_0}^{n-1} b(s)u(s) + \sum_{s=n_0}^{n-1} k(s)u^p(s), \quad n \in \mathbb{N}(n_0), \tag{2.7}$$

where $p \ge 0$, $p \ne 1$ and u_0 is a positive constant. Then one has

$$u(n) \le \frac{1}{e_{-b}(n)} \left(u_0^{1-p} + (1-p) \sum_{s=n_0}^{n-1} k(s) e_{-b}^{1-p}(s) \right)^{1/(1-p)}, \quad n \in \mathbb{N}(n_0, \beta), \tag{2.8}$$

where

$$e_{-b}(n) = \prod_{s=n_0}^{n-1} (1 - b(s)), \quad \beta = \sup\left\{ n \in \mathbb{N}(n_0) : u_0^{1-p} + (1-p) \sum_{s=n_0}^{n-1} k(s) e_{-b}^{1-p}(s) > 0 \right\}. \tag{2.9}$$

Proof. Let the right hand of (2.7) denote by

$$w(n) = u_0 + \sum_{s=n_0}^{n-1} b(s)u(s) + \sum_{s=n_0}^{n-1} k(s)u^p(s). \tag{2.10}$$

Then, we have

$$\begin{aligned} \Delta w(n) &= b(n)u(n) + k(n)u^p(n) \\ &\le b(n)w(n) + k(n)w^p(n) \\ &\le b(n)w(n+1) + k(n)w^p(n), \quad n \in \mathbb{N}(n_0), \end{aligned} \tag{2.11}$$

since $w(n)$ is nondecreasing. Multiplying (2.11) by the factor $e_{-b}(n)$, we obtain

$$\begin{aligned} \Delta(e_{-b}(n)w(n)) &= e_{-b}(n)\Delta w(n) - b(n)e_{-b}(n)w(n+1) \\ &\le k(n)e_{-b}(n)w^p(n) \\ &= k(n)e_{-b}^{1-p}(n)(e_{-b}(n)w(n))^p, \quad n \in \mathbb{N}(n_0), \end{aligned} \tag{2.12}$$

since $e_{-b}(n)$ is a positive sequence on $\mathbb{N}(n_0)$. From Corollary 2.2, we obtain

$$e_{-b}(n)w(n) \le \left[u_0^{1-p} + (1-p) \sum_{s=n_0}^{n-1} k(s) e_{-b}^{1-p}(s) \right]^{1/(1-p)}, \quad n \in \mathbb{N}(n_0, \beta), \tag{2.13}$$

where

$$\beta = \sup\left\{ n \in \mathbb{N}(n_0) : u_0^{1-p} + (1-p)\sum_{s=n_0}^{n-1} k(s) e_{-b}^{1-p}(s) > 0 \right\}. \tag{2.14}$$

Since $u(n) \le w(n)$ and $e_{-b}(n) > 0$, this implies that our inequality holds. □

Remark 2.4. Note that (2.7) with $p = 1$ in Theorem 2.3 implies

$$\begin{aligned} u(n) &\le u_0 \prod_{s=n_0}^{n-1} (1 + b(s) + k(s)) \\ &\le u_0 \exp\left[\sum_{s=n_0}^{n-1} (b(s) + k(s))\right], \quad n \in \mathbb{N}(n_0). \end{aligned} \tag{2.15}$$

Hence, we can obtain a comparison result for linear difference inequalities.

The following theorem can be regarded as an extension of the inequality given by Willett and Wong in [12] which is the discrete analogue of the inequality given by Choi et al. in [17, Theorem 2.7].

Theorem 2.5. *Let $u(n)$ and $b(n) < 1$ be nonnegative sequences defined on $\mathbb{N}(n_0)$, and let $k(n,m)$ be a nonnegative function for $n, m \in \mathbb{N}(n_0)$ with $n \ge m$. Suppose that*

$$u(n) \le c + \sum_{s=n_0}^{n-1} b(s)\left[u(s) + \sum_{\tau=n_0}^{s-1} k(s,\tau) u^p(\tau)\right], \quad n \in \mathbb{N}(n_0), \tag{2.16}$$

where c is a positive constant and $p > 0$, $p \ne 1$ is a constant. Then, one has

$$u(n) \le c + \sum_{s=n_0}^{n-1} \frac{b(s)}{e_{-b}(s)}\left[c^q + q\sum_{\tau=n_0}^{s-1} e_{-b}^q(\tau)\left(k(\tau+1,\tau) + \sum_{\sigma=n_0}^{\tau-1} |\Delta_\tau k(\tau,\sigma)|\right)\right]^{1/q}, \quad n \in \mathbb{N}(n_0,\beta), \tag{2.17}$$

where $q = 1-p$, $e_{-b}(n) = \prod_{s=n_0}^{n-1}(1-b(s))$, $\Delta_n k(n,m) = k(n+1,m) - k(n,m)$, and

$$\beta = \sup\left\{ n \in \mathbb{N}(n_0) : c^q + q\sum_{s=n_0}^{n-1} e_{-b}^q(s)\left[k(s+1,s) + \sum_{\tau=n_0}^{s-1} |\Delta_s k(s,\tau)|\right] > 0 \right\}. \tag{2.18}$$

Proof. Define $v(n)$ by the right member of (2.16). Then,

$$\begin{aligned} \Delta v(n) &= b(n)u(n) + b(n)\sum_{s=n_0}^{n-1} k(n,s) u^p(s) \\ &\le b(n)v(n) + b(n)\sum_{s=n_0}^{n-1} k(n,s) v^p(s), \end{aligned} \tag{2.19}$$

by $u(n) \le v(n)$ and $u^p(n) \le v^p(n)$ for $0 \le p \ne 1$. Letting

$$w(n) = v(n) + \sum_{s=n_0}^{n-1} k(n,s)v^p(s), \qquad w(n_0) = v(n_0) = c, \tag{2.20}$$

we obtain

$$\begin{aligned}\Delta w(n) &= \Delta v(n) + k(n+1,n)v^p(n) + \sum_{s=n_0}^{n-1} \Delta_n k(n,s)v^p(s) \\ &\le b(n)w(n) + k(n+1,n)w^p(n) + \sum_{s=n_0}^{n-1} |\Delta_n k(n,s)|w^p(s) \\ &\le b(n)w(n) + \left[k(n+1,n) + \sum_{s=n_0}^{n-1} |\Delta_n k(n,s)|\right]w^p(n),\end{aligned} \tag{2.21}$$

for each $n \in \mathbb{N}(n_0)$. By Theorem 2.3, we have

$$w(n) \le \frac{1}{e_{-b}(n)}\left[c^{1-p} + (1-p)\sum_{s=n_0}^{n-1} e_{-b}^{1-p}(s)\left(k(s+1,s) + \sum_{\tau=n_0}^{s-1} |\Delta_s k(s,\tau)|\right)\right]^{1/(1-p)}, \quad n \in \mathbb{N}(n_0,\beta), \tag{2.22}$$

where $e_{-b}(n) = \prod_{s=n_0}^{n-1}(1-b(s))$ and

$$\beta = \sup\left\{n \in \mathbb{N}(n_0) : c^q + q\sum_{s=n_0}^{n-1} e_{-b}^{1-p}(s)\left[k(s+1,s) + \sum_{\tau=n_0}^{s-1} |\Delta_s k(s,\tau)|\right] > 0\right\}. \tag{2.23}$$

Substituting (2.22) into (2.19) and then summing it from n_0 to n, we have

$$v(n) \le c + \sum_{s=n_0}^{n-1} \frac{b(s)}{e_{-b}(s)}\left[c^q + q\sum_{\tau=n_0}^{s-1} e_{-b}^{q}(\tau)\left(k(\tau+1,\tau) + \sum_{\sigma=n_0}^{\tau-1} |\Delta_\tau k(\tau,\sigma)|\right)\right]^{1/q}, \quad n \in \mathbb{N}(n_0,\beta), \tag{2.24}$$

where $q = 1-p$. Hence, the proof is complete. □

Remark 2.6. We suppose further that $\Delta_n k(n,m)$ is a nonnegative function for $n, m \in \mathbb{N}(n_0)$ with $n \ge m$ in Theorem 2.5. Then, we have

$$u(n) \le c + \sum_{s=n_0}^{n-1} \frac{b(s)}{e_{-b}(s)}\left[c^q + q\sum_{\tau=n_0}^{s-1} e_{-b}^{q}(\tau)\left(k(\tau+1,\tau) + \sum_{\sigma=n_0}^{\tau-1} \Delta_\tau k(\tau,\sigma)\right)\right]^{1/q}, \quad n \in \mathbb{N}(n_0,\beta), \tag{2.25}$$

where

$$\beta = \sup\left\{ n \in \mathbb{N}(n_0) : c^q + q \sum_{s=n_0}^{n-1} e_{-b}^q(s) \left[k(s+1,s) + \sum_{\tau=n_0}^{s-1} \Delta_s k(s,\tau) \right] > 0 \right\}. \tag{2.26}$$

If we set $k(n,m) = c(n)d(m)$ in Theorem 2.5, then we obtain the following corollary from Theorem 2.5. This is an analogue of the nonlinear difference inequality in [17, Corollary 2.8].

Corollary 2.7. *Let $u(n), b(n) < 1, c(n), d(n)$ be nonnegative sequences defined on $\mathbb{N}(n_0)$, and let $u_0 = u(n_0)$ be a positive constant. Suppose that*

$$u(n) \leq u_0 + \sum_{s=n_0}^{n-1} b(s) \left[u(s) + c(s) \sum_{\tau=n_0}^{s-1} d(\tau) u^p(\tau) \right], \quad n \in \mathbb{N}(n_0), \tag{2.27}$$

where $p \geq 0$, $p \neq 1$ is a constant. Then, one has

$$u(n) \leq u_0 + \sum_{s=n_0}^{n-1} \frac{b(s)}{e_{-b}(s)} \left[u_0^q + q \sum_{\tau=n_0}^{s-1} e_{-b}^q(\tau) \left(c(\tau+1)d(\tau) + |\Delta c(\tau)| \sum_{\sigma=n_0}^{\tau-1} d(\sigma) \right) \right]^{1/q}, \quad n \in \mathbb{N}(n_0, \beta), \tag{2.28}$$

where $q = 1 - p$, $e_{-b}(n) = \prod_{s=n_0}^{n-1}(1 - b(s))$, and

$$\beta = \sup\left\{ n \in \mathbb{N}(n_0) : u_0^q + q \sum_{s=n_0}^{n-1} e_{-b}^q(s) \left(c(s+1)d(s) + |\Delta c(s)| \sum_{\tau=n_0}^{s-1} d(\tau) \right) > 0 \right\}. \tag{2.29}$$

If we use Lemma 2.1 in the proof of Theorem 2.5, then we obtain the following bound of $u(n)$ which contains double fold summations.

Corollary 2.8. *Let $u(n)$ and $b(n)$ be nonnegative sequences defined on $\mathbb{N}(n_0)$, and let $k(n,m)$ be a nonnegative function for $n, m \in \mathbb{N}(n_0)$ with $n \geq m$. Suppose that*

$$u(n) \leq c + \sum_{s=n_0}^{n-1} b(s) \left[u(s) + \sum_{\tau=n_0}^{s-1} k(s,\tau) u^p(\tau) \right], \quad n \in \mathbb{N}(n_0), \tag{2.30}$$

where c is a positive constant, and $p > 0$, $p \neq 1$ is a constant. Then, one has

$$u(n) \leq \frac{1}{e(n)}\left[c^{1-p} + (1-p)\sum_{s=n_0}^{n-1}\left(b(s)\sum_{\tau=n_0}^{s-1}k(s,\tau)\right)e^{1-p}(s+1)\right]^{1/(1-p)}, \quad n \in \mathbb{N}(n_0,\beta), \tag{2.31}$$

where $e(n) = \prod_{s=n_0}^{n-1}[1+b(s)]^{-1}$ and

$$\beta = \sup\left\{n \in \mathbb{N}(n_0) : c^{1-p} + (1-p)\sum_{s=n_0}^{n-1}\left(b(s)\sum_{\tau=n_0}^{s-1}k(s,\tau)\right)e^{1-p}(s+1) > 0\right\}. \tag{2.32}$$

Proof. Define $v(n)$ by the right member of (2.30). Then, we have

$$\begin{aligned}
\Delta v(n) &= b(n)u(n) + b(n)\sum_{s=n_0}^{n-1}k(n,s)u^p(s) \\
&\leq b(n)v(n) + b(n)\sum_{s=n_0}^{n-1}k(n,s)v^p(s) \\
&\leq b(n)v(n) + \left(b(n)\sum_{s=n_0}^{n-1}k(n,s)\right)v^p(n),
\end{aligned} \tag{2.33}$$

since $u(n) \leq v(n)$ and $v^p(n)$ is nondecreasing in n. From Lemma 2.1, we obtain

$$v(n) \leq \frac{1}{e(n)}\left[c^{1-p} + (1-p)\sum_{s=n_0}^{n-1}\left(b(s)\sum_{\tau=n_0}^{s-1}k(s,\tau)\right)e^{1-p}(s+1)\right]^{1/(1-p)}, \quad n \in \mathbb{N}(n_0,\beta), \tag{2.34}$$

where $e(n) = \prod_{s=n_0}^{n-1}[1+b(s)]^{-1}$ and

$$\beta = \sup\left\{n \in \mathbb{N}(n_0) : c^{1-p} + (1-p)\sum_{s=n_0}^{n-1}\left(b(s)\sum_{\tau=n_0}^{s-1}k(s,\tau)\right)e^{1-p}(s+1) > 0\right\}. \tag{2.35}$$

Since $u(n) \leq v(n)$, the proof is complete. □

If we set $p = 1$ in Theorem 2.5, then we can obtain the following discrete analogue of Theorem 2.2 in [17] which improve in [16, Theorem 2.1].

Corollary 2.9. *Let $u(n)$ and $b(n)$ be nonnegative sequences defined on $\mathbb{N}(n_0)$ and $k(n,m)$ be a nonnegative function for each $n, m \in \mathbb{N}(n_0)$ with $n \geq m$. Suppose that*

$$u(n) \leq c + \sum_{s=n_0}^{n-1}b(s)\left(u(s) + \sum_{\tau=n_0}^{s-1}k(s,\tau)u(\tau)\right), \quad n \in \mathbb{N}(n_0), \tag{2.36}$$

where c is a positive constant. Then, one has

$$u(n) \le c\left[1 + \sum_{s=n_0}^{n-1} b(s) \prod_{\tau=n_0}^{s-1} (1 + p(\tau))\right]$$
$$\le c\left[1 + \sum_{s=n_0}^{n-1} b(s) \exp\left(\sum_{\tau=n_0}^{s-1} p(\tau)\right)\right], \quad n \in \mathbb{N}(n_0), \tag{2.37}$$

where $p(n) = b(n) + k(n+1, n) + \sum_{s=n_0}^{n-1} |\Delta_n k(n,s)|$.

The proof of this corollary follows by the similar argument as in the proof of Theorem 2.5. We omit the details.

3. An Application

In this section, we present an application of nonlinear difference inequalities established in Theorem 2.5 to study the boundedness of the solutions of nonlinear Volterra difference equations.

Consider the difference equation of Volterra type

$$\Delta x(n) = A(n)x(n) + \sum_{s=n_0}^{n-1} K(n,s)x(s) + F(n), \qquad x(n_0) = x_0, \tag{3.1}$$

where $A(n)$ and $K(n,s)$ are $d \times d$ matrices for each $n, s \in \mathbb{N}(n_0)$ and $F : \mathbb{N}(n_0) \to \mathbb{R}^d$.

Lemma 3.1 (see [18, Theorem 2.9.1]). *Assume that there exists a* $d \times d$ *matrix* $L(n,s)$ *defined on* $\mathbb{N}(n_0) \times \mathbb{N}(n_0)$ *and satisfying*

$$K(n,s) + \Delta_s L(n,s) + L(n,s+1)A(s) + \sum_{\tau=s+1}^{n-1} L(n,\tau+1)K(\tau,s) = 0, \tag{3.2}$$

where $\Delta_s L(n,s) = L(n,s+1) - L(n,s)$.

Then, (3.1) is equivalent to the ordinary linear difference equation

$$\Delta y(n) = B(n)y(n) + L(n,n_0)x_0 + H(n), \qquad y(n_0) = x_0, \tag{3.3}$$

where $B(n) = A(n) - L(n,n)$ and

$$H(n) = F(n) + \sum_{s=n_0}^{n-1} L(n,s+1)F(s). \tag{3.4}$$

Consider the linear nonhomogeneous difference equation

$$\Delta x(n) = B(n)x(n) + P(n), \qquad x(n_0) = x_0, \tag{3.5}$$

where $B(n)$ is a $d\times d$ matrix over $\mathbb{R}$ and $P : \mathbb{N}(n_0) \to \mathbb{R}^d$. We present the variation of constants formula of difference equations.

Lemma 3.2. *The solution $x(n) = x(n, n_0, x_0)$ of (3.5) is given by the variation of constants formula*

$$x(n) = \Phi(n, n_0)x_0 + \sum_{s=n_0}^{n-1}\Phi(n, s+1)P(s), \tag{3.6}$$

where $\Phi(n, n_0)$ is a fundamental matrix solution of the difference equation $\Delta x(n) = B(n)x(n)$ such that $\Phi(n_0, n_0)$ is the identity matrix.

Now, we give an application of our results. We consider the perturbation of linear Volterra difference equation (3.1) with $F(n) = 0$

$$\Delta x(n) = A(n)x(n) + \sum_{s=n_0}^{n-1}K(n, s)x(s) + G(n, x(n)), \tag{3.7}$$

with initial condition $x(n_0) = x_0$, where $G : \mathbb{N}(n_0) \times \mathbb{R}^d \to \mathbb{R}^d$.

Theorem 3.3. *Suppose that the following conditions hold for $n \geq n_0$, $0 < \alpha < 1$:*

(i) $|\Phi(n, n_0)| \leq M_1$,

(ii) $|L(n, s)| \leq M_2\alpha^{(n-s)}$,

(iii) $|G(s, x)| \leq b(s)|x|$, $n \geq s \geq n_0$,

where M_1 and M_2 are some positive constants and $b(n)$ is a nonnegative sequence defined on $\mathbb{N}(n_0)$ with $\sum_{s=n_0}^{\infty}\alpha^{-s}b(s) < \infty$. Then, all solutions of (3.7) are bounded in $\mathbb{N}(n_0)$.

Proof. By Lemma 3.1, (3.7) is equivalent to

$$\Delta x(n) = B(n)x(n) + L(n, n_0)x_0 + G(n, x(n)) + \sum_{s=n_0}^{n-1}L(n, s+1)G(s, x(s)), \tag{3.8}$$

with $x(n_0) = x_0$, where $B(n) = A(n) - L(n, n)$ and $L(n, s)$ is a solution of (3.2). It follow from Lemma 3.2 that the solution $x(n, n_0, x_0)$ of (3.8) is given by

$$x(n) = \Phi(n, n_0)x_0 + \sum_{s=n_0}^{n-1}\Phi(n, s+1)\left[L(s, n_0)x_0 + G(s, x(s)) + \sum_{\tau=n_0}^{s-1}L(s, \tau+1)G(\tau, x(\tau))\right], \quad n \geq n_0. \tag{3.9}$$

By using the conditions (i)–(iii), we obtain

$$
\begin{aligned}
|x(n)| &= |\Phi(n,n_0)||x_0| + \sum_{s=n_0}^{n-1}|\Phi(n,s+1)||L(s,n_0)||x_0| \\
&\quad + \sum_{s=n_0}^{n-1}|\Phi(n,s+1)|\left[|G(s,x(s))| + \sum_{\tau=n_0}^{s-1}|L(s,\tau+1)||G(\tau,x(\tau))|\right] \\
&\le M_1|x_0| + M_1\sum_{s=n_0}^{n-1}M_2\alpha^{s-n_0}|x_0| + M_1\sum_{s=n_0}^{n-1}b(s)\left[|x(s)| + \sum_{\tau=n_0}^{s-1}|L(s,\tau+1)|b(\tau)|x(\tau)|\right] \\
&\le \frac{(1+M_2-\alpha)}{1-\alpha}M_1|x_0| + \sum_{s=n_0}^{n-1}M_1b(s)\left[|x(s)| + \sum_{\tau=n_0}^{s-1}|L(s,\tau+1)|b(\tau)|x(\tau)|\right], \quad n \ge n_0.
\end{aligned}
\tag{3.10}
$$

Letting $u(n) = |x(n)|$, $c = ((1+M_2-\alpha)/(1-\alpha))M_1|x_0|$ and

$$
|L(s,\tau+1)|b(\tau) \le M_2\alpha^{(s-\tau-1)}b(\tau) = k(s,\tau), \tag{3.11}
$$

and employing the above estimate by Corollary 2.9, then we have

$$
u(n) \le c\left[1 + \sum_{s=n_0}^{n-1}M_1b(s)\exp\left(\sum_{\tau=n_0}^{s-1}p(\tau)\right)\right] \le M, \quad n \in \mathbb{N}(n_0), \tag{3.12}
$$

where $M = c[1+\sum_{s=n_0}^{\infty}M_1b(s)\exp(\sum_{\tau=n_0}^{\infty}p(\tau))]$, because

$$
\begin{aligned}
p(\tau) &= b(\tau) + k(\tau+1,\tau) + \sum_{\sigma=n_0}^{\tau-1}|\Delta_\tau k(\tau,\sigma)| \\
&= b(\tau) + M_2\alpha b(\tau) + M_2\frac{(1-\alpha)}{\alpha}\alpha^{\tau}\sum_{\sigma=n_0}^{\tau-1}\alpha^{-\sigma}b(\sigma) \\
&\le b(\tau)(1+M_2\alpha) + \alpha^{\tau}M_2\frac{(1-\alpha)}{\alpha}\sum_{\sigma=n_0}^{\infty}\alpha^{-\sigma}b(\sigma),
\end{aligned}
\tag{3.13}
$$

and $\sum_{\tau=n_0}^{\infty}p(\tau) < \infty$. Hence, the solutions $x(n)$ of (3.7) are bounded in $\mathbb{N}(n_0)$, and the proof is complete. □

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