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Research Article

First-Order Twistor Lifts

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The use of twistor methods in the study of Jacobi fields has proved quite fruitful, leading to a series of results. L. Lemaire and J. C. Wood proved several properties of Jacobi fields along harmonic maps from the two-sphere to the complex projective plane and to the three- and four-dimensional spheres, by carefully relating the infinitesimal deformations of the harmonic maps to those of the holomorphic data describing them. In order to advance this programme, we prove a series of relations between infinitesimal properties of the map and those of its twistor lift. Namely, we prove that isotropy and harmonicity to first order of the map correspond to holomorphicity to first order of its lift into the twistor space, relatively to the standard almost complex structures $\mathcal{J}^1$ and $\mathcal{J}^2$. This is done by obtaining first-order analogues of classical twistorial constructions.

1. Introduction

Harmonic maps are mappings φ between Riemannian manifolds M and N which extremize the energy functional

$$E(\varphi) = \frac{1}{2}\int_M \|d\varphi\|^2 \omega_g. \tag{1.1}$$

Letting TM denote the tangent bundle of M, one can (locally) characterize harmonic maps as solutions of the nonlinear second-order differential equation

$$\tau(\varphi) = 0, \tag{1.2}$$

where $\tau(\varphi)$ denotes the *tension field* of φ,

$$\tau(\varphi) = \text{trace}\nabla d\varphi = \sum_i \nabla d\varphi(X_i, X_i), \quad \{X_i\} \text{ orthonormal (local) frame of } TM. \tag{1.3}$$

A bibliography can be found in [1] and for some useful summaries on this topic, see [2, 3].

The infinitesimal deformations of a harmonic map are called *Jacobi fields*. More precisely, let $\varphi : M \to N$ be a smooth map and denote by $\Gamma(\varphi^{-1}TN)$ the set of smooth sections of the pull-back bundle $\varphi^{-1}TN$. If φ is harmonic and $v \in \Gamma(\varphi^{-1}TN)$ is a vector field along it, v is said to be a *Jacobi field* (along φ) if it satisfies the linear *Jacobi equation* $J_\varphi(v) = 0$, where the *Jacobi operator* J_φ is defined by

$$J_\varphi(v) = \Delta v - \text{trace}\, R^N(d\varphi, v)d\varphi. \tag{1.4}$$

Here, Δ is the Laplacian on $\varphi^{-1}TN$,

$$\Delta v = -\sum_i \left(\nabla_{X_i}\nabla_{X_i} v - \nabla_{\nabla_{X_i}X_i} v\right) \tag{1.5}$$

and, letting R^N denote the curvature tensor of N,

$$\text{trace}\, R^N(d\varphi, v)d\varphi = \sum_i R^N(d\varphi X_i, v)(d\varphi X_i). \tag{1.6}$$

Jacobi fields are characterized as lying in the kernel of the second variation of the energy functional. Indeed, if $\varphi_{t,s}$ is a two-parameter variation of a harmonic map $\varphi_{(0,0)}$, then, writing $v = \partial\varphi/\partial t|_{(0,0)}$ and $w = \partial\varphi/\partial s|_{(0,0)}$, the *Hessian* H_φ of φ is the bilinear operator on $\Gamma(\varphi^{-1}TN)$ given by

$$H_\varphi(v, w) := \left.\frac{\partial^2 E(\varphi_{t,s})}{\partial t \partial s}\right|_{(0,0)} = \int_M \langle J_\varphi(v), w\rangle \omega_g \tag{1.7}$$

so that a Jacobi field v (along φ) is characterized by the condition

$$H_\varphi(v, w) = 0, \quad \forall w. \tag{1.8}$$

A Jacobi field is called *integrable* if it is tangent to a deformation through harmonic maps. In [4, 5], the question of whether all Jacobi fields are integrable is answered for the case where the domain is the two-sphere and the codomain the two-dimensional complex projective space or the three- and four-dimensional sphere. This was done by relating the deformations of the map associated with the Jacobi field and those of the twistor lift of the map. More precisely, given an oriented even-dimensional manifold N^{2n}, we can construct its (positive) twistor space Σ^+N. This manifold admits two natural almost complex structures $\mathcal{J}^1$ and $\mathcal{J}^2$. Given a map $\varphi : M^2 \to N^{2n}$ from a Riemann surface M^2, harmonicity is intimately related with the existence of a $\mathcal{J}^2$-holomorphic lift $\psi : M^2 \to \Sigma^+N$, whereas isotropy is related with the existence of a $\mathcal{J}^1$-holomorphic lift ψ (see [6]). On the other hand, Jacobi vector fields induce families of maps which are harmonic to first-order and, in some cases, isotropic to first order. The translation of these first order properties in terms of twistor lifts plays an important role on the study of the Jacobi fields and we shall exhibit how this translation can be established in general.

This work is divided as follows: in the next two sections, we recall some well-known results concerning twistor lifts of harmonic and isotropic maps. In Section 4, we show how this constructions generalize to their parametric versions and examine more closely the construction when the codomain is a four-dimensional manifold. We leave to the last section some technical proofs.

2. The Setup

2.1. Twistor Spaces

Let E^{2k} be an oriented even-dimensional Euclidean space, equipped with a metric $\langle , \rangle$. A *Hermitian structure* J on E is $J \in \mathfrak{gl}(E)$ with $J^2 = -Id$ and $\langle JX, JY \rangle = \langle X, Y \rangle$ for all $X, Y \in E$. We say that J is *positive* if there is a positive basis of E of the form $\{e_1, Je_1, \ldots, e_k, Je_k\}$ and *negative* otherwise. The set of all positive Hermitian structures (resp., all negative Hermitian structures) on E is denoted by $\Sigma^+ E$ (resp., $\Sigma^- E$). The Lie group $\mathbf{SO}(E)$ acts transitively on $\Sigma^+ E$ by the formula $S \cdot J = SJS^{-1}$ and the isotropy subgroup at J is given by

$$\mathbf{U}_J(E) = \{S \in \mathbf{SO}(E) : SJ = JS\}. \tag{2.1}$$

Letting $\mathfrak{u}_J(E) = \{\lambda \in \mathfrak{so}(E) : \lambda J = J\lambda\}$ and $\mathfrak{m}_J(E) = \{\lambda \in \mathfrak{so}(E) : \lambda J = -J\lambda\}$, we easily conclude that

$$\mathfrak{so}(E) = \mathfrak{u}_J(E) \oplus \mathfrak{m}_J(E). \tag{2.2}$$

In particular, the tangent space of $\Sigma^+ E = \mathbf{SO}(E)/\mathbf{U}(E)$ at J is given by $\mathfrak{m}_J(E)$. In this vector space, we can introduce a complex structure $\mathcal{J}^v$ defining

$$\mathcal{J}^v \lambda := J\lambda. \tag{2.3}$$

When equipped with $\mathcal{J}^v$, the manifold $\Sigma^+ E$ is a complex manifold.

Given E with a Hermitian structure J, we can consider on $E^{\mathbb{C}} = E \otimes \mathbb{C}$ its $(1,0)$ and $(0,1)$-parts given as usual by

$$E^{10} = \{X - iJX, X \in E\}, \qquad E^{01} = \{X + iJX, X \in E\}. \tag{2.4}$$

These are *isotropic* subspaces, in the sense that $\langle X, Y \rangle = 0$ for all X, Y in E^{10} (or in E^{01}). Associating an Hermitian structure J on E with its $(1,0)$-space s gives a 1–1 correspondence between Hermitian structures and maximal isotropic subspaces. We say that a maximal isotropic subspace is *positive* if the corresponding orthogonal complex structure is positive and we denote the set of all such subspaces by $\mathrm{Gr}^+_{\mathrm{iso}}(E^{\mathbb{C}})$.

Let (V^{2k}, g, ∇) be an oriented even-dimensional vector bundle over a manifold M equipped with a connection ∇ and a parallel metric g. Then, we may take the bundle

$$\Sigma^+ V = \mathbf{SO}(V) \times_{\mathbf{SO}(2k)} \Sigma^+ \mathbb{R}^{2k} \simeq \mathrm{Gr}^+_{\mathrm{iso}}\left(V^{\mathbb{C}}\right) \tag{2.5}$$

whose fibre at $x \in M$ is precisely $\Sigma^+ V_x$. If M^2 is Riemann surface, the vector bundle $V^{\mathbb{C}}$ has vanishing $(2,0)$-part of its curvature tensor and therefore admits a unique structure as a holomorphic bundle over M^2, by a well-known theorem of Koszul and Malgrange [7]. This induces a holomorphic structure $\mathcal{J}^{\mathrm{KM}}$ on the bundle $\mathrm{Gr}^+_{\mathrm{iso}}(V^{\mathbb{C}}) \simeq \Sigma^+ V$ for which a section s of $\mathrm{Gr}^+_{\mathrm{iso}}(V^{\mathbb{C}})$ is holomorphic if and only if [6] (see [8]).

$$\nabla_X s \subseteq s \quad \forall X \in T^{01} M. \tag{2.6}$$

Let N be an oriented Riemannian manifold with dimension $2n$. We call *(positive) twistor space* of N the bundle whose fibre at $y \in N$ is precisely $\Sigma^+ T_y N$; that is,

$$\Sigma^+ N = \mathbf{SO}(N) \times_{\mathbf{SO}(2n)} \Sigma^+ \mathbb{R}^{2n} = \frac{\mathbf{SO}(N) \times_{\mathbf{SO}(2n)} \mathbf{SO}(2n)}{\mathbf{U}(2n)}. \tag{2.7}$$

The Riemannian connection on N induces a splitting of the tangent space to $\Sigma^+ N$ into *vertical* and *horizontal* parts, $T\Sigma^+ N = \mathcal{V} \oplus \mathcal{H}$. Namely, if $\pi : \Sigma^+ N \to N$ denotes the canonical projection defined by $\pi(y, J_y) = y$, then

$$\begin{gathered} \mathcal{V}_{y,J_y} = \ker d\pi_{y,J_y} \simeq T_{J_y} \Sigma^+ T_y N, \\ \mathcal{H}_{y,J_y} = \left\{ d\sigma_y(X_y) : X_y \in T_y N, \sigma \text{ section of } \Sigma^+ N \text{ with } \nabla_{X_y} \sigma = 0 \text{ and } \sigma(y) = (y, J_y) \right\}. \end{gathered} \tag{2.8}$$

With respect to this decomposition, $d\pi_{y,J_y}$ maps $\mathcal{H}_{y,J_y}$ isomorphically into $T_y N$ and allows to define an almost complex structure $\mathcal{J}^{\mathcal{H}}$ on $\mathcal{H}$ as the pull-back of J_y on $T_y N$. Together with (2.3), this allows to define two almost complex structures $\mathcal{J}^1$ and $\mathcal{J}^2$ on $\Sigma^+ N$ by the formulae

$$\mathcal{J}^1 = \begin{cases} \mathcal{J}^{\mathcal{H}} & \text{on } \mathcal{H}, \\ \mathcal{J}^{\mathcal{V}} & \text{on } \mathcal{V}, \end{cases} \qquad \mathcal{J}^2 = \begin{cases} \mathcal{J}^{\mathcal{H}} & \text{on } \mathcal{H}, \\ -\mathcal{J}^{\mathcal{V}} & \text{on } \mathcal{V}. \end{cases} \tag{2.9}$$

When equipped with $\mathcal{J}^2$, Σ^+ is never a complex manifold; as for $\mathcal{J}^1$, it is integrable if and only if N^{2n} is conformally flat ($n \geq 3$) or anti-self-dual ($n = 2$) (for more details, see [6, 9, 10]; a discussion on this topic can also be found in [11] and references therein).

Notice that π is a holomorphic map for any of these complex structures; that is, for $a = 1$ or $a = 2$, one has that

$$d\pi_{y,J_y}(\mathcal{J}^a X) = J_y d\pi_{y,J_y}(X), \quad \forall X \in T_{y,J_y} \Sigma^+ N. \tag{2.10}$$

Definition 2.1. Let (M, J) and $(Z, \mathcal{J})$ be two almost complex manifolds. Let $TZ = \mathcal{H} \oplus \mathcal{V}$ be a decomposition of TZ into *$\mathcal{J}$-stable subbundles*; that is, $\mathcal{J}\mathcal{H} \subseteq \mathcal{H}$ and $\mathcal{J}\mathcal{V} \subseteq \mathcal{V}$. We shall call such a decomposition a *$\mathcal{J}$-stable decomposition*. Let $\psi : M \to Z$ be a smooth map. We shall say that ψ is *$\mathcal{H}$-holomorphic* if

$$(d\psi(JX))^{\mathcal{H}} = \mathcal{J}(d\psi X)^{\mathcal{H}}, \quad \forall X \in TM. \tag{2.11}$$

Analogously we define *$\mathcal{V}$-holomorphic maps*.

A smooth map $\psi : M \to Z$ is holomorphic if and only if it is both $\mathcal{H}$ and $\mathcal{V}$-holomorphic for some, and so any, stable decomposition $TZ = \mathcal{H} \oplus \mathcal{V}$. Taking $Z = \Sigma^+N$, the decomposition $T\Sigma^+N = \mathcal{H} \oplus \mathcal{V}$ is clearly stable for both the almost complex structures $\mathcal{J}^1$ and $\mathcal{J}^2$ on Σ^+N.

Remark 2.2. We can easily introduce a metric on the twistor space Σ^+N: let $(y, J_y) \in \Sigma^+N$ and consider the tangent space at this point, $T_{(y,J_y)}\Sigma^+N = \mathcal{H} \oplus \mathcal{V}$. We know that we have the identifications $\mathcal{H} \simeq T_yN$ and $\mathcal{V} \simeq \mathfrak{m}_J(T_yN)$. To get a metric on $\mathcal{H}$, transport the metric from that on T_yN; that is, $h(X, Y) = \langle d\pi X, d\pi Y\rangle$, for all $X, Y \in \mathcal{H}$, where $\langle , \rangle$ denotes the metric on N at $y = \pi(y, J_y)$. For the vertical space $\mathcal{V} \simeq \mathfrak{m}_{J_y}(T_yN) \subseteq \mathfrak{gl}(T_yN)$, we can consider the restriction of the metric on the space $\mathfrak{gl}(T_yN)$. Finally, we declare $\mathcal{H}$ and $\mathcal{V}$ to be orthogonal under the metric h; that is, $h(X, V) = 0$, for all $X \in \mathcal{H}$, $V \in \mathcal{V}$. With this metric, the decomposition $\mathcal{H} \oplus \mathcal{V}$ is orthogonal and $\mathcal{J}^a$-stable $(a = 1, 2)$, $(\Sigma^+N, h, \mathcal{J}^a)$ $(a = 1, 2)$ are almost Hermitian manifolds and the projection map π is a Riemannian submersion.

2.2. Conformal and Isotropic Maps

Given a smooth map $\varphi : M^2 \to N$, φ is said to be *weakly conformal* at $x \in M$ if there is $\Lambda_x \in \mathbb{R}$ with

$$\langle d\varphi_x X, d\varphi_x Y\rangle = \Lambda_x \langle X, Y\rangle, \quad \forall X, Y \in T_xM. \tag{2.12}$$

If $\Lambda_x \neq 0$, then x is said to be a *regular point* (of φ) and the map φ is called *conformal* at x. Moreover, a map which is conformal (resp., weakly conformal) at all points $x \in M$ is said to be a *conformal map* (resp., a *weakly conformal map*).

If $(N; \langle , \rangle, J)$ is an almost Hermitian manifold, any holomorphic (resp., antiholomorphic) map $\varphi : M^2 \to N$ is (weakly) conformal as it maps $T^{10}M$ to $T^{10}N$ (resp., to $T^{01}N$). A stronger property than conformality is isotropy: if $\varphi : M^2 \to N$ is a smooth map from a Riemann surface, φ is *isotropic* if [12]

$$\langle \partial_z^r \varphi, \partial_z^s \varphi\rangle = 0, \quad \forall r, s \geq 1, \tag{2.13}$$

where $\partial_z^r \varphi = \nabla_{\partial_z}(\partial_z^{r-1}\varphi)$. Actually, the condition (2.13) can be weakened to

$$\langle \partial_z^r \varphi, \partial_z^r \varphi\rangle = 0, \quad \forall r \geq 1. \tag{2.14}$$

To check this, establish an induction on $j = |r - s|$: if $j = 0$ the result is trivial. Assuming now that (2.14) implies (2.13) for all $j \le n$ and taking $r, s \ge 1$ with $|r - s| = n + 1$, we may assume without loss of generality that $r \ge s$, $r = s + n + 1$ and we get

$$\left\langle \partial_z^{s+n+1}\varphi, \partial_z^{s}\varphi \right\rangle = \partial_z\left\langle \partial_z^{s+n}\varphi, \partial_z^{s}\varphi \right\rangle - \left\langle \partial_z^{s+n}\varphi, \partial_z^{s+1}\varphi \right\rangle. \tag{2.15}$$

Since $|s + n - s| = n$ and $|s + n - s - 1| = |n - 1| \le n$, both terms in the above expression vanish.

Moreover, letting $r = s = 1$ in (2.13), it is easy to check that an isotropic map from a Riemann surface is a (weakly) conformal map.

Let $\varphi : M^2 \to N$ be a smooth map from a Riemann surface M^2. We shall say that $z \in M^2$ is an *umbilic point* (of φ) if $\{\partial_z\varphi(z), \partial_z^2\varphi(z)\}$ is a $\mathbb{C}$-linearly dependent set. If $\varphi : M^2 \to N$ is such that all points $z \in M^2$ are umbilic, we shall say that φ is *totally umbilic* (see [6]).

3. Nonparametric Twistorial Constructions

The following are well-known twistorial constructions [13] (see also [6]).

Theorem 3.1. *If $\psi : M^2 \to (\Sigma^+N, \mathcal{J}^1)$ is holomorphic, the projection map $\varphi = \pi \circ \psi$ is isotropic. Conversely, if $\varphi : M^2 \to N$ is a conformal totally umbilic immersion, there is (locally) a holomorphic map $\psi : M^2 \to (\Sigma^+N, \mathcal{J}^1)$ such that $\varphi = \pi \circ \psi$.*

If $\psi : M^2 \to (\Sigma^+N, \mathcal{J}^2)$ is holomorphic, the projection map $\varphi = \pi \circ \psi$ is harmonic. Conversely, if $\varphi : M^2 \to N$ is a conformal harmonic map, there is (locally) a holomorphic map $\psi : M^2 \to (\Sigma^+N, \mathcal{J}^2)$ such that $\varphi = \pi \circ \psi$.

We shall sketch the proof of this result. We start by noticing that given a smooth map $\varphi : M \to N$ obtained as the projection of $\psi : M^2 \to \Sigma^+N$, $\varphi = \pi \circ \psi$, without requiring further conditions *à priori* on ψ, nothing guarantees that φ is holomorphic relatively to the induced almost Hermitian structure J_ψ on TN; if it is, we shall say that the structure J_ψ is *strictly compatible* with φ (or that the map ψ is a *strictly compatible twistor lift* of φ). Such a structure J_ψ can exist if and only if $d\varphi(T^{10}M) \subseteq T^{10}_{J_\psi}N$ is isotropic: in other words, if and only if φ is (weakly) conformal. If J_ψ preserves $d\varphi(TM)$ but does not necessarily render φ holomorphic, we shall say that J_ψ (or the map ψ) is *compatible* with φ.

If φ is given as the projection $\varphi = \pi \circ \psi$ of an $\mathcal{H}$-holomorphic map $\psi : (M, J^M) \to (\Sigma^+N, \mathcal{J}^a)$ ($a = 1$ or 2), then φ is holomorphic with respect to the induced almost Hermitian structure J_ψ on TN:

$$\begin{aligned} &J_\psi d\varphi X \\ &= d\varphi J^M X \Longleftrightarrow d\pi\left(\mathcal{J}^{\mathcal{H}}(d\psi X)^{\mathcal{H}}\right) = d\pi\left(\left(d\psi\left(J^M X\right)\right)^{\mathcal{H}}\right) \Longleftrightarrow \mathcal{J}^{\mathcal{H}}(d\psi X)^{\mathcal{H}} = \left(d\psi\left(J^M X\right)\right)^{\mathcal{H}}. \end{aligned} \tag{3.1}$$

In particular, φ is (weakly) conformal. Moreover, the above equivalence shows that any strictly compatible lift of φ is $\mathcal{H}$-holomorphic. On the opposite direction, let $\varphi : M^2 \to N$ be a conformal map. Let $V := TM^\perp \subseteq \varphi^{-1}TN$ denote the normal bundle of TM in TN. We may decompose the connection on $\varphi^{-1}TN$ into its tangent and normal parts, $\nabla = \nabla^\top + \nabla^\perp$. Hence, on $TM^\perp$ we have a metric $\langle , \rangle$ and connection $\nabla^\perp$ inherited from those on $\varphi^{-1}TN$. Moreover, we may take the bundle Σ^+V over M which has the $\mathcal{J}^{\mathrm{KM}}$-holomorphic structure. Since φ is conformal, we may transfer the Hermitian structure J^M of M^2 into $d\varphi(TM)$. Hence, we have a natural map

$$\eta : \Sigma^+V \longrightarrow \Sigma^+N,$$
$$\widehat{J} \longrightarrow J = \begin{cases} \widehat{J} & \text{on } TM^{\perp}, \\ J^M & \text{on } d\varphi(TM). \end{cases} \tag{3.2}$$

Taking any holomorphic section of $(\Sigma^+V, \mathcal{J}^{\mathrm{KM}})$ and composing with η, we obtain a strictly compatible twistor lift ψ of φ. Since any such lift is $\mathcal{H}$-holomorphic, we have just proved the following result.

Proposition 3.2. *Given $\varphi : M^2 \to N$, φ is conformal if and only if φ is (locally) the projection of an $\mathcal{H}$-holomorphic map $\psi : M^2 \to \Sigma^+N$.*

To proceed, we need the following result [13].

Proposition 3.3. *Let $\psi : M \to \Sigma^+N$ and let $\varphi = \pi \circ \psi$. Take s the section of $\mathrm{Gr}^+_{\mathrm{iso}}(TN^{\mathbb{C}})$ corresponding to ψ. Then, the map ψ is $\mathcal{J}^1$-holomorphic if and only if φ is holomorphic with respect to J_ψ and*

$$\nabla_X s \subset s \quad \textit{for any } X \in T^{10}M. \tag{3.3}$$

The map ψ is $\mathcal{J}^2$-holomorphic if and only if φ is holomorphic with respect to J_ψ and

$$\nabla_{\overline{X}} s \subset s \quad \textit{for any } X \in T^{10}M. \tag{3.4}$$

Let $\psi : M^2 \to \Sigma^+N$ be a $\mathcal{J}^1$-holomorphic map and let s be the corresponding section of $\mathrm{Gr}^+_{\mathrm{iso}}(TN^{\mathbb{C}})$. If $\varphi = \pi \circ \psi$ is the projection map, we have that φ is J_ψ-holomorphic. In particular, $\partial_z\varphi \in s$. Since ψ is $\mathcal{J}^1$-holomorphic, we can write

$$\nabla_{\partial_z}\partial_z\varphi \subseteq s \tag{3.5}$$

and, inductively, it follows that $\partial_z^r\varphi \subseteq s$ for all $r \geq 1$ so that φ is isotropic. Conversely, let $\varphi : M^2 \to N$ be a conformal totally umbilic map. In this case, we consider the manifold M^2 equipped with the opposite holomorphic structure-J^M and again construct the holomorphic bundle $(\Sigma^+V, \mathcal{J}^{\mathrm{KM}})$. For this structure, a section $\widehat{s}$ is holomorphic if and only if $\nabla^{\perp}_{\partial_z}\widehat{s} \subseteq \widehat{s}$. Since φ is conformal, we may define the map η as in (3.2). On the other hand, because φ is totally umbilic, we know that $\partial_z^2\varphi$ lies in the span of $\partial_z\varphi$; in other words, if J is any almost Hermitian structure on TN strictly compatible with φ, $\partial_z^2\varphi$ lies in $T_J^{10}N$. We may therefore conclude that η is $\mathcal{J}^1$-holomorphic. As a matter of fact, given a holomorphic section $\widehat{s}$ of Σ^+V and considering $s = \eta \circ \widehat{s}$, we have that $s = \widehat{s} \oplus d\varphi(T^{10}M)$. We may write

$$\left\langle \nabla^{\top}_{\partial_z}\widehat{s}, d\varphi\left(T^{10}M\right)\right\rangle = -\left\langle \widehat{s}, \nabla_{\partial_z} d\varphi\left(T^{10}M\right)\right\rangle = 0, \tag{3.6}$$

since $\widehat{s}$ and $\partial_z^2\varphi$ lie in s. In particular,$\nabla^{\top}_{\partial_z}\widehat{s} \subseteq s$ and so

$$\nabla_{\partial_z} s \subseteq \left(\nabla^{\top}_{\partial_z} + \nabla^{\perp}_{\partial_z}\right)\widehat{s} + \nabla_{\partial_z} d\varphi\left(T^{10}M\right) \subseteq s + \widehat{s} + \operatorname{span}\left\{\partial_z\varphi, \partial_z^2\varphi\right\} \subseteq s \tag{3.7}$$

and therefore we conclude that η is $\mathcal{J}^1$-holomorphic.

With a similar argument, let $\psi : M^2 \to \Sigma^+ N$ be a $\mathcal{J}^2$-holomorphic map and let s be the section of $\mathrm{Gr}^+_{\mathrm{iso}}(TN^{\mathbb{C}})$ corresponding to ψ. Then, taking the projection map $\varphi = \pi \circ \psi$, we get

$$\left(\tau(\varphi)\right)^{01} = \left(J_\psi \operatorname{trace}_g \nabla J^{\psi}\right)^{01} = \left(4\nabla_{\partial_{\bar{z}}} d\varphi(\partial_z)\right)^{01}. \tag{3.8}$$

Since φ is J_ψ-holomorphic, $d\varphi(\partial_z)$ lies in s. From (iv) in Proposition 3.3, we conclude that $\nabla_{\partial_{\bar{z}}} d\varphi(\partial_z)$ lies in s and therefore has vanishing $(0,1)$-part. From the reality of $\tau(\varphi)$, we have that $\tau(\varphi) = 0$ and so φ is harmonic. Conversely, if $\varphi : M^2 \to N$ is conformal and harmonic, we may take the bundle $(\Sigma^+ V, \mathcal{J}^{\mathrm{KM}})$ and the map η in (3.2). Since φ is harmonic, η is holomorphic with respect to the $\mathcal{J}^{\mathrm{KM}}$ and $\mathcal{J}^2$ structures [6, 8]: letting $\widehat{s}$ denote a holomorphic section of $\Sigma^+ V$ and s the composition $\eta \circ \widehat{s}$, then, $s = d\varphi(T^{10}M) \oplus \widehat{s}$ and

$$\nabla_{\partial_{\bar{z}}} s = \left(\nabla^{\perp}_{\partial_{\bar{z}}} + \nabla^{\top}_{\partial_{\bar{z}}}\right)\widehat{s} + \nabla_{\partial_{\bar{z}}} d\varphi\left(T^{10}M\right). \tag{3.9}$$

Since φ is harmonic, $\nabla_{\partial_{\bar{z}}} d\varphi(T^{10}M) \subseteq d\varphi(T^{10}M) \subseteq s$. On the other hand, since $\widehat{s}$ is holomorphic, $\nabla^{\perp}_{\partial_{\bar{z}}}\widehat{s} \subseteq \widehat{s} \subseteq s$. Finally,

$$\left\langle \nabla^{\top}_{\partial_{\bar{z}}}\widehat{s}, d\varphi(\partial_z) \right\rangle = -\left\langle \widehat{s}, \nabla_{\partial_{\bar{z}}} d\varphi(\partial_z)\right\rangle = 0 \tag{3.10}$$

shows that $\nabla_{\partial_{\bar{z}}} s \subseteq s$ and so that η is holomorphic. Therefore, taking any holomorphic section of $\Sigma^+ V$ and composing with η give a $\mathcal{J}^2$-holomorphic lift of φ.

Remark 3.4. Notice that to guarantee the existence of the $\mathcal{J}^1$-holomorphic lift for φ, the important fact was that $\partial_z^2\varphi$ belongs to the $(1,0)$-part of TN for any almost Hermitian structure strictly compatible with φ. This is guaranteed if φ is a totally umbilic map, but it is not strictly necessary. For instance, if $\varphi : M^2 \to N^4$ is an isotropic map, the vectors $\partial_z\varphi$, $\partial_z^2\varphi$ span an isotropic subspace. If this vectors are linearly independent, taking this space as the $(1,0)$-space of ψ, then ψ is a $\mathcal{J}^1$-holomorphic lift of φ, although ψ may be a map into $\Sigma^- N$; on the other hand, if φ is totally umbilic, then we may take ψ either as the unique strictly compatible map into $\Sigma^+ N$ or into $\Sigma^- N$ and both these maps are $\mathcal{J}^1$-holomorphic.

4. First-Order Twistorial Constructions

4.1. Harmonicity and Isotropy to First Order

Let I denote an interval of the real line containing 0. Given a (family of) map(s) $\varphi : I \times M \to N$, $(t, x) \to \varphi_t(x)$, we say that φ *is harmonic to first order* if

$$\varphi_0 \text{ is harmonic and } \partial_t|_0 \tau(\varphi_t) = 0, \tag{4.1}$$

where $\partial_t|_0 \tau(\varphi_t) = \nabla^{\varphi^{-1}TN}_{\partial_t|_0} \tau(\varphi_t)$ and $\tau(\varphi_t) = \text{trace} \nabla d\varphi_t \in \varphi^{-1}TN$.

Let $\varphi_0 : M \to N$ be a harmonic map, $v \in \Gamma(\varphi_0^{-1}TN)$ a vector field along φ_0, and $\varphi : I \times M \to N$ a one-parameter variation of φ_0. We say that φ is *tangent* to v if $v = \partial_t|_0 \varphi_t$. The following result is a key ingredient in what follows [4]:

Proposition 4.1. *Let $\varphi_0 : M \to N$ be a harmonic map between compact manifolds M and N. Let $v \in \Gamma(\varphi_0^{-1}TN)$ be a vector field along φ_0 and $\varphi : I \times M \to N$ a one-parameter variation of φ_0 tangent to v. Then,*

$$\partial_t|_0 \tau(\varphi_t) = -J_\varphi(v). \tag{4.2}$$

In particular, v is Jacobi if and only if any tangent one-parameter variation is harmonic to first order.

We have seen in Theorem 3.1 that harmonicity was not enough to establish a relation with possible twistor lifts of a map conformality and was also a key ingredient, as maps obtained as projections of twistorial maps must be holomorphic with respect to some almost Hermitian structure along the map. On the other hand, when the domain is the 2-sphere, harmonicity implies (weak) conformality or even isotropy, the last case occurring if the target manifold is itself also a sphere or the complex projective space [12, 14].

Let M^2 be a Riemann surface and $\varphi : I \times M \to N$ a smooth map. The map φ is said to be *conformal to first order* if [15]

$$\varphi_0 \text{ is conformal and } \partial_t|_0 \langle \partial_z \phi_t, \partial_z \phi_t \rangle = 0. \tag{4.3}$$

Analogously, φ is said to be *isotropic to first order,*

$$\phi_0 \text{ is real isotropic and } \partial_t|_0 \langle \partial_z^r \phi_t, \partial_z^s \phi_t \rangle = 0, \quad \forall r, s \geq 1. \tag{4.4}$$

As for the nonparametric case, one can prove [16] that condition (4.4) can be weakened to the following

$$\phi_0 \text{ is real isotropic and } \partial_t|_0 \langle \partial_z^r \phi_t, \partial_z^r \phi_t \rangle = 0, \quad \forall r \geq 1. \tag{4.5}$$

As in the nonparametric case, harmonicity to first order implies conformality to first order for maps defined on the two-sphere and even isotropy when the codomain is itself a real or complex space form [15].

4.2. Twistorial Constructions

As we have seen, Jacobi fields induce variations that are harmonic (and, in some cases, conformal or even isotropic) to first order. On the other hand, in Section 2 we have seen that conformality, harmonicity, and isotropy of the map φ correspond to $\mathscr{H}$, $\mathcal{J}^2$ and $\mathcal{J}^1$-holomorphicity of the twistor lift ψ. As we shall see, these results do have a first-order version as follows. We start with a definition.

Definition 4.2. Let (M, J) be an almost complex manifold and $(Z, h, \mathcal{J})$ an almost Hermitian manifold. Given a smooth map $\psi : I \times M \to Z$, we say that ψ is *holomorphic to first order* if $\psi_0 : M \to Z$ is holomorphic and

$$\nabla_{\partial_t|_0}(d\psi_t JX - \mathcal{J} d\psi_t X) = 0 \quad \forall X \in TM, \tag{4.6}$$

where ∇ is the Levi-Civita connection on Z induced by the metric h. Moreover, if $TZ = \mathscr{H} \oplus \mathcal{V}$ is a $\mathcal{J}$-stable decomposition of TZ, orthogonal with respect to h, we shall say that ψ is $\mathscr{H}$-*holomorphic to first order* if ψ_0 is $\mathscr{H}$-holomorphic and

$$\nabla_{\partial_t|_0}\Big((d\psi_t(JX))^{\mathscr{H}} - \mathcal{J}^{\mathscr{H}}(d\psi_t X)^{\mathscr{H}}\Big) = 0 \quad \forall X \in TM, \tag{4.7}$$

where $\mathcal{J}^{\mathscr{H}}$ is the restriction of $\mathcal{J}$ to $\mathscr{H}$. Changing $\mathscr{H}$ to $\mathcal{V}$ gives the definition of $\mathcal{V}$-*holomorphicity to first order*.

In contrast with the nonparametric case, it is not obvious that $\mathcal{J}$-holomorphicity to first order implies $\mathscr{H}$-holomorphicity to first order. As a matter of fact, from (4.6), it only follows that $(\nabla_{\partial_t|_0}(d\psi_t(JX) - \mathcal{J} d\psi_t X))^{\mathscr{H}} = 0$. However, we do have the following.

Lemma 4.3. *Let $\psi : (M, J) \to (Z, h, \mathcal{J})$ be a smooth map and let $TZ = \mathscr{H} \oplus \mathcal{V}$ be a $\mathcal{J}$-stable decomposition of Z, orthogonal with respect to h. Then, ψ is holomorphic to first order if and only if ψ is both $\mathscr{H}$ and $\mathcal{V}$-holomorphic to first order.*

Proof. Assume that ψ is holomorphic to first order. Then, ψ_0 is $\mathscr{H}$-holomorphic. As for (4.7), letting Y denote an arbitrary section of TZ and $Y^{\mathscr{H}}$ its projection into $\mathscr{H}$, we have

$$\begin{aligned}
(4.7) \iff & \Big\langle \nabla_{\partial_t|_0}\Big((d\psi_t JX)^{\mathscr{H}} - \mathcal{J}^{\mathscr{H}}(d\psi_t X)^{\mathscr{H}}\Big), Y\Big\rangle = 0 \\
\iff & \partial_t|_0\Big\langle (d\psi_t JX)^{\mathscr{H}} - \mathcal{J}^{\mathscr{H}}(d\psi_t X)^{\mathscr{H}}, Y\Big\rangle - \Big\langle (d\psi_0 JX)^{\mathscr{H}} - \mathcal{J}^{\mathscr{H}}(d\psi_0 X)^{\mathscr{H}}, \nabla_{\partial_t|_0} Y\Big\rangle = 0 \\
\iff & (\text{since } \psi_0 \text{ is } \mathscr{H}\text{-holomorphic})\, \partial_t|_0\Big\langle d\psi_t JX - \mathcal{J} d\psi_t X, Y^{\mathscr{H}}\Big\rangle = 0 \\
\iff & (\psi_0 \text{ is holomorphic})\big(\nabla_{\partial_t|_0}(d\psi_t JX - \mathcal{J} d\psi_t X)\big)^{\mathscr{H}} = 0,
\end{aligned} \tag{4.8}$$

which is true since ψ is $\mathcal{J}$-holomorphic to first order. Hence, ψ is $\mathscr{H}$-holomorphic to first order. Changing $\mathscr{H}$ to $\mathcal{V}$ shows that ψ is $\mathcal{V}$-holomorphic to first order. The converse also follows using analogous arguments. □

Remark 4.4. The importance of choosing the Levi-Civita connection on Z is *illusory*. In particular, we can define the concept of *holomorphicity to first order* (*or* $\mathcal{H}$, $\mathcal{V}$*-holomorphicity to first order*) for maps defined between almost complex manifolds, not necessarily equipped with any metric.

Indeed, let $\psi : I \times (M, J) \to (Z, h, \mathcal{J})$ be a holomorphic to first-order map with respect to ∇, so that (4.6) holds. Let $\{Y_j\}$ denote a (local) frame for TZ. Then,

$$\begin{aligned}
&\nabla_{\partial_t|_0}\left(d\psi_t JX - \mathcal{J}d\psi_t X\right) = 0 \\
&\quad\Longleftrightarrow \sum_j \partial_t|_0\Big(\left(d\psi_t JX - \mathcal{J}d\psi_t X\right)_j Y_j\Big) = 0 \\
&\quad\Longleftrightarrow \sum_j \Big(\partial_t|_0\Big(\left(d\psi_t JX - \mathcal{J}d\psi_t X\right)_j\Big) \cdot Y_j + \left(d\psi_0 JX - \mathcal{J}d\psi_0 X\right)_j \cdot \nabla^{\psi^{-1}}_{\partial_t|_0} Y_j\Big) = 0.
\end{aligned} \tag{4.9}$$

Since ψ is holomorphic to first order, ψ_0 is holomorphic, the above equation is equivalent to

$$\partial_t|_0\Big(\left(d\psi_t JX - \mathcal{J}d\psi_t X\right)_j\Big) = 0, \quad \forall j. \tag{4.10}$$

Now, since ψ_0 holomorphicity does not depend on the chosen connection, we can deduce that holomorphicity with respect to $\widetilde{\nabla}$ reduces to the same condition (4.10). Thus, ψ being holomorphic to first order does not depend on the chosen connection. For $\mathcal{H}$ (resp., $\mathcal{V}$) holomorphicity to first order, we use similar arguments, replacing $\{Y_j\}$ for a horizontal (resp., vertical) frame.

4.3. The $\mathcal{H}$-Holomorphic Case

In the nonparametric case, given a conformal map $\varphi : M^2 \to N$, we can always find a lift $\psi : M^2 \to \Sigma^+ N$ such that φ is holomorphic with respect to J_ψ. In other words, (locally defined) strictly compatible lifts always exist. In general, this lift may not be $\mathcal{J}^1$ or $\mathcal{J}^2$-holomorphic but it is $\mathcal{H}$-holomorphic. Let φ_t be a variation of φ, conformal to first order. Then, if a lift ψ_t to the twistor space that makes φ_t holomorphic for all small t exists, φ_t is necessarily conformal for all small t, which may not be the case. So, even if conformality is preserved to first order, there might be no strictly compatible twistor lift for all t; hence, we should relax the condition on conformality. We shall say that a twistor lift ψ of a conformal to first order map φ is *compatible to first order* (with φ) if

$$\psi_0 \text{ is strictly compatible with } \varphi_0, \qquad \psi_t \text{ is compatible with } \varphi_t, \quad \forall t. \tag{4.11}$$

We start with a technical lemma, whose proof the reader can find in Section 5.

Lemma 4.5. *Let $\varphi : I \times M^2 \to N$ be a conformal to first-order map. Let ψ be a twistor lift compatible to first order with φ. Then for all $X \in \Gamma(TM)$ there is a function a_t^X and a vector field $v_t^X \in \varphi_t^{-1}(TN)$ with $a_0^X = 1$, $v_0^X = 0$ and $\partial_t|_0 a_t^X = 0$, $\nabla_{\partial_t|_0} v_t^X = 0$ such that*

$$J_{\psi_t} d\varphi_t X = a_t^X d\varphi_t JX + v_t^X. \tag{4.12}$$

In particular, φ is J_ψ-holomorphic to first order in the sense that

$$\varphi_0 \text{ is holomorphic with respect to } J_{\psi_0}, \qquad \nabla_{\partial_t|_0}\left(d\varphi_t JX - J_{\psi_t} d\varphi_t X\right) = 0. \tag{4.13}$$

Lemma 4.6. *Let $\psi : I \times M^2 \to \Sigma^+ N$ be $\mathcal{H}$-holomorphic to first order. Then, $\varphi = \pi \circ \psi$ is J_ψ-holomorphic to first order.*

Proof. Since ψ is $\mathcal{H}$-holomorphic to first order, $\nabla_{\partial_t|_0}((d\psi_t JX)^{\mathcal{H}} - \mathcal{J}^{\mathcal{H}}(d\psi_t X)^{\mathcal{H}}) = 0$. Therefore, for all $Y^{\mathcal{H}} \in \mathcal{H}$, since ψ_0 is $\mathcal{H}$-holomorphic,

$$0 = \left\langle \nabla_{\partial_t|_0}\left\{ (d\psi_t JX)^{\mathcal{H}} - \mathcal{J}^{\mathcal{H}}(d\psi_t X)^{\mathcal{H}} \right\}, Y^{\mathcal{H}} \right\rangle = \partial_t|_0 \left\langle (d\psi_t JX)^{\mathcal{H}} - \mathcal{J}^{\mathcal{H}}(d\psi_t X)^{\mathcal{H}}, Y^{\mathcal{H}} \right\rangle. \tag{4.14}$$

Since π is a J_ψ-holomorphic Riemannian submersion, the above equation can be written as

$$0 = \partial_t|_0 \left\langle d\pi(d\psi_t JX) - J_{\psi_t} d\pi(d\psi_t X), d\pi\left(Y^{\mathcal{H}}\right) \right\rangle. \tag{4.15}$$

Hence, for all $Y \in TN$, using the fact that φ_0 is J_{ψ_0} holomorphic,

$$0 = \partial_t|_0 \langle d\varphi_t JX - J_{\psi_t} d\varphi_t X, Y \rangle = \langle \nabla_{\partial_t|_0}(d\varphi_t JX - J_{\psi_t} d\varphi_t X), Y \rangle, \tag{4.16}$$

showing that $\nabla_{\partial_t|_0} d\varphi_t JX = \nabla_{\partial_t|_0} J_{\psi_t} d\varphi X$ and concluding our proof. □

Proposition 4.7. *Let $\psi : I \times M^2 \to \Sigma^+ N$ be $\mathcal{H}$-holomorphic to first order map. Then, the projected map $\varphi = \pi \circ \psi : I \times M^2 \to N$ is conformal to first order. Conversely, let $\varphi : I \times M^2 \to N$ be a conformal to first order map. Then there is a (local) $\mathcal{H}$-holomorphic to first order map $\psi : I \times M^2 \to \Sigma^+ N$ which is compatible to first order with φ.*

Proof. Take $\psi : I \times M^2 \to \Sigma^+ N$ an $\mathcal{H}$-holomorphic to first order map and let $\varphi = \pi \circ \psi$. We know that φ_0 is conformal (Proposition 3.2). As for the first-order variation, using the preceding Lemma 4.6,

$$\begin{aligned} \partial_t|_0 \|d\varphi_t JX\|^2 &= 2\langle \nabla_{\partial_t|_0} d\varphi_t JX, d\varphi_0 JX \rangle = 2\langle \nabla_{\partial_t|_0} J_{\psi_t} d\varphi_t X, J_{\psi_0} d\varphi_0 X \rangle \\ &= \partial_t|_0 \langle J_{\psi_t} d\varphi_t X, J_{\psi_t} d\varphi_t X \rangle = \partial_t|_0 \langle d\varphi_t X, d\varphi_t X \rangle. \end{aligned} \tag{4.17}$$

Using similar arguments, we can show that

$$\partial_t|_0 \langle d\varphi_t JX, d\varphi_t X \rangle = 0, \tag{4.18}$$

concluding the first part of the proof.

Conversely, let ψ be any twistor lift of φ compatible to first order. Then, there is a function a_t^X and a vector field v_t^X verifying the conditions on Lemma 4.5 with

$$J_{\psi_t} d\varphi_t X = a_t^X d\varphi_t JX + v_t^X. \tag{4.19}$$

Now, ψ is $\mathscr{H}$-holomorphic to first order if and only if ψ_0 is $\mathscr{H}$-holomorphic (which follows from Proposition 3.2) and (4.6) holds. Using the same argument as in Lemma 4.3, (4.6) is equivalent to

$$\partial_t|_0 \langle d\varphi_t JX - J_{\psi_t} d\varphi_t X, Y \rangle = 0, \quad \forall Y \in TN. \tag{4.20}$$

But

$$\begin{aligned} &\partial_t|_0 \langle d\varphi_t JX - J_{\psi_t} d\varphi_t X, Y \rangle \\ &\quad = \left\langle \nabla_{\partial_t|_0} \left(d\varphi_t JX - a_t^X d\varphi_t JX - v_t^X \right), Y \right\rangle - \left\langle d\varphi_0 JX - a_0^X d\varphi_0 JX - v_0^X, \nabla_{\partial_t|_0} Y \right\rangle = 0, \end{aligned} \tag{4.21}$$

from the given conditions on a_t^X and v_t^X and thus concluding the proof. □

4.4. *The $\mathscr{Q}_1$-Holomorphic Case*

Next, we give a useful characterization for maps to be $\mathscr{Q}^a$-holomorphic to first order ($a = 1$ or 2), whose proof is in Section 5 (compare with Proposition 3.3).

Lemma 4.8. *Let $\psi : I \times M^2 \to \Sigma^+ N$ be a smooth map and let $\varphi = \pi \circ \psi$ be its projection. Then, ψ is $\mathscr{Q}^a$-holomorphic to first order ($a = 1$ or 2) if and only if*

$$\varphi \text{ is } J_\psi\text{-holomorphic to first order}, \tag{4.22}$$

$$\begin{aligned} &\forall Y_t^{10} \in \varphi_t^{-1}\left(T_{J_{\psi_t}}^{10} N\right) \quad \exists Z_t^{10} \in \varphi_t^{-1}\left(T_{J_{\psi_t}}^{10} N\right) \text{ such that} \\ &\nabla_{\partial_t|_0} \nabla_{\partial_z} Y_t^{10} = \nabla_{\partial_t|_0} Z_t^{10}, \quad \nabla_{\partial_z} Y_0^{10} = Z_0^{10}, \end{aligned} \qquad (a = 1) \tag{4.23}$$

or

$$\begin{aligned} &\forall Y_t^{10} \in \varphi_t^{-1}\left(T_{J_{\psi_t}}^{10} N\right) \quad \exists Z_t^{10} \in \varphi_t^{-1}\left(T_{J_{\psi_t}}^{10} N\right) \text{ such that} \\ &\nabla_{\partial_t|_0} \nabla_{\partial_{\bar z}} Y_t^{10} = \nabla_{\partial_t|_0} Z_t^{10}, \quad \nabla_{\partial_{\bar z}} Y_0^{10} = Z_0^{10}. \end{aligned} \qquad (a = 2) \tag{4.24}$$

From the preceding lemma, we can also deduce the following.

Lemma 4.9. *Let $\psi : I \times M^2 \to \Sigma^+ N$ be a map $\mathscr{Q}^1$-holomorphic to first order and consider the projected map $\varphi = \pi \circ \psi$. Then for all $r \geq 1$ there is $Z_t^{10} \in \varphi_t^{-1}(T_{J_{\psi_t}}^{10} N)$ with*

$$\partial_z^r \varphi_0 = Z_0^{10}, \qquad \nabla_{\partial_t|_0} \partial_z^r \varphi_t = \nabla_{\partial_t|_0} Z_t^{10}. \tag{4.25}$$

Proof. For $r = 1$, we have $\partial_z \varphi_t = d\varphi_t X - i d\varphi_t JX$ so that using (4.22) we have

$$\nabla_{\partial_t|_0} \partial_z \varphi_t = \nabla_{\partial_t|_0} \{ d\varphi_t X - i d\varphi_t JX \} = \nabla_{\partial_t|_0} \{ d\varphi_t X - i J_{\psi_t} d\varphi_t X \}. \tag{4.26}$$

Taking $Z_t^{10} = d\varphi_t X - iJ_{\varphi_t} d\varphi_t X \in \varphi_t^{-1}(T^{10}_{J_{\varphi_t}} N)$, we obtain the result. To establish an induction, assume now that the result is valid for $r = k$; that is, there is $Z_t^{10,k}$ such that

$$\nabla_{\partial_t|_0} \partial_z^k \varphi_t = \nabla_{\partial_t|_0} Z_t^{10,k}, \qquad \partial_z^k \varphi_0 = Z_0^{10,k}. \tag{4.27}$$

Taking $r = k+1$ and noticing that $[\partial_t|_0 \varphi_t, \partial_z \varphi_t] = d\varphi_t[\partial_z, \partial_t|_0] = 0$,

$$\begin{aligned}\nabla_{\partial_t|_0} \partial_z^{k+1} \varphi &= \nabla_{\partial_t|_0} \nabla_{\partial_z} \partial_z^k \varphi_t = R^N\left(\partial_t|_0 \varphi_t, \partial_z \varphi_t\right) \partial_z^k \varphi_t + \nabla_{\partial_z} \nabla_{\partial_t|_0} \partial_z^k \varphi_t + \nabla_{[\partial_t|_0 \varphi_t, \partial_z \varphi_t]} \partial_z^k \varphi_t \\ &= R^N\left(\partial_t|_0 \varphi_t, \partial_z \varphi_0\right) Z_0^{10,k} + \nabla_{\partial_z} \nabla_{\partial_t|_0} Z_t^{10,k} \\ &= R^N\left(\partial_t|_0 \varphi_t, \partial_z \varphi_0\right) Z_0^{10,k} + \nabla_{\partial_t|_0} \nabla_{\partial_z} Z_t^{10,k} + R^N\left(\partial_z \varphi_0, \partial_t|_0 \varphi_t\right) Z_0^{10,k} = \nabla_{\partial_t|_0} \nabla_{\partial_z} Z_t^{10,k},\end{aligned} \tag{4.28}$$

as R^N is antisymmetric on the first two arguments. Now, since ψ is $\mathcal{J}^1$-holomorphic, (4.23) holds so that there is $Z_t^{10,k+1}$ such that

$$\nabla_{\partial_t|_0} \nabla_{\partial_z} Z_t^{10,k} = \nabla_{\partial_t|_0} Z_t^{10,k+1}, \qquad \nabla_{\partial_z} Z_0^{10,k} = Z_0^{10,k+1}. \tag{4.29}$$

But the second condition gives $\partial_z^{k+1} \varphi_0 = \partial_z \partial_z^k \varphi_0 = \nabla_{\partial_z} Z_0^{10,k} = Z_0^{10,k+1}$, whereas the first holds precisely that $\nabla_{\partial_t|_0} \partial_z^{k+1} \varphi_t = \nabla_{\partial_t|_0} \nabla_{\partial_z} Z_t^{10,k} = \nabla_{\partial_t|_0} Z_t^{10,k+1}$, as we wanted to show. □

Proposition 4.10 (projections of maps $\mathcal{J}^1$-holomorphic to first order). *Let $\psi : I \times M^2 \to \Sigma^+ N$ be a map $\mathcal{J}^1$-holomorphic to first order, where M^2 is any Riemann surface. Then, the projection map $\varphi = \pi \circ \psi$ is isotropic to first order.*

Notice that we could replace $\Sigma^+ N$ with $\Sigma^- N$, as real isotropy (to first order) does not depend on the fixed orientation on N.

Proof. That φ_0 is isotropic follows from the nonparametric case. Therefore, we are left with proving that

$$\partial_t|_0 \langle \partial_z^r \varphi, \partial_z^r \varphi \rangle = 0, \quad \forall r \geq 1. \tag{4.30}$$

Using Lemma 4.9, for fixed $r \geq 1$, choose $Z_t^{10} \in \varphi_t^{-1}(T^{10}_{J_{\varphi_t}} N)$ with $\partial_z^r \varphi_0 = Z_0^{10}$ and $\nabla_{\partial_t|_0} \partial_z^r \varphi_t = \nabla_{\partial_t|_0} Z_t^{10}$. Then, the conclusion follows from

$$\partial_t|_0 \langle \partial_z^r \varphi, \partial_z^r \varphi \rangle = 2 \langle \nabla_{\partial_t|_0} \partial_z^r \varphi_t, \partial_z^r \varphi_0 \rangle = 2 \left\langle \nabla_{\partial_t|_0} Z_t^{10}, Z_0^{10} \right\rangle = \partial_t|_0 \left\langle Z_t^{10}, Z_t^{10} \right\rangle = 0. \tag{4.31}$$

□

We now turn our attention to the existence of lifts $\mathcal{J}^1$-holomorphic to first order for a given isotropic to first-order map $\varphi : I \times M^2 \to N^4$. Recall that in the nonparametric case such lift exists (see Remark 3.4).

Theorem 4.11. *Let $\varphi : I \times M^2 \to N^4$ be an isotropic to first order with $\partial_z\varphi_0$ and $\partial_z^2\varphi_0$ linearly independent. Then, there is either a (local) map $\psi^+ : I\times M^2 \to \Sigma^+N^4$ or a map $\psi^- : I\times M^2 \to \Sigma^-N^4$ which is $\mathcal{J}^1$-holomorphic to first order and compatible to first order with φ.*

Before proving Theorem 4.11, we give the following lemma, which we prove in Section 5

Lemma 4.12. *Let φ be as in the preceding Theorem 4.11.*

(i) *Suppose that the $\mathcal{J}^1$-holomorphic lift of φ_0 is $\psi_0^+ \in \Sigma^+N$ (resp., $\psi_0^- \in \Sigma^-N$). Take J_{φ_t} the unique positive (resp., negative) almost Hermitian structure on $T_{\varphi_t}N$ compatible with φ_t. Then, taking*

$$u_t^X = \nabla_X d\varphi_t X - \nabla_{JX} d\varphi_t JX, \qquad v_t^X = -\nabla_X d\varphi_t JX - \nabla_{JX} d\varphi_t X, \tag{4.32}$$

we have that

$$\nabla_{\partial_t|_0} J_{\varphi_t} u_t^X = -\nabla_{\partial_t|_0} v_t^X. \tag{4.33}$$

(ii) *There are a_t, b_t such that*

$$\nabla_{\partial_t|_0} \partial_z^3 \varphi_t = \nabla_{\partial_t|_0}\left(a_t \partial_z \varphi_t + b_t \partial_z^2 \varphi_t\right). \tag{4.34}$$

We are now ready to prove Theorem 4.11.

Proof of Theorem 4.11. As before, take ψ_0^+ or ψ_0^- the $\mathcal{J}^1$-holomorphic lift of φ_0. Assume, without loss of generality, that it is ψ_0^+. Then, at each t take J_{φ_t} the unique positive almost Hermitian structure compatible with φ_t and let us prove that this map ψ is $\mathcal{J}^1$-holomorphic to first order. Using Lemma 4.5, φ is J_φ-holomorphic to first order and we are left with proving that (4.23) holds. It is enough to prove that there is a basis $\{Y_{1_t}^{10}, Y_{2_t}^{10}\}$ of $\varphi_t^{-1}(T_{J_{\varphi_t}}^{10}N)$ for which (4.23) holds. Now, take $Y_{1_t}^{10} = d\varphi_t X - iJ_{\varphi_t} d\varphi_t X$ and $Y_{2_t}^{10} = u_t^X - iJ_{\varphi_t} u_t^X$ where u_t^X is as in (4.32). Then,

$$\begin{aligned}\nabla_{\partial_t|_0}\nabla_{\partial_z} Y_{1_t}^{10} &= R(\partial_t|_0\varphi_t, \partial_z\varphi_t)Y_{1_0}^{10} + \nabla_{\partial_z}\nabla_{\partial_t|_0}(d\varphi_t X - iJ_{\varphi_t} d\varphi_t X)\\ &= \nabla_{\partial_t|_0}\nabla_{\partial_z}(d\varphi_t X - id\varphi JX) = \nabla_{\partial_t|_0}\left(u_t^X + iv_t^X\right) = \nabla_{\partial_t|_0}\left(u_t^X - iJ_{\varphi_t} u_t^X\right) = \nabla_{\partial_t|_0} Y_{2_t}^{10}.\end{aligned} \tag{4.35}$$

Analogously,

$$\begin{aligned}
\nabla_{\partial_t|_0}\nabla_{\partial_z}Y^{10}_{2_t} &= R(\partial_t|_0\varphi_t,\partial_z\varphi_0)Y^{10}_{2_0}+\nabla_{\partial_z}\nabla_{\partial_t|_0}\left(u^X_t - iJ_{\varphi_t}u^X_t\right)\\
&= R(\partial_t|_0\varphi_t,\partial_z\varphi_0)\partial^2_z\varphi_0 + R(\partial_z\varphi_0,\partial_t|_0\varphi_t)\left(u^X_0+iv^X_0\right)+\nabla_{\partial_t|_0}\nabla_{\partial_z}\left(u^X_t+iv^X_t\right)\\
&= \nabla_{\partial_t|_0}\left(a\partial_z\varphi + b\partial^2_z\varphi_t\right) = \partial_t|_0 aY^{10}_{1_0}+\partial_t|_0 bY^{10}_{2_0}+a\nabla_{\partial_t|_0}\partial_z\varphi_t+\nabla_{\partial_t|_0}\partial^2_z\varphi_t\\
&= \partial_t|_0 aY^{10}_{1_0}+\partial_t|_0 bY^{10}_{2_0}+a\nabla_{\partial_t|_0}Y^{10}_{1_t}+\nabla_{\partial_t|_0}Y^{10}_{2_t} = \nabla_{\partial_t|_0}\left(aY^{10}_{1_t}+bY^{10}_{2_t}\right),
\end{aligned}\tag{4.36}$$

where we have used $Y^{10}_{2_0}=\partial^2_z\varphi_0$, $\nabla_{\partial_t|_0}J_{\varphi_t}u^X_t=\nabla_{\partial_t|_0}v^X_t$, $u^X_0+iv^X_0=\partial^2_z\varphi_0$, and $\nabla_{\partial_z}(u^X_t+iv^X_t)=\partial^3_z\varphi_t$. Hence, $Y^{10}_{1_t}$ and $Y^{10}_{2_t}$ satisfy equation (4.23), concluding our proof. □

4.5. The $\mathcal{J}_2$ Holomorphic Case

We prove the following.

Theorem 4.13. *Let $\psi : I\times M^2 \to \Sigma^+N$ be a map $\mathcal{J}^2$-holomorphic to first order. Then, $\varphi=\pi\circ\psi : I\times M^2 \to N$ is harmonic to first order (and conformal to first order from Lemma 4.3 and Proposition 4.7).*

We first give the following characterization of $\mathcal{V}$-holomorphic to first order maps.

Lemma 4.14. *Let $\psi : I\times M^2 \to (\Sigma^+N,\mathcal{J}^a)$ ($a=1$ or $a=2$) be a smooth map. Then, ψ is a $\mathcal{V}$-holomorphic to first order map if and only if ψ_0 is $\mathcal{V}$-holomorphic and*

$$\nabla_{\partial_t|_0}\left(\nabla_{JX}J_{\psi_t}+(-1)^aJ_{\psi_t}\nabla_XJ_{\psi_t}\right)=0.\tag{4.37}$$

Proof. If $\psi : M\to\Sigma^+N$ is any smooth map, then [17] $(d\psi X)^{\mathcal{V}}=\nabla_XJ_\psi$. Hence, from (2.9),

$$\mathcal{J}^a(d\psi X)^{\mathcal{V}}=(-1)^{a+1}J_\psi\nabla_XJ_\psi \quad (a=1,2).\tag{4.38}$$

Thus, we can rephrase equation (4.37) as

$$\nabla_{\partial_t|_0}\left(d\psi_t(JX)-\mathcal{J}^ad\psi_tX\right)^{\mathcal{V}}=0,\tag{4.39}$$

which is the condition for $\mathcal{V}$-holomorphicity. □

Proof of Theorem 4.13. That φ_0 is harmonic follows from Theorem 3.1. Hence, we are left with proving that $\partial_t|_0\tau(\varphi_t) = 0$. Since ψ is $\mathcal{J}^2$-holomorphic to first order, we deduce that ψ is both $\mathcal{H}$ and $\mathcal{V}$-holomorphic to first order (Lemma 4.3). From $\mathcal{V}$-holomorphicity (4.37), we have

$$\begin{aligned}
&\nabla_{\partial_t|_0}(\nabla_{JX}J_{\varphi_t} + J_{\varphi_t}\nabla_X J_{\varphi_t}) = 0 \\
&\quad\Longrightarrow (\nabla_{\partial_t|_0}(\nabla_{JX}J_{\varphi_t} + J_{\varphi_t}\nabla_X J_{\varphi_t}))(d\varphi_0 X) = 0 \\
&\quad\Longrightarrow \nabla_{\partial_t|_0}(\nabla_{JX}d\varphi_t JX + \nabla_X d\varphi_t X) = \nabla_{\partial_t|_0}(J_{\varphi_t}(\nabla_{JX}d\varphi_t X - \nabla_X(J_{\varphi_t}d\varphi_t X))) \\
&\quad\Longrightarrow \nabla_{\partial_t|_0}\tau(\varphi_t) = \nabla_{\partial_t|_0}(J_{\varphi_t}(\nabla_{JX}d\varphi_t X - \nabla_X(J_{\varphi_t}d\varphi_t X))).
\end{aligned} \tag{4.40}$$

Using Lemma 4.6 and (4.13) together with symmetry of the second fundamental form of φ_0, the right-hand side of the above identity becomes

$$\begin{aligned}
&(\nabla_{\partial_t|_0}J_{\varphi_t})(\nabla_{JX}d\varphi_0 X - \nabla_X J_{\varphi_0}d\varphi_0 X) + J_{\varphi_0}(\nabla_{\partial_t|_0}(\nabla_{JX}d\varphi_t X - \nabla_X J_{\varphi_t}d\varphi_t X)) \\
&\quad= \nabla_{\partial_t|_0}J_{\varphi_t}(\nabla d\varphi_0(JX,X) - \nabla d\varphi_0(X,JX) + d\varphi_0(\nabla_{JX}X - \nabla_X JX)) \\
&\qquad+ J_{\varphi_0}(\nabla_{\partial_t|_0}(\nabla_{JX}d\varphi_t X - \nabla_X d\varphi_t JX)) \\
&\quad= J_{\varphi_0}(\nabla_{\partial_t|_0}(\nabla d\varphi_t(JX,X) - \nabla d\varphi_t(X,JX) + d\varphi_t(\nabla_{JX}X - \nabla_X JX))) = 0,
\end{aligned} \tag{4.41}$$

so that $\partial_t|_0\tau(\varphi_t) = 0$, concluding the proof. □

Theorem 4.15. *Let $\varphi : I \times M^2 \to N$ be a harmonic and conformal to first order map. Then, there is (locally) a map $\psi : I \times M^2 \to \Sigma^+ N$ which is $\mathcal{J}^2$-holomorphic to first order and with $\varphi = \pi \circ \psi$.*

Since harmonicity (to first order) does not depend on the orientation on N, we could have replaced $\Sigma^+ N$ by $\Sigma^- N$ in both Theorems 4.13 and 4.15

Proof. For each t consider $V_t = d\varphi_t(TM)^\perp \subseteq \varphi_t^{-1}(TN)$, bundle over M^2. Since M^2 is a Riemann surface, $R^{20}_{V_t} = 0$ and we can conclude that for each t there is a Koszul-Malgrange holomorphic structure on $\Sigma^+ V_t$. Hence ([16, Theorem I.5.1.]), there is a smooth section $\widehat{s}$ with $\widehat{s}_t$ a Koszul-Malgrange holomorphic section of $\Sigma^+ V_t$: $\nabla^\perp_{\partial_{\bar z}} s_t \subseteq s_t$. So,

$$J_{\varphi_t}\left(\nabla^\perp_{X+iJX}(v_t - iJ_{\varphi_t}v_t)\right) = i\nabla^\perp_{X+iJX}(v_t - iJ_{\varphi_t}v_t), \quad \forall v_t \in d\varphi_t(TM)^\perp, \tag{4.42}$$

equivalently,

$$\begin{aligned}
J_{\varphi_t}\left(\nabla^\perp_X v_t + \nabla^\perp_{JX}J_{\varphi_t}v_t\right) &= -\nabla^\perp_{JX}v_t + \nabla^\perp_X J_{\varphi_t}v_t, \\
J_{\varphi_t}\left(\nabla^\perp_{JX}v_t - \nabla^\perp_X J_{\varphi_t}v_t\right) &= \nabla^\perp_X v_t + \nabla^\perp_{JX}J_{\varphi_t}v_t.
\end{aligned} \tag{4.43}$$

Take $s = \widehat{s} \oplus T^{10}$ where T^{10}_t is the $(1,0)$-part on $d\varphi_t(TM)^{\mathbb{C}}$ determined by $J^\top_t$ = rotation by $+\pi/2$ on $d\varphi_t(TM)$. (Notice that as φ_t is not conformal we might not get a Hermitian structure by setting $T^{10}_t = d\varphi_t(T^{10}M)$; on the other hand, positive rotation by $\pi/2$ comes from the natural orientation on $d\varphi_t(TM)$ imported from TM *via* $d\varphi_t$.) Then s defines a compatible twistor lift of φ. Let us check that ψ is $\mathcal{J}^2$-holomorphic to first order. That ψ_0 is holomorphic is immediate. From the proof of Proposition 4.7, we deduce that ψ is $\mathcal{H}$-holomorphic to first order as it is compatible to first order with φ and the latter is conformal to first order. Hence, we are left with proving that (4.37)

$$\nabla_{\partial_t|_0}\nabla_{JX}J_{\psi_t} = -\nabla_{\partial_t|_0}J_{\psi_t}\nabla_X J_{\psi_t} \tag{4.44}$$

holds. We shall establish this equation by showing that both sides agree when applied to any vector $v \in TN$. For that, we consider, in turn, the three cases $v = d\varphi_0 X$, $d\varphi_0 JX$, and $v \in d\varphi_0(TM)^\perp$. The first two have similar arguments so that we prove only the first.

(i) $v = d\varphi_0 X$. From ψ_0 holomorphicity, we have $\nabla_{JX}J_{\psi_0} = -J_{\psi_0}\nabla_X J_{\psi_0}$. On the other hand, as ψ is $\mathcal{H}$-holomorphic to first order, (4.13) is satisfied. Finally, for all t,

$$\nabla_{JX}d\varphi_t X - \nabla_X d\varphi_t JX = \nabla d\varphi_t(JX,X) - \nabla d\varphi_t(X,JX) + d\varphi_t\big(\nabla_{JX}X - \nabla_X JX\big) = 0. \tag{4.45}$$

Since φ is harmonic to first order, our condition follows from

$$\begin{aligned}
\big(\nabla_{\partial_t|_0}\nabla_{JX}J_{\psi_t}\big)d\varphi_0 X &= -\big(\nabla_{\partial_t|_0}J_{\psi_t}\nabla_X J_{\psi_t}\big)d\varphi_0 X\\
&\Longleftrightarrow \nabla_{\partial_t|_0}\nabla_{JX}\big(J_{\psi_t}d\varphi_t X\big) - \nabla_{\partial_t|_0}\big(J_{\psi_t}\nabla_{JX}d\varphi_t X\big)\\
&\Longleftrightarrow \nabla_{\partial_t|_0}\big(\nabla_{JX}d\varphi_t JX + \nabla_X d\varphi_t X\big)\\
&= \big(\nabla_{\partial_t|_0}J_{\psi_t}\big)\big(\nabla_{JX}d\varphi_0 X - \nabla_X d\varphi_0 JX\big)\\
&= -\nabla_{\partial_t|_0}\big(J_{\psi_t}\nabla_X\big(J_{\psi_t}d\varphi_t X\big)\big) - \nabla_{\partial_t|_0}\nabla_X d\varphi_t X\\
&\quad + J_{\psi_0}\nabla_{\partial_t|_0}\big(\nabla_{JX}d\varphi_t X - \nabla_X d\varphi_t JX\big)\\
&\Longleftrightarrow \partial_t|_0\tau(\varphi_t) = 0.
\end{aligned} \tag{4.46}$$

(ii) $v \in d\varphi_0(TM)^\perp$. We now have

$$\begin{aligned}
&\big(\nabla_{\partial_t|_0}\nabla_{JX}J_{\psi_t}\big)v = -\big(\nabla_{\partial_t|_0}J_{\psi_t}\nabla_X J_{\psi_t}\big)v\\
&\quad\Longleftrightarrow \nabla_{\partial_t|_0}\nabla_{JX}\big(J_{\psi_t}v\big) - \nabla_{\partial_t|_0}\big(J_{\psi_t}\nabla_{JX}v\big) = -\nabla_{\partial_t|_0}\big(J_{\psi_t}\nabla_X\big(J_{\psi_t}v\big)\big) - \nabla_{\partial_t|_0}\nabla_X v\\
&\quad\Longleftrightarrow \begin{cases} \nabla_{\partial_t|_0}\nabla^\perp_{JX}\big(J_{\psi_t}v\big) - \nabla_{\partial_t|_0}\Big(J_{\psi_t}\nabla^\perp_{JX}v\Big) + \nabla_{\partial_t|_0}\big(J_{\psi_t}\nabla^\perp_X\big(J_{\psi_t}v\big)\big) + \nabla_{\partial_t|_0}\nabla^\perp_X v = 0,\\ \nabla_{\partial_t|_0}\nabla^\top_{JX}\big(J_{\psi_t}v\big) - \nabla_{\partial_t|_0}\Big(J_{\psi_t}\nabla^\top_{JX}v\Big) + \nabla_{\partial_t|_0}\big(J_{\psi_t}\nabla^\top_X\big(J_{\psi_t}\mathrm{v}\big)\big) + \nabla_{\partial_t|_0}\nabla^\top_X v = 0. \end{cases}
\end{aligned} \tag{4.47}$$

Now, the first condition follows from (4.43) since s is Koszul-Malgrange holomorphic for each t. As for the second, letting L denote its left-hand side, we shall prove that $\langle L, w\rangle = 0$ for all $w \in TN$. We do this by establishing the three cases $w = d\varphi_0 X$, $w = d\varphi_0 JX$ and $w \in d\varphi_0(TM)^\perp$ (since the first two cases have similar arguments, we prove only the first).

(ii$_a$) When $w = d\varphi_0 X$ we have

$$
\begin{aligned}
\langle L, d\varphi_0 X\rangle &= -\partial_t|_0\left(\langle J_{\varphi_t} v_t, \nabla_{JX} d\varphi_t X\rangle + \langle v_t, \nabla_X d\varphi_t X\rangle\right) \\
&\quad + \partial_t|_0\left(-\langle v_t, \nabla_{JX} d\varphi_t JX\rangle + \langle J_{\varphi_t} v_t, \nabla_X d\varphi_t JX\rangle\right) \\
&= -\partial_t|_0\langle v_t, \nabla_X d\varphi_t X + \nabla_{JX} d\varphi_t JX\rangle + \partial_t|_0\langle J_{\varphi_t} v_t, \nabla_X d\varphi_t JX - \nabla_{JX} d\varphi_t X\rangle \\
&= -\langle \nabla_{\partial_t|_0} v_t, \tau(\varphi_0)\rangle - \langle v_0, \nabla_{\partial_t|_0}\tau(\varphi_t)\rangle = 0.
\end{aligned}
\tag{4.48}
$$

(ii$_b$) Let $w_t \in d\varphi_t(TM)^\perp$. Then,

$$
\begin{aligned}
\langle L, w\rangle = \partial_t|_0 &\left\langle \nabla^\top_{JX}(J_{\varphi_t} v) - J_{\varphi_t}\nabla^\top_{JX} v + J_{\varphi_t}\nabla^\top_X(J_{\varphi_t} v) + \nabla^\top_X v, w\right\rangle \\
&- \left\langle \nabla^\top_{JX}(J_{\varphi_0} v) - J_{\varphi_0}\nabla^\top_{JX} v + J_{\varphi_0}\nabla^\top_X(J_{\varphi_0} v) + \nabla^\top_X v, \nabla_{\partial_t|_0} w\right\rangle.
\end{aligned}
\tag{4.49}
$$

The first term on the right side of the above equation vanishes as w_t lies in $d\varphi_t(TM)^\perp$, whereas the second is zero from ψ_0-holomorphicity, concluding our proof. □

4.6. The 4-Dimensional Case

Theorem 4.16. *Let $\varphi : I \times M^2 \to N^4$ be harmonic and isotropic to first-order map and with $\partial_z\varphi_0$ and $\partial_z^2\varphi_0$ being linearly independent. Then, (locally) there is either a map $\psi^+ : I \times M^2 \to \Sigma^+ N$ or a map $\psi^- : I \times M^2 \to \Sigma^- N$ which is simultaneously $\mathcal{J}^1$ and $\mathcal{J}^2$-holomorphic to first order and with $\varphi = \pi \circ \psi$. Conversely, if $\psi : I \times M^2 \to \Sigma^+ N^4$ (or $\psi : I \times M^2 \to \Sigma^- N^4$) is $\mathcal{J}^1$ and $\mathcal{J}^2$-holomorphic to first order, the projected map $\varphi = \pi \circ \psi : I \times M^2 \to N^4$ is harmonic and isotropic to first order.*

Proof. The converse is obvious from Proposition 4.10 and Theorem 4.13. As for the first part, in Theorem 4.11 we saw that we can lift the map φ to a map $\mathcal{J}^1$-holomorphic to first order. Moreover, this lift could be defined as the unique positive or negative almost complex structure compatible with φ. On the other hand, in Theorem 4.15 we have seen that there is a map $\mathcal{J}^2$-holomorphic to first order with $\varphi = \pi \circ \psi$ and for which φ is compatible. From the comment after Theorem 4.15, there is also a twistor lift of φ into $\Sigma^- N$. Therefore, from the dimension of N, we conclude that the lifts constructed in both cited results are the same and, therefore, simultaneously $\mathcal{J}^1$ and $\mathcal{J}^2$-holomorphic to first order. □

We would now like to guarantee the *uniqueness to first order* of our twistor lift. Before stating such a result, we start with a lemma, proved in Section 5.

Lemma 4.17. *Let $\psi : I \times M^2 \to \Sigma^+ N$ be a map $\mathcal{J}^1$-holomorphic to first order. Consider the twistor projection $\varphi_t = \pi \circ \psi$ and the vectors u_t and v_t defined by*

$$
\partial_z^2 \varphi_t = u_t + i v_t \tag{4.50}
$$

so that

$$u_t = \nabla_X d\varphi_t X - \nabla_{JX} d\varphi_t JX, \qquad v_t = -\nabla_X d\varphi_t JX - \nabla_{JX} d\varphi_t X. \tag{4.51}$$

Then, for all z_0 *for which* $\partial_z \varphi_0(z_0)$ *and* $\partial_z^2 \varphi_0(z_0)$ *are linearly independent*

$$\nabla_{\partial_t|_0} d\varphi_t JX = \nabla_{\partial_t|_0} J_{\psi_t} d\varphi_t X, \qquad \nabla_{\partial_t|_0} J_{\psi_t} u_t = -\nabla_{\partial_t|_0} v_t, \qquad \nabla_{\partial_t|_0} J_{\psi_t} v_t = \nabla_{\partial_t|_0} u_t. \tag{4.52}$$

Notice that in Lemma 4.12, we were given φ and *defined* the twistor lift as the unique lift compatible with φ. Now, we are given the twistor map ψ but nothing guarantees that projecting the map to φ_t makes φ compatible; that is, J_{ψ_t} may not preserve $d\varphi_t(TM)$.

Proposition 4.18. *Let* $\psi^1, \psi^2 : I \times M^2 \to \Sigma^+ N^4$ *be two* $\mathcal{J}^1$*-holomorphic to first-order maps such that* $\psi_0^1 = \psi_0^2$ *and the variational vector fields induced on* N^4 *are the same; that is, writing* $a_i := \partial_t|_0(\pi \circ \psi^i)$, $i = 1, 2$, *we have* $a_1 = a_2$. *Then, at all points* z_0 *for which* $\partial_z \varphi_0(z_0)$ *and* $\partial_z^2 \varphi_0(z_0)$ *are linearly independent, writing* $w_i = \partial_t|_0 \psi_t^i$, $i = 1, 2$, *we have* $w_1 = w_2$.

Proof. Let $\varphi^i = \pi \circ \psi_i$ $(i = 1, 2)$ denote the projection maps. From our hypothesis, it follows that $w_1^{\mathscr{H}} = w_2^{\mathscr{H}}$. Hence, the only thing left is to prove that the vertical parts coincide. Now, from the proof of Lemma 4.14, $(\partial_t|_0 \psi_t^i)^{\mathcal{V}} = \nabla_{\partial_t|_0} J_{\psi_t^i}$ so that our result follows if $\nabla_{\partial_t|_0} J_{\psi_t^1} = \nabla_{\partial_t|_0} J_{\psi_t^2}$. We prove this identity showing that $Q(Y) = 0$ for all Y, where

$$Q(Y) = \left(\nabla_{\partial_t|_0} J_{\psi_t^1}\right) Y - \left(\nabla_{\partial_t|_0} J_{\psi_t^2}\right) Y. \tag{4.53}$$

We consider the four possible cases for Y; namely, when Y is equal to $d\varphi_0 X$, $d\varphi_0 JX$, u_0 or v_0, where u_0 and v_0 are as in the preceding lemma (notice that, since $\psi_0^1 = \psi_0^2$, then $u_0^1 = u_0^2$ and $v_0^1 = v_0^2$).

(i) When $Y = d\varphi_0 X$ ($Y = d\varphi_0 JX$ uses similar arguments), we have

$$\begin{aligned} Q(d\varphi_0 X) &= \nabla_{\partial_t|_0}\left(J_{\psi_t^1} d\varphi_t^1\right) - J_{\psi_0^1}(\nabla_{\partial_t|_0} d\varphi_t X) - \nabla_{\partial_t|_0}\left(J_{\psi_t^2} d\varphi_t^2\right) + J_{\psi_0^2}\left(\nabla_{\partial_t|_0} d\varphi_t^2 X\right) \\ &= \nabla_{\partial_t|_0} d\varphi_t^1 JX - \nabla_{\partial_t|_0} d\varphi_t^2 JX - J_{\psi_0}(\nabla_X a_1 - \nabla_X a_2) = \nabla_{JX} a_1 - \nabla_{JX} a_2 = 0, \end{aligned} \tag{4.54}$$

where we have used Lemma 4.17, as well as the fact that $J_{\psi_0^2} = J_{\psi_0^1}$ and $\nabla_{\partial_t|_0} d\varphi_t^i X = \nabla_X a_i$.

(ii) Taking $Y = u_0$ ($Y = v_0$ uses similar arguments), we have

$$\begin{aligned} Q(u_0) &= \nabla_{\partial_t|_0}\left(J_{\psi_t^1} u_t^1\right) - J_{\psi_0^1}\left(\nabla_{\partial_t|_0} u_t^1\right) - \nabla_{\partial_t|_0}\left(J_{\psi_t^2} u_t^2\right) + J_{\psi_0^2}\left(\nabla_{\partial_t|_0} u_t^2\right) \\ &= -\nabla_{\partial_t|_0} v_t^1 + \nabla_{\partial_t|_0} v_t^2 - J_{\psi_0}\left(\nabla_{\partial_t|_0} u_t^1 - \nabla_{\partial_t|_0} u_t^2\right). \end{aligned} \tag{4.55}$$

But

$$\begin{aligned}\nabla_{\partial_t|_0} v_t^1 &= -\nabla_{\partial_t|_0}\nabla_X d\varphi_t^1 JX - \nabla_{\partial_t|_0}\nabla_{JX} d\varphi_t^1 X \\ &= -R^N\left(\partial_t|_0\varphi_t^1, d\varphi_t^1 X\right) d\varphi_t^1 JX + \nabla_X \nabla_{\partial_t|_0} d\varphi_t^1 JX \\ &\quad + R^N\left(\partial_t|_0\varphi_t^1, d\varphi_t^1 JX\right) d\varphi_t^1 X + \nabla_{JX}\nabla_{\partial_t|_0} d\varphi_t^1 X \\ &= -R^N(a_1, d\varphi_0 X) d\varphi_0 JX + \nabla_X \nabla_{JX} a_1 + R^N(a_1, d\varphi_0 JX) d\varphi_0 X + \nabla_{JX}\nabla_X a_1.\end{aligned} \tag{4.56}$$

As $a_1 = a_2$, we deduce $\nabla_{\partial_t|_0} v_t^1 = \nabla_{\partial_t|_0} v_t^2$; analogously, $\nabla_{\partial_t|_0} u_t^1 = \nabla_{\partial_t|_0} u_t^2$ so that $Q(u_0) = 0$. □

Hence, the twistor lifts constructed in Theorem 4.16 are *unique to first order*, in the sense that the vector field w induced on $\Sigma^+ N^4$ (or $\Sigma^- N^4$) by the map φ, $w = \partial_t|_0 \varphi_t$ depends only on the initial projected map φ_0 and on the Jacobi field v along φ_0. Moreover, taking N^4 the 4-sphere or the complex projective plane, letting $\varphi : M^2 \to N^4$ be a harmonic map, and $v \in \varphi^{-1}(TN)$ a Jacobi field, isotropy to first order is immediately guaranteed. Hence, the previous construction allows a (local) unified proof of the twistor correspondence between Jacobi fields and twistor vector fields that are tangent to variations on $\Sigma^+ N^4$ which are simultaneously $\mathcal{J}^1$ and $\mathcal{J}^2$-holomorphic (*infinitesimal horizontal holomorphic deformations* in [5]). We can also conclude which different properties (namely, conformality, real isotropy or harmonicity) are related with those of the twistor lift (resp., $\mathcal{H}$, $\mathcal{J}^1$ or $\mathcal{J}^2$-holomorphicity).

5. Additional Proofs

Proof of Lemma 4.5. Since φ is compatible to first order, J_{φ_t} preserves $d\varphi_t(TM)$ for all t. Hence, there are a_t^X and b_t^X such that

$$J_{\varphi_t} d\varphi_t X = a_t^X d\varphi_t JX + b_t^X d\varphi_t X. \tag{5.1}$$

Take $v_t^X = b_t^X d\varphi_t X$. Since, at $t = 0$, $J_{\varphi_0} d\varphi_0 X = d\varphi_0 JX$ we deduce $v_0^X = 0$ and $a_0^X = 1$. Now, since $d\varphi_t X$ and $J_{\varphi_t} d\varphi_t X$ form an orthogonal basis for $d\varphi_t(TM)$, we have

$$\begin{aligned} d\varphi_t JX &= \frac{\langle d\varphi_t JX, d\varphi_t X\rangle}{\|d\varphi_t X\|^2} d\varphi_t X + \frac{\langle d\varphi_t JX, J_{\varphi_t} d\varphi_t X\rangle}{\|J_{\varphi_t} d\varphi_t X\|^2} J_{\varphi_t} d\varphi_t X \\ &\Longrightarrow \langle J_{\varphi_t} d\varphi_t X, d\varphi_t JX\rangle^2 = \|d\varphi_t JX\|^2 \|d\varphi_t X\|^2 - \langle d\varphi_t JX, d\varphi_t X\rangle^2. \end{aligned} \tag{5.2}$$

Differentiating with respect to t at the point $t = 0$, the above identity yields

$$\partial_t|_0 \langle J_{\varphi_t} d\varphi_t X, d\varphi_t JX\rangle = \partial_t|_0 \langle d\varphi_t X, d\varphi_t X\rangle. \tag{5.3}$$

Computing the inner product of (5.1) with $J_{\varphi_t} d\varphi_t X$ and using the fact that $\langle d\varphi_t X, J_{\varphi_t} d\varphi_t X\rangle$ vanishes for all t, we get $\langle d\varphi_t X, d\varphi_t X\rangle = a_t^X \langle d\varphi_t JX, J_{\varphi_t} d\varphi_t X\rangle$. Hence,

$$\partial_t|_0 \langle d\varphi_t X, d\varphi_t X\rangle = \partial_t|_0 a_t^X \langle d\varphi_0 JX, J_{\varphi_0} d\varphi_0 X\rangle + a_0^X \partial_t|_0 \langle d\varphi_t JX, J_{\varphi_t} d\varphi_t X\rangle \tag{5.4}$$

and we deduce $\partial_t|_0 a_t^X = 0$, as $a_0^X = 1$ and (5.3) hold. Using (5.1) again, we can now write

$$\begin{aligned} \nabla_{\partial_t|_0} v_t^X &= \nabla_{\partial_t|_0} J_{\varphi_t} d\varphi_t X - \partial_t|_0 a_t^X d\varphi_0 JX - \nabla_{\partial_t|_0} d\varphi_t JX \\ &\Longrightarrow \left\langle \nabla_{\partial_t|_0} v_t^X, d\varphi_0 X \right\rangle = \partial_t|_0 \langle J_{\varphi_t} d\varphi_t X, d\varphi_t X\rangle - \langle J_{\varphi_0} d\varphi_0 X, \nabla_{\partial_t|_0} d\varphi_t X\rangle \\ &\quad - \partial_t|_0 \langle d\varphi_t JX, d\varphi_t X\rangle + \langle d\varphi_0 JX, \nabla_{\partial_t|_0} d\varphi_t X\rangle, \end{aligned} \tag{5.5}$$

which vanishes since $\langle J_{\varphi_t} d\varphi_t X, d\varphi_t X\rangle = 0$ for all t, the second and last terms cancelling as φ_0 is J_{φ_0}-holomorphic and φ is conformal to first order. Analogously, $\langle \nabla_{\partial_t|_0} v_t^X, J_{\varphi_0} d\varphi_0 X\rangle$ vanishes so that $\langle \nabla_{\partial_t|_0} v_t^X, d\varphi_0(TM)\rangle = 0$. For the orthogonal part, taking $r_t \in (d\varphi_t TM)^\perp$,

$$\left\langle \nabla_{\partial_t|_0} v_t^X, r_0 \right\rangle = \partial_t|_0 \left\langle v_t^X, r_t \right\rangle - \left\langle v_0^X, \nabla_{\partial_t|_0} r_t \right\rangle = 0, \tag{5.6}$$

showing that $\nabla_{\partial_t|_0} v_t^X = 0$ and concluding the proof. □

Proof of Lemma 4.8. We shall do the proof only for the $\mathcal{J}^1$ case, with the $\mathcal{J}^2$ case being similar. Assume that φ is $\mathcal{J}^1$-holomorphic to first order. Then, using Lemmas 4.3 and 4.6, φ is J_φ-holomorphic to first order. On the other hand, φ satisfies equation (4.37), which implies that for all Y_t

$$\nabla_{\partial_t|_0} \left(\nabla_{JX} (J_{\varphi_t} Y_t) - \nabla_X Y_t \right) = \nabla_{\partial_t|_0} \left(J_{\varphi_t} \left(\nabla_{JX} Y_t + \nabla_X (J_{\varphi_t} Y_t) \right) \right). \tag{5.7}$$

Take Y_t^{10} in $T^{10}_{J_{\varphi_t}} N$. Let Y_t be such that $Y_t^{10} = (1/2)(Y_t - iJ_{\varphi_t} Y_t)$ and write $2iZ_t = \nabla_{JX} Y_t + \nabla_X (J_{\varphi_t} Y_t)$. Then,

$$\begin{aligned} \nabla_{\partial_t|_0} \nabla_{\partial_z} Y_t^{10} &= \frac{1}{4} \nabla_{\partial_t|_0} \left(-J_{\varphi_t} \left(\nabla_{JX} Y_t + \nabla_X (J_{\varphi_t} Y_t) \right) - i \left(\nabla_X (J_{\varphi_t} Y_t) + \nabla_{JX} Y_t \right) \right) \\ &= \frac{1}{4} \nabla_{\partial_t|_0} \left(-2iJ_{\varphi_t} Z_t + 2Z_t \right) = \nabla_{\partial_t|_0} \frac{1}{2} \left(Z_t - iJ_{\varphi_t} Z_t \right) = \nabla_{\partial_t|_0} Z_t^{10}. \end{aligned} \tag{5.8}$$

Moreover, since φ_0 is holomorphic, $\nabla_X Y_0 - \nabla_{JX}(J_{\varphi_0} Y_0) = -J_{\varphi_0}(\nabla_{JX} Y_0 + \nabla_X (J_{\varphi_0} Y_0))$ and

$$\nabla_{\partial_z} Y_0^{10} = \frac{1}{4} \left(\nabla_X Y_0 - \nabla_{JX} (J_{\varphi_0} Y_0) - i \left(\nabla_X (J_{\varphi_0} Y_0) + \nabla_{JX} Y_0 \right) \right) = \frac{1}{4} \left(-2iJ_{\varphi_0} Z_0 + 2Z_0 \right) = Z_0^{10}, \tag{5.9}$$

finishing the first part of our proof.

Conversely, suppose that (4.22) and (4.23) hold. Take $Y_t^{10} \in T^{10}_{J_{\varphi_t}}N$ and Z_t^{10} with $\nabla_{\partial_z} Y_0^{10} = Z_0^{10}$ and $\nabla_{\partial_t|_0}(\nabla_{\partial_z} Y_t^{10} - Z_t^{10}) = 0$. Then

$$\frac{1}{2}\nabla_{\partial_t|_0}\left(\nabla_X Y_t - \nabla_{JX}\left(J_{\varphi_t} Y_t\right) - i\left(\nabla_X\left(J_{\varphi_t} Y_t\right) + \nabla_{JX} Y_t\right) - Z_t + iJ_{\varphi_t} Z_t\right) = 0 \tag{5.10}$$

and we can now easily conclude that (5.7) holds. Together with the fact that ψ_0 is $\mathcal{J}^1$-holomorphic (Proposition 3.3), we can conclude that (4.37) is verified. As for the horizontal part, we have that

$$\nabla_{\partial_t|_0}\left(d\psi_t JX - \mathcal{J} d\psi_t X\right)^{\mathcal{H}} = 0 \Longleftrightarrow \partial_t|_0\langle d\varphi_t JX - J_{\varphi_t} d\varphi_t X, Y\rangle = 0, \quad \forall Y \in TN, \tag{5.11}$$

as ψ_0 is holomorphic. Since (4.22) holds, the last condition is trivially satisfied and we can conclude that our map is $\mathcal{J}^1$-holomorphic to first order, as desired. □

Proof of Lemma 4.12. (i) Since φ is isotropic to first order, $\partial_t|_0\langle \partial_z^2\varphi_t, \partial_z\varphi_t\rangle = 0$, equivalently, $\partial_t|_0\langle u_t^X + iv_t^X, d\varphi_t X - id\varphi_t JX\rangle$. Thus, we have

$$\begin{aligned}\partial_t|_0\left\langle u_t^X, d\varphi X\right\rangle &= -\partial_t|_0\left\langle v_t^X, d\varphi_t JX\right\rangle,\\ \partial_t|_0\left\langle u_t^X, d\varphi_t JX\right\rangle &= \partial_t|_0\left\langle v_t^X, d\varphi_t X\right\rangle.\end{aligned} \tag{5.12}$$

Similarly, $\partial_t|_0\langle \partial_z^2\varphi_t, \partial_z^2\varphi_t\rangle = 0$ is equivalent to $\partial_t|_0\langle u_t^X + iv_t^X, u_t^X + iv_t^X\rangle = 0$ and implies

$$\begin{aligned}\partial_t|_0\left\langle u_t^X, u_t^X\right\rangle &= \partial_t|_0\left\langle v_t^X, v_t^X\right\rangle,\\ \partial_t|_0\left\langle u_t^X, v_t^X\right\rangle &= 0.\end{aligned} \tag{5.13}$$

As ψ is compatible with φ, Lemma 4.5 guarantees that φ is J_ψ-holomorphic to first order. On the other hand, since $\partial_z\varphi_0$ and $\partial_z^2\varphi_0$ are linearly independent, we deduce that $d\varphi_0 X, d\varphi_0 JX$, u_0^X and v_0^X form a basis for $T_{\varphi_0}N$. Hence, (4.33) will be satisfied if and only on evaluating the inner product of $\nabla_{\partial_t|_0} J_{\varphi_t} u_t^X$ and $-\nabla_{\partial_t|_0} v_t^X$ with which one of these four vectors one obtains the same result. We shall only prove for the first and fourth vectors, the other two cases being similar.

(a) Since $J_{\varphi_0} u_0^X = -v_0^X$,

$$\begin{aligned}\left\langle \nabla_{\partial_t|_0} J_{\varphi_t} u_t^X, d\varphi_0 X\right\rangle &= \partial_t|_0\left\langle J_{\varphi_t} u_t^X, d\varphi_t X\right\rangle - \left\langle J_{\varphi_0} u_0^X, \nabla_{\partial_t|_0} d\varphi_t X\right\rangle\\ &= -\partial_t|_0\left\langle u_t^X, d\varphi_t JX\right\rangle + \left\langle v_0^X, \nabla_{\partial_t|_0} d\varphi_t X\right\rangle\\ &= (\text{using}(5.12)) - \partial_t|_0\left\langle v_t^X, d\varphi_t X\right\rangle + \left\langle v_0^X, \nabla_{\partial_t|_0} d\varphi_t X\right\rangle = \left\langle \nabla_{\partial_t|_0} v_t^X, d\varphi_0 X\right\rangle.\end{aligned} \tag{5.14}$$

(b) Using (5.13),

$$\begin{aligned}\left\langle \nabla_{\partial_t|_0} J_{\varphi_t} u_t^X, u_0^X \right\rangle &= \partial_t|_0 \left\langle J_{\varphi_t} u_t^X, u_t^X \right\rangle - \left\langle J_{\varphi_0} u_0^X, \nabla_{\partial_t|_0} u_t^X \right\rangle \\ &= \partial_t|_0 \left\langle v_t^X, u_t^X \right\rangle - \left\langle \nabla_{\partial_t|_0} v_t^X, u_0^X \right\rangle = -\left\langle \nabla_{\partial_t|_0} v_t^X, u_0^X \right\rangle.\end{aligned} \tag{5.15}$$

(ii) We know that $\partial_z\varphi_t$, $\partial_z^2\varphi_t$, $\overline{\partial_z\varphi_t}$, $\overline{\partial_z^2\varphi_t}$ span $T^{\mathbb{C}}N^4$. Hence, there are a_t, b_t, c_t, and d_t with

$$\partial_z^3\varphi_t = a_t\partial_z\varphi_t + b_t\partial_z^2\varphi_t + c_t\overline{\partial_z\varphi_t} + d_t\overline{\partial_z^2\varphi_t}, \tag{5.16}$$

where $c_0 = d_0 = 0$ since $\partial_z^3\varphi_0 \in \operatorname{span}\{\partial_z\varphi_0, \partial_z^2\varphi_0\} = T^{10}_{J_{\varphi_0}}N$. Therefore,

$$\nabla_{\partial_t|_0}\partial_z^3\varphi_t = a_0\nabla_{\partial_t|_0}\partial_z\varphi_t + \left.\frac{\partial a_t^X}{\partial t}\right|_0 \partial_z\varphi_0 + b_0\nabla_{\partial_t|_0}\partial_z^2\varphi_t + \left.\frac{\partial b_t^X}{\partial t}\right|_0 \partial_z^2\varphi_0 + \left.\frac{\partial c_t^X}{\partial t}\right|_0 \overline{\partial_z\varphi_0} + \left.\frac{\partial a_t^X}{\partial t}\right|_0 \overline{\partial_z^2\varphi_0}. \tag{5.17}$$

Now, using the fact that φ is isotropic to first order, we have

$$\left\langle \nabla_{\partial_t|_0}\partial_z^3\varphi_t, \partial_z\varphi_0 \right\rangle = -\left\langle \partial_z^3\varphi_0, \nabla_{\partial_t|_0}\partial_z\varphi_t \right\rangle. \tag{5.18}$$

Together with

$$\begin{aligned}\left\langle \nabla_{\partial_t|_0}\partial_z\varphi_t, \partial_z\varphi_0 \right\rangle &= \left\langle \partial_z^2\varphi_0, \partial_z\varphi_0 \right\rangle = \left\langle \partial_z\varphi_0, \partial_z\varphi_0 \right\rangle = 0, \\ \left\langle \nabla_{\partial_t|_0}\partial_z^2\varphi_t, \partial_z\varphi_0 \right\rangle &= -\left\langle \partial_z^2\varphi_0, \nabla_{\partial_t|_0}\partial_z\varphi_t \right\rangle,\end{aligned} \tag{5.19}$$

we deduce

$$\left.\frac{\partial c_t^X}{\partial t}\right|_0 \left\|\partial_z\varphi_0\right\|^2 + \left.\frac{\partial d_t^X}{\partial t}\right|_0 \left\langle \overline{\partial_z^2\varphi_0}, \partial_z\varphi_0 \right\rangle = 0. \tag{5.20}$$

Similarly, from

$$\left\langle \nabla_{\partial_t|_0}\partial_z^3\varphi_t, \partial_z^2\varphi_0 \right\rangle = -\left\langle \partial_z^3\varphi_0, \nabla_{\partial_t|_0}\partial_z^2\varphi_t \right\rangle, \tag{5.21}$$

we have

$$\left.\frac{\partial c_t^X}{\partial t}\right|_0 \left\langle \overline{\partial_z\varphi_0}, \partial_z^2\varphi_0 \right\rangle + \left.\frac{\partial d_t^X}{\partial t}\right|_0 \left\|\partial_z^2\varphi_0\right\|^2 = 0. \tag{5.22}$$

Equations (5.20) and (5.22) imply that either $\|\langle\partial_z\varphi_0,\overline{\partial_z^2\varphi_0}\rangle\| = \|\partial_z\varphi_0\|\|\partial_z^2\varphi_0\|$ (which is absurd as $\partial_z^2\varphi_0$ does not lie in span$\{\partial_z\varphi_0\}$) or $\partial d_t^X/\partial t|_0 = \partial c_t^X/\partial t|_0 = 0$ and consequently (4.34) holds, as wanted. □

Proof of Lemma 4.17. That $\nabla_{\partial_t|_0} d\varphi_t JX = \nabla_{\partial_t|_0} J_{\varphi_t} d\varphi_t X$ follows from the proof of Proposition 4.7. Since $\partial_z\varphi_0(z_0)$, and $\partial_z^2\varphi_0(z_0)$ are linearly independent vectors, we can deduce that $d\varphi_t X$, $d\varphi_t JX$, u_t and v_t form a basis for $T_{\varphi_t(z)}N$ for (t,z) is a neighbourhood of $(0,z_0)$. On the other hand, as φ is the projection of a map $\mathcal{Q}^1$-holomorphic to first order, we know that it must be isotropic to first order from Proposition 4.10. Hence,

$$\partial_t|_0\left\langle \partial_z^2\varphi_t, \partial_z\varphi_t\right\rangle = \partial_t|_0\left\langle \partial_z^2\varphi_t, \partial_z^2\varphi_t\right\rangle = 0. \tag{5.23}$$

The argument to establish the second and third identities in (4.52), will now be similar to the one in Lemma 4.12(i). □

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