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Research Article

Periodic Boundary Value Problems for Second-Order Functional Differential Equations

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This note considers a periodic boundary value problem for a second-order functional differential equation. We extend the concept of lower and upper solutions and obtain the existence of extreme solutions by using upper and lower solution method.

1. Introduction

Upper and lower solution method plays an important role in studying boundary value problems for nonlinear differential equations; see [1] and the references therein. Recently, many authors are devoted to extend its applications to boundary value problems of functional differential equations [2–5]. Suppose α is one upper solution or lower solution of periodic boundary value problems for second-order differential equation; the condition $\alpha(0) = \alpha(T)$ is required. A neutral problem is that whether we can define upper and lower solution without assuming $\alpha(0) = \alpha(T)$. The aim of the present paper is to discuss the following second order functional differential equation

$$\begin{aligned} -u''(t) &= f(t, u(t), u(\theta(t))), \quad t \in J, \\ u(0) &= u(T), \quad u'(0) = u'(T), \end{aligned} \tag{1.1}$$

where $J = [0, T]$, $f \in C(J \times R^2, R)$, $0 \le \theta(t) \le t$, $t \in J$.

In this paper, we extended the concept of lower and upper solutions for (1.1). By using the method of upper and lower solutions and monotone iterative technique, we obtained the existence of extreme solutions for the boundary value problem (1.1).

Through this paper, we assume that $M > 0,\ N \geq 0$, and

$$T^2(M+N) \leq 2, \tag{1.2}$$

$$\begin{aligned}
\Lambda_1 &= \left\{u \in C^2(J) : u(t) \geq 0, 1+u(0) = u(T), u'(0) \geq u'(T)\right\}, \\
\Lambda_2 &= \left\{u \in C^2(J) : u(t) \geq 0, 1+u(T) = u(0), u'(0) \leq u'(T)\right\}.
\end{aligned} \tag{1.3}$$

Definition 1.1. Functions $\alpha \in C^2(J)$ and $\beta \in C^2(J)$ are called lower solution and upper solution of the boundary value problem (1.1), respectively if

$$\begin{gathered}
-\alpha''(t) \leq f(t, \alpha(t), \alpha(\theta(t))) - H_\alpha(t), \quad t \in J, \\
\alpha'(0) \geq \alpha'(T),
\end{gathered} \tag{1.4}$$

$$\begin{gathered}
-\beta''(t) \geq f\left(t, \beta(t), \beta(\theta(t))\right) + h_\beta(t), \quad t \in J, \\
\beta'(0) \leq \beta'(T),
\end{gathered} \tag{1.5}$$

where

$$H_\alpha(t) = \begin{cases} 0, & \alpha(0) = \alpha(T), \\ (-c''(t) + Mc(t) + Nc(\theta(t)))(\alpha(0) - \alpha(T)), & \alpha(0) > \alpha(T), \\ (-b''(t) + Mb(t) + Nb(\theta(t)))(\alpha(T) - \alpha(0)), & \alpha(0) < \alpha(T), \end{cases} \tag{1.6}$$

$$h_\beta(t) = \begin{cases} 0, & \beta(0) = \beta(T), \\ (-c''(t) + Mc(t) + Nc(\theta(t)))\left(\beta(0) - \beta(T)\right), & \beta(0) < \beta(T), \\ (-b''(t) + Mb(t) + Nb(\theta(t)))\left(\beta(T) - \beta(0)\right), & \beta(0) > \beta(T) \end{cases} \tag{1.7}$$

and $c \in \Lambda_1, b \in \Lambda_2$.

Remark 1.2. Clearly, $\Lambda_1 \neq \emptyset$ and $\Lambda_2 = \emptyset$. For example,

$$\frac{t}{T} \in \Lambda_1, \qquad \sin\frac{\pi t}{2T} \in \Lambda_1, \qquad 1 - \frac{t}{T} \in \Lambda_2, \qquad \sin\frac{\pi}{2}\left(1 - \frac{t}{T}\right) \in \Lambda_2. \tag{1.8}$$

2. Comparison Results

We now present the main results of this section.

Lemma 2.1. *Assume that $u \in C^2(J)$ satisfies*

$$-u''(t) + Mu(t) + Nu(\theta(t)) \le 0, \quad t \in J,$$
$$u(0) = u(T), \qquad u'(0) \ge u'(T), \tag{2.1}$$

then $u(t) \le 0$ for all $t \in J$.

Proof. Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$. We consider the following two cases.

Case 1. $u(t) \ge 0$, $u(t) \not\equiv 0$ on J. It is easy to obtain that $u''(t) \ge 0$ on J. Thus $u(t) \equiv constant = K > 0$ from $u'(0) \ge u'(T)$. Consequently, we obtain that

$$(M+N)K = -u''(t) + Mu(t) + Nu(\theta(t)) \le 0, \tag{2.2}$$

which contradicts $K > 0$.

Case 2. There exist $t_1, t_2 \in J$ such that $u(t_1) > 0$ and $u(t_2) < 0$. Hence, two cases are possible.

Subcase 1. $u(0) = u(T) < 0$. There exists a $t_3 \in (0, T)$ such that

$$u(t_3) = \max_{t\in J} u(t) > 0, \qquad u'(t_3) = 0. \tag{2.3}$$

Let $u(t_*) = \min_{t\in[0,t_3)} u(t) < 0$. Then

$$u''(t) \ge (M+N)u(t_*), \quad t \in [0, t_3). \tag{2.4}$$

Integrating the above inequality from $s(t_* \le s \le t_3)$ to t_3, we obtain

$$-u'(s) \ge (t_3 - s)(M+N)u(t_*), \quad t_* \le s \le t_3, \tag{2.5}$$

and then integrate from t_* to t_3 to obtain

$$-u(t_*) < u(t_3) - u(t_*) \le \int_{t_*}^{t_3} (s - t_3)(M+N)u(t_*)ds$$
$$\le -\frac{M+N}{2}u(t_*)(t_3 - t_*)^2 \le -\frac{M+N}{2}u(t_*)T^2 \tag{2.6}$$

that implies $1 < (T^2/2)(M+N)$. This is a contradiction.

Subcase 2. $u(0) = u(T) \geq 0$. There exists a $t_3 \in J$ such that

$$u(t_3) = \max_{t \in J} u(t) > 0. \tag{2.7}$$

If $t_3 \in (0, T)$, then $u'(t_3) = 0$. If $t_3 = 0$ or $t_3 = T$, then $u'(0) \leq 0 \leq u'(T)$. So $u'(0) = u'(T) = 0$.

Let $u(t_*) = \min_{t \in (0,T)} u(t) < 0$. Then

$$u''(t) \geq (M + N)u(t_*), \quad t \in J. \tag{2.8}$$

When $t_* < t_3$, same as Subcase 1, we obtain that $1 < T^2(M + N)/2$.

when $t_* > t_3$, integrating the inequality (2.8) from t_3 to $s(t_3 \leq s \leq t_*)$, we obtain

$$u'(s) \geq (s - t_3)(M + N)u(t_*), \tag{2.9}$$

and then integrate from t_3 to t_* to obtain

$$u(t_*) > u(t_*) - u(t_3) \geq \int_{t_3}^{t_*} (s - t_3)(M + N)u(t_*)ds \geq \frac{M + N}{2} u(t_*)(t_* - t_3)^2 \geq \frac{M + N}{2} u(t_*)T^2 \tag{2.10}$$

that implies $1 < (T^2/2)(M + N)$. This is a contradiction. The proof is complete. □

Lemma 2.2. *Assume that $u \in C^2(J)$ satisfies*

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) + \left(-c''(t) + Mc(t) + Nc(\theta(t))\right)(u(0) - u(T)) \leq 0, \quad t \in J, \\ u(0) > u(T), \qquad u'(0) \geq u'(T). \end{gathered} \tag{2.11}$$

Then $u(t) \leq 0$ for all $t \in J$.

Proof. Put

$$y(t) = u(t) + c(t)(u(0) - u(T)), \quad t \in J, \tag{2.12}$$

then $y \in E$ and $u(t) \leq y(t)$ for all $t \in J$ and

$$y'(t) = u'(t) + c'(t)(u(0) - u(T)). \tag{2.13}$$

We have

$$\begin{aligned} -y''(t) + My(t) + Ny(\theta(t)) &= -u''(t) + Mu(t) + Nu(\theta(t)) \\ &\quad + \left(-c''(t) + Mc(t) + Nc(\theta(t))\right)(u(0) - u(T)) \leq 0, \\ y(0) &= (1 + c(0))u(0) - c(0)u(T) = y(T), \\ y'(0) &= u'(0) + c'(0)(u(0) - u(T)) \geq y'(T). \end{aligned} \tag{2.14}$$

Hence by Lemma 2.1, $y(t) \leq 0$ for all $t \in J$, which implies that $u(t) \leq 0$ for $t \in J$. This completes the proof. □

Lemma 2.3. *Assume that* $u \in C^2(J)$ *satisfies*

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) + \left(-b''(t) + Mb(t) + Nb(\theta(t))\right)(u(T) - u(0)) \leq 0, \quad t \in J, \\ u(0) < u(T), \quad u'(0) \geq u'(T). \end{gathered} \tag{2.15}$$

Then $u(t) \leq 0$ *for all* $t \in J$.

The proof of Lemma 2.3 is similar to that of Lemma 2.2, here we omit it.

Lemma 2.4. *Assume that* $u \in C^2(J)$ *satisfies*

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) \leq 0, \quad t \in J, \\ u(0) \leq 0, \qquad u(T) \leq 0. \end{gathered} \tag{2.16}$$

Then $u(t) \leq 0$ *for all* $t \in J$.

Proof. Suppose that $u(t) > 0$ for some $t \in J$. Then from boundary conditions, we have that there exists a $t^* \in (0,1)$ such that

$$u(t^*) = \max_{t \in J} u(t) > 0, \qquad u'(t^*) = 0. \tag{2.17}$$

Suppose that $u(t) \geq 0$ for $t \in J$. It is easy to see that $u(0) = 0$ and $u''(t) \geq 0$ for $t \in J$. From $u(0) = 0$ and $u(t) \geq 0$ for $t \in J$, we obtain that $u'(0) \geq 0$. Therefore, $u'(t) \geq u'(0) \geq 0$. It follows that $u(T) = \max_{t \in J} u(t) > 0$, a contradiction.

Suppose that there exist $t_1,\ t_2 \in J$ such that $u(t_1) > 0$ and $u(t_2) < 0$. Let $t_* \in [0, t^*)$ be such that $u(t_*) = \min_{t \in [0,t^*)} u(t) \leq 0$. From the first inequality of (2.16), we have

$$u''(t) \geq (M + N)u(t_*), \quad t \in [0, t^*). \tag{2.18}$$

Integrating the above inequality from $s(t_* \leq s \leq t^*)$ to t^*, we obtain

$$-u'(s) \geq (t^* - s)(M + N)u(t_*), \quad t_* \leq s \leq t^*, \tag{2.19}$$

and then integrate from t_* to t^* to obtain

$$\begin{aligned} -u(t_*) < u(t^*) - u(t_*) &\le \int_{t_*}^{t^*} (s - t^*)(M+N)u(t_*)ds \\ &\le -\frac{M+N}{2}u(t_*)(t^* - t_*)^2 \le -\frac{M+N}{2}u(t_*)T^2. \end{aligned} \tag{2.20}$$

Hence, $u(t_*)[2 - (M+N)T^2] > 0$, a contradiction. The proof is complete. □

3. Linear Problem

In this section, we consider the boundary value problem

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \\ u(0) = u(T), \qquad u'(0) = u'(T), \end{gathered} \tag{3.1}$$

where $\sigma \in C(J, R)$.

Theorem 3.1. *Assume that there exist α, β which are lower and upper solutions of* (3.1) *and $\alpha \le \beta$ on J. Then there exists one unique solution u to problem* (3.1) *and $\alpha \le u \le \beta$ on J.*

Proof. We first show that the solution of (3.1) is unique. Let u_1 and u_2 be solutions of (3.1) and set $v = u_1 - u_2$. Thus

$$\begin{gathered} -v''(t) + Mv(t) + Nv(\theta(t)) = 0, \quad t \in J, \\ v(0) = v(T), \qquad v'(0) = v'(T). \end{gathered} \tag{3.2}$$

By Lemma 2.1, we have that $v \le 0$ for $t \in J$, that is, $u_1 \le u_2$ on J. Similarly, one can obtain that $u_2 \le u_1$ on J. Hence $u_1 = u_2$.

Next, we prove that if u is a solution of (3.1), then $\alpha \le u \le \beta$.

Let $m = \alpha - u$. If $\alpha(0) = \alpha(T)$, then $H_\alpha(t) = 0$ on J. So we have

$$\begin{gathered} -m''(t) + Mm(t) + Nm(\theta(t)) \le 0, \quad t \in J, \\ m(0) = m(T), \qquad m'(0) \ge m'(T). \end{gathered} \tag{3.3}$$

By Lemma 2.1, we have that $m = \alpha - u \le 0$ on J.

If $\alpha(0) > \alpha(T)$, then $H_\alpha(t) = (-c''(t) + Mc(t) + Nc(\theta(t)))(\alpha(0) - \alpha(T))$. Thus

$$\begin{aligned} -m''(t) + Mm(t) + Nm(\theta(t)) = {}& -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) \\ & + u''(t) - Mu(t) - Nu(\theta(t)) \\ \le {}& \sigma(t) - H_\alpha(t) - \sigma(t) = -H_\alpha(t) \\ = {}& -\big(-c''(t) + Mc(t) + Nc(\theta(t))\big)(m(0) - m(T)). \end{aligned} \tag{3.4}$$

It is easy to see that $m'(0) \geq m'(T)$. By Lemma 2.2, we have that $m = \alpha - u \leq 0$ on J. Analogously, $u \leq \beta$ on J.

If $\alpha(0) < \alpha(T)$, then $H_\alpha(t) = (-b''(t) + Mb(t) + Nb(\theta(t)))(\alpha(T) - \alpha(0))$. Thus

$$\begin{aligned}-m''(t) + Mm(t) + Nm(\theta(t)) &= -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) \\ &\quad + u''(t) - Mu(t) - Nu(\theta(t)) \\ &\leq \sigma(t) - H_\alpha(t) - \sigma(t) = -H_\alpha(t) \\ &= -\bigl(-b''(t) + Mb(t) + Nb(\theta(t))\bigr)(m(T) - m(0)).\end{aligned} \tag{3.5}$$

It is easy to see that $m'(0) \geq m'(T)$. By Lemma 2.2, we have that $m = \alpha - u \leq 0$ on J. Analogously, $u \leq \beta$ on J.

Finally, we show that (3.1) has a solution by several steps.

Step 1. Consider the equation

$$\begin{gathered}-u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \\ u(0) = u(T) = \lambda,\end{gathered} \tag{3.6}$$

where M, N and σ are defined in (3.1). For any $\lambda \in R$, we show that (3.6) has a unique solution $u(t, \lambda)$.

It is easy to check that (3.6) is equivalent to the integral equation

$$u(t) = \lambda + \int_0^T G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]ds, \tag{3.7}$$

where

$$G(t,s) = \begin{cases} \dfrac{t}{T}(T-s), & 0 \leq t \leq s \leq T, \\ \dfrac{s}{T}(T-t), & 0 \leq s \leq t \leq T. \end{cases} \tag{3.8}$$

Define a mapping $\Phi : C(J,R) \to C(J,R)$ by

$$(\Phi u)(t) = \lambda + \int_0^T G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]ds. \tag{3.9}$$

Obviously $C(J,R)$ is a Banach space with norm $\|u\| = \sup_{t\in J}|u(t)|$. For any $x, y \in C(J,R)$, we have

$$(\Phi x)(t) - (\Phi y)(t) = \int_0^T G(t,s)\bigl[M\bigl(y(s) - x(s)\bigr) + N\bigl(y(\theta(s)) - x(\theta(s))\bigr)\bigr]ds. \tag{3.10}$$

Noting that $\max_{t\in J}\int_0^T G(t,s)ds = T^2/8$, condition (1.2) implies that $\Phi : C(J,R) \to C(J,R)$ is a contraction mapping. There exists unique $u \in C(J,R)$ such that $\Phi u = u$. Thus (3.6) has a unique solution $u(t,\lambda)$. Moreover, $u(t,\lambda) \in C^2(J,R)$.

Step 2. We show that for any $t \in J$, $u(t,\lambda)$ and $u_t(t,\lambda)$ are continuous in λ, where $u(t,\lambda)$ is a unique solution of the problem (3.6). Let $u(t,\lambda_i), i = 1,2$, be the solution of

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J, \\ u(0) = u(T) = \lambda_i, \quad i = 1,2. \end{gathered} \tag{3.11}$$

Then

$$u(t,\lambda_i) = \lambda_i + \int_0^T G(t,s)[\sigma(s) - Mu(s,\lambda_i) - Nu(\theta(s),\lambda_i)]ds, \quad i = 1,2. \tag{3.12}$$

From (3.12), we have that

$$\begin{aligned} \|u(t,\lambda_1) - u(t,\lambda_2)\| &\le |\lambda_1 - \lambda_2| + (M+N)\|u(t,\lambda_1) - u(t,\lambda_2)\|\max_{t\in J}\int_0^T G(t,s)ds \\ &\le |\lambda_1 - \lambda_2| + \frac{T^2}{8}(M+N)\|u(t,\lambda_1) - u(t,\lambda_2)\|. \end{aligned} \tag{3.13}$$

Hence

$$\|u(t,\lambda_1) - u(t,\lambda_2)\| \le \frac{8}{8 - T^2(M+N)}|\lambda_1 - \lambda_2|. \tag{3.14}$$

Since $u(t,\lambda) \in C^2(J,R)$, $u_t(t,\lambda)$ exists for any $t \in J$ and $\lambda \in R$. From (3.12) and (3.14), we have

$$\begin{aligned} \|u_t(t,\lambda_1) - u_t(t,\lambda_2)\| &\le \frac{M+N}{T}\|u(t,\lambda_1) - u(t,\lambda_2)\|\max_{t\in J}\left(\int_0^t s\,ds + \int_t^T (T-s)ds\right) \\ &\le \frac{(M+N)T}{2}\|u(t,\lambda_1) - u(t,\lambda_2)\| \\ &\le \frac{4(M+N)T}{8 - T^2(M+N)}|\lambda_1 - \lambda_2|. \end{aligned} \tag{3.15}$$

Step 3. We show that there exists one λ such that $u_t(0,\lambda) = u_t(T,\lambda)$, where $u(t,\lambda)$ is the unique solution of the problem (3.6).

Put

$$\overline{\alpha}(t) = \begin{cases} \alpha(t), & \alpha(0) = \alpha(T), \\ \alpha(t) + c(t)(\alpha(0) - \alpha(T)), & \alpha(0) > \alpha(T), \\ \alpha(t) + b(t)(\alpha(T) - \alpha(0)), & \alpha(0) < \alpha(T), \end{cases} \tag{3.16}$$

$$\overline{\beta}(t) = \begin{cases} \beta(t), & \beta(0) = \beta(T), \\ \beta(t) - c(t)\big(\beta(T) - \beta(0)\big), & \beta(0) < \beta(T), \\ \beta(t) - b(t)\big(\beta(0) - \beta(T)\big), & \beta(0) > \beta(T); \end{cases} \tag{3.17}$$

then $\alpha(t) \le \overline{\alpha}(t), \overline{\beta}(t) \le \beta(t)$ for any $t \in J$ and

$$\begin{gathered} -\overline{\alpha}''(t) + M\overline{\alpha}(t) + N\overline{\alpha}(\theta(t)) \le \delta(t), \quad t \in J, \\ \overline{\alpha}(0) = \overline{\alpha}(T), \qquad \overline{\alpha}'(0) \ge \overline{\alpha}'(T), \end{gathered} \tag{3.18}$$

$$\begin{gathered} -\overline{\beta}''(t) + M\overline{\beta}(t) + N\overline{\beta}(\theta(t)) \ge \delta(t), \quad t \in J, \\ \overline{\beta}(0) = \overline{\beta}(T), \qquad \overline{\beta}'(0) \le \overline{\beta}'(T). \end{gathered} \tag{3.19}$$

Using (3.18) and (3.19), one easily obtains that $\overline{\alpha}(t) \le \overline{\beta}(t)$ for any $t \in J$.

Put $\lambda \in [\overline{\alpha}(0), \overline{\beta}(0)]$, then $\overline{\alpha}(0) = \overline{\alpha}(T) \le u(0,\lambda) = u(T,\lambda) \le \overline{\beta}(0) = \overline{\beta}(T)$. Using Lemma 2.3, we easily obtain that $\overline{\alpha}(t) \le u(t,\lambda) \le \overline{\beta}(t)$ on J. Hence

$$u_t(0, \overline{\alpha}(0)) \ge \overline{\alpha}'(0), \qquad u_t(T, \overline{\alpha}(T)) \le \overline{\alpha}'(T), \tag{3.20}$$

$$u_t\Big(0, \overline{\beta}(0)\Big) \le \overline{\beta}'(0), \qquad u_t\Big(T, \overline{\beta}(T)\Big) \ge \overline{\beta}'(T). \tag{3.21}$$

Define a function

$$P(\lambda) = u_t(0,\lambda) - u_t(T,\lambda), \tag{3.22}$$

where $u(t,\lambda)$ is the unique solution of the problem (3.6). Since P is continuous and

$$P(\overline{\alpha}(0))P\Big(\overline{\beta}(0)\Big) \le 0, \tag{3.23}$$

there exists one $\lambda_0 \in [\overline{\alpha}(0), \overline{\beta}(0)]$ such that $P(\lambda_0) = 0$, that is, $u_t(0,\lambda_0) = u_t(T,\lambda_0)$. Obviously, $u(t,\lambda_0)$ is a unique solution of the problem (3.1). This completes the proof. □

4. Main Result

Theorem 4.1. *Let the following conditions hold.*

(H_1) *The functions α, β are lower and upper solutions of* (1.1)*, respectively, and $\alpha \le \beta$ on J.*

(H_2) *The constants M, N in definition of upper and lower solutions satisfy*

$$f(t,x,y) - f(t,\overline{x},\overline{y}) \ge -M(x-\overline{x}) - N(y-\overline{y}) \tag{4.1}$$

for $\alpha(t) \le \overline{x} \le x \le \beta(t), \alpha(\theta(t)) \le \overline{y} \le y \le \beta(\theta(t))$, $t \in J$.

Then, there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty}\alpha_n(t) = \rho(t), \lim_{n\to\infty}\beta_n(t) = r(t)$ uniformly on J, and ρ, r are the minimal and the maximal solutions of (1.1)*, respectively, such that*

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \rho \le x \le r \le \beta_n \le \cdots \le \beta_2 \le \beta_1 \le \beta_0 \tag{4.2}$$

on J, where x is any solution of (1.1) *such that $\alpha(t) \le x(t) \le \beta(t)$ on J.*

Proof. Let $[\alpha,\beta] = \{u \in C^2(J) : \alpha(t) \le u(t) \le \beta(t), t \in J\}$. For any $\gamma \in [\alpha,\beta]$, we consider the equation

$$\begin{gathered} -u''(t) + Mu(t) + Nu(\theta(t)) = f(t,\gamma(t),\gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)), \quad t \in J, \\ u(0) = u(T), \qquad u'(0) = u'(T). \end{gathered} \tag{4.3}$$

Theorem 3.1 implies that the problem (4.3) has a unique solution $u \in C^2(J)$. We define an operator A by $u = A\gamma$, then A is an operator from $[\alpha,\beta]$ to $[\alpha,\beta]$.

We shall show that

(a) $\alpha \le A\alpha$, $\quad A\beta \le \beta$;

(b) A is nondecreasing in $[\alpha,\beta]$.

From $A\alpha \in [\alpha,\beta]$ and $A\beta \in [\alpha,\beta]$, we have that (a) holds. To prove (b), we show that $A\mu_1 \le A\mu_2$ if $\alpha \le \mu_1 \le \mu_2 \le \beta$.

Let $\rho_1^* = A\mu_1$, $\rho_2^* = A\mu_2$, and $p = \rho_1^* - \rho_2^*$; then by (H_2), we have

$$\begin{aligned} -p''(t) + Mp(t) + Np(\theta(t))) &= f(t,\mu_1(t),\mu_1(\theta(t))) + M\mu_1(t) + N\mu_1(\theta(t)) \\ &\quad - f(t,\mu_2(t),\mu_2(\theta(t))) - M\mu_2(t) - N\mu_2(\theta(t)) \\ &\le 0, \end{aligned} \tag{4.4}$$

and $p(0) = p(T)$, $\quad p'(0) = p'(T)$. By Lemma 2.1, $p(t) \le 0$, which implies $A\mu_1 \le A\mu_2$.

Define the sequence $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\alpha_{n+1} = A\alpha_n$, $\beta_{n+1} = A\beta_n$ for $n = 0,1,2,\ldots$. From (a) and (b), we have

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_2 \le \beta_1 \le \beta_0 \tag{4.5}$$

on $t \in J$. Therefore, there exist $\rho,\ r$ such that $\lim_{n\to\infty}\alpha_n(t) = \rho(t)$, $\lim_{n\to\infty}\beta_n(t) = r(t)$ uniformly on J. Clearly, $\rho,\ r$ are solutions of (1.1).

Finally, we prove that if $x \in [\alpha_0, \beta_0]$ is one solution of (1.1), then $\rho(t) \le x(t) \le r(t)$ on J. Since $\alpha_0(t) \le x(t) \le \beta_0(t)$ and $Ax = x$, by property (b), we obtain that $\alpha_1(t) \le x(t) \le \beta_1(t)$ for $t \in J$. Using property (b) repeatedly, we have

$$\alpha_n(t) \le x(t) \le \beta_n(t) \tag{4.6}$$

for all n. Passing to the limit as $n \to \infty$, we obtain $\rho(t) \le x(t) \le r(t),\ t \in J$. This completes the proof. □

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