Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 905679, 6 pages
doi:10.1155/2010/905679

Research Article

Optimal Power Mean Bounds for the Weighted Geometric Mean of Classical Means

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Received 13 November 2009; Accepted 25 February 2010

Academic Editor: Andrea Laforgia

For $p \in \mathbb{R}$, the power mean of order p of two positive numbers a and b is defined by $M_p(a,b) = ((a^p + b^p)/2)^{1/p}$, for $p \neq 0$, and $M_p(a,b) = \sqrt{ab}$, for $p = 0$. In this paper, we answer the question: what are the greatest value p and the least value q such that the double inequality $M_p(a,b) \leq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leq M_q(a,b)$ holds for all $a,b > 0$ and $\alpha,\beta > 0$ with $\alpha + \beta < 1$? Here $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ denote the classical arithmetic, geometric, and harmonic means, respectively.

1. Introduction

For $p \in \mathbb{R}$, the power mean of order p of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.1}$$

Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ can be found in literatures [1–12]. It is well known that $M_p(a,b)$ is continuous and increasing with respect to $p \in \mathbb{R}$ for fixed a and b.

Let $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the classical arithmetic, geometric, and harmonic means of two positive numbers a and b, respectively. Then

$$\begin{aligned}\min\{a,b\} &\leq H(a,b) = M_{-1}(a,b) \leq G(a,b) = M_0(a,b)\\ &\leq A(a,b) = M_1(a,b) \leq \max\{a,b\}.\end{aligned} \tag{1.2}$$

In [13], Alzer and Janous established the following sharp double inequality (see also [14, page 350]):

$$M_{\log 2/\log 3}(a,b) \leq \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leq M_{2/3}(a,b) \tag{1.3}$$

for all $a,b > 0$.

In [15], Mao proved

$$M_{1/3}(a,b) \leq \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \leq M_{1/2}(a,b) \tag{1.4}$$

for all $a,b > 0$, and $M_{1/3}(a,b)$ is the best possible lower power mean bound for the sum $(1/3)A(a,b) + (2/3)G(a,b)$.

The following sharp bounds for $(2/3)G + (1/3)H$ and $(1/3)G + (2/3)H$ in terms of power mean are proved in [16]:

$$\begin{aligned}M_{-1/3}(a,b) &\leq \frac{2}{3}G(a,b) + \frac{1}{3}H(a,b) \leq M_0(a,b),\\ M_{-2/3}(a,b) &\leq \frac{1}{3}G(a,b) + \frac{2}{3}H(a,b) \leq M_0(a,b)\end{aligned} \tag{1.5}$$

for all $a,b > 0$.

The purpose of this paper is to answer the question: what are the greatest value p and the least value q such that the double inequality

$$M_p(a,b) \leq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leq M_q(a,b) \tag{1.6}$$

holds for all $a,b > 0$ and $\alpha,\beta > 0$ with $\alpha + \beta < 1$?

2. Main Result

In order to establish our main results we need the following lemma.

Lemma 2.1. *If* $\lambda \in (-1,0) \bigcup (0,1)$, $t \geq 1$ *and* $f(t) = (1/\lambda)\log((t^{\lambda}+1)/2) - \lambda\log((t+1)/2) - ((1-\lambda)/2)\log t$, *then*

(1) $f(t) > 0$ *for* $\lambda \in (0,1)$ *and* $t > 1$;

(2) $f(t) < 0$ *for* $\lambda \in (-1,0)$ *and* $t > 1$.

Proof. Simple computations lead to

$$f(1) = 0, \tag{2.1}$$

$$f'(t) = \frac{g(t)}{t(t+1)\left(t^{\lambda}+1\right)}, \tag{2.2}$$

where $g(t) = ((1-\lambda)/2)t^{\lambda+1} + ((1+\lambda)/2)t^{\lambda} - ((1+\lambda)/2)t - ((1-\lambda)/2)$:

$$g(1) = 0, \tag{2.3}$$

$$g'(t) = \frac{(1-\lambda)(1+\lambda)}{2}t^{\lambda} + \frac{\lambda(1+\lambda)}{2}t^{\lambda-1} - \frac{1+\lambda}{2}, \tag{2.4}$$

$$g'(1) = 0, \tag{2.5}$$

$$g''(t) = \frac{\lambda(1-\lambda)(1+\lambda)}{2}(t-1)t^{\lambda-2}. \tag{2.6}$$

(1) If $\lambda \in (0,1)$ and $t > 1$, then (2.6) implies

$$g''(t) > 0. \tag{2.7}$$

Therefore, Lemma 2.1(1) follows from (2.1)–(2.3) and (2.5) together with (2.7).

(2) If $\lambda \in (-1,0)$ and $t > 1$, then (2.6) yields

$$g''(t) < 0. \tag{2.8}$$

Therefore, Lemma 2.1(2) follows from (2.1)–(2.3) and (2.5) together with (2.8). □

Theorem 2.2. *For all $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$, one has*

(1) $M_{2\alpha+\beta-1}(a,b) = M_0(a,b) = A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ *for* $2\alpha + \beta = 1$;

(2) $M_{2\alpha+\beta-1}(a,b) \geq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \geq M_0(a,b)$ *for* $2\alpha + \beta > 1$, *and* $M_{2\alpha+\beta-1}(a,b) \leq A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leq M_0(a,b)$ *for* $2\alpha + \beta < 1$, *each equality occurs if and only if* $a = b$, *and* $M_0(a,b)$ *and* $M_{2\alpha+\beta-1}(a,b)$ *are the best possible power mean bounds for the product* $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$.

Proof. (1) If $2\alpha + \beta = 1$, then simple computations lead to

$$\begin{aligned} A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) &= \left(\frac{a+b}{2}\right)^{2\alpha+\beta-1}(ab)^{1-(\alpha+(\beta/2))} \\ &= \sqrt{ab} = M_0(a,b) = M_{2\alpha+\beta-1}(a,b). \end{aligned} \tag{2.9}$$

(2) If $2\alpha + \beta \neq 1$ and $a = b$, then we clearly see that

$$A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) = M_{2\alpha+\beta-1}(a,b) = M_0(a,b) = a. \tag{2.10}$$

If $2\alpha+\beta\neq 1$ and $a \neq b$, without loss of generality, we assume that $a > b$. Let $t = (a/b) > 1$ and $\lambda = 2\alpha + \beta - 1$, then $\lambda \in (-1,0) \bigcup (0,1)$, and simple computations lead to

$$\begin{aligned} &\log M_{2\alpha+\beta-1}(a,b) - \log\left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right] \\ &= \frac{1}{2\alpha+\beta-1}\log\frac{t^{2\alpha+\beta-1}+1}{2} - (2\alpha+\beta-1)\log\frac{1+t}{2} - \left(1-\alpha-\frac{\beta}{2}\right)\log t \\ &= \frac{1}{\lambda}\log\frac{t^{\lambda}+1}{2} - \lambda\log\frac{t+1}{2} - \frac{1-\lambda}{2}\log t, \end{aligned} \tag{2.11}$$

$$\frac{A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)}{M_0(a,b)} = \left(\frac{\sqrt{t}+\left(1/\sqrt{t}\right)}{2}\right)^{2\alpha+\beta-1}. \tag{2.12}$$

Therefore, $M_{2\alpha+\beta-1}(a,b) > A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) > M_0(a,b)$ for $2\alpha+\beta > 1$ follows from (2.11) and Lemma 2.1(1) together with (2.12), and $M_{2\alpha+\beta-1}(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < M_0(a,b)$ for $2\alpha+\beta < 1$ follows from (2.11) and Lemma 2.1(2) together with (2.12). □

Next, we prove that $M_0(a,b)$ and $M_{2\alpha+\beta-1}(a,b)$ are the best possible power mean bounds for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$.

Firstly, we prove that $M_{2\alpha+\beta-1}(a,b)$ is the best possible upper power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta > 1$.

For any $\epsilon \in (0, 2\alpha+\beta-1)$ and $x > 0$, one has

$$\begin{aligned} &\left[M_{2\alpha+\beta-1-\epsilon}(1,1+x)\right]^{2\alpha+\beta-1-\epsilon} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{2\alpha+\beta-1-\epsilon} \\ &= \frac{(1+x)^{2\alpha+\beta-1-\epsilon}+1}{2} - \left(1+\frac{x}{2}\right)^{(2\alpha+\beta-1)(2\alpha+\beta-1-\epsilon)}(1+x)^{(1-\alpha-(\beta/2))(2\alpha+\beta-1-\epsilon)}. \end{aligned} \tag{2.13}$$

Let $x \to 0$, then the Taylor expansion leads to

$$\begin{aligned} &\frac{(1+x)^{2\alpha+\beta-1-\epsilon}+1}{2} - \left(1+\frac{x}{2}\right)^{(2\alpha+\beta-1)(2\alpha+\beta-1-\epsilon)}(1+x)^{(1-\alpha-(\beta/2))(2\alpha+\beta-1-\epsilon)} \\ &\quad = -\frac{1}{8}\epsilon(2\alpha+\beta-1-\epsilon)x^2 + o\left(x^2\right). \end{aligned} \tag{2.14}$$

Equations (2.13) and (2.14) imply that if $2\alpha+\beta > 1$, then for any $\epsilon \in (0, 2\alpha+\beta-1)$ there exists $\delta_1 = \delta_1(\epsilon,\alpha,\beta) > 0$, such that $M_{2\alpha+\beta-1-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x \in (0,\delta_1)$.

Secondly, we prove that $M_0(a,b)$ is the best possible lower power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta>1$.

For any $\epsilon>0$ and $t>1$, one has

$$\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{\epsilon}(t,1)}=\frac{\left((1+t^{-1})/2\right)^{2\alpha+\beta-1}}{\left((1+t^{-\epsilon})/2\right)^{1/\epsilon}}t^{\alpha+(\beta/2)-1}. \tag{2.15}$$

From (2.15) and $\alpha+(\beta/2)<1$, we clearly see that

$$\lim_{t\to+\infty}\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{\epsilon}(t,1)}=0. \tag{2.16}$$

Equation (2.16) implies that if $2\alpha+\beta>1$, then for any $\epsilon\in(0,2\alpha+\beta-1)$ there exists $T_1=T_1(\epsilon,\alpha,\beta)>1$, such that $A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)<M_{\epsilon}(t,1)$ for $t\in(T_1,+\infty)$.

Thirdly, we prove that $M_{2\alpha+\beta-1}(a,b)$ is the best possible lower power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta<1$.

For any $\epsilon\in(0,1-2\alpha-\beta)$ and $x>0$, one has

$$\begin{aligned}&\left[M_{2\alpha+\beta-1+\epsilon}(1,1+x)\right]^{1-2\alpha-\beta-\epsilon}-\left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{1-2\alpha-\beta-\epsilon}\\&\quad=\frac{g(x)}{\left[1+(1+x)^{1-2\alpha-\beta-\epsilon}\right](1+(x/2))^{(1-2\alpha-\beta)(1-2\alpha-\beta-\epsilon)}},\end{aligned} \tag{2.17}$$

where $g(x)=2(1+x)^{1-2\alpha-\beta-\epsilon}(1+(x/2))^{(1-2\alpha-\beta)(1-2\alpha-\beta-\epsilon)}-(1+x)^{(1-\alpha-(\beta/2))(1-2\alpha-\beta-\epsilon)}[1+(1+x)^{1-2\alpha-\beta-\epsilon}]$.

Let $x\to 0$, then the Taylor expansion leads to

$$g(x)=\frac{1}{4}\epsilon(1-2\alpha-\beta-\epsilon)x^2+o\left(x^2\right). \tag{2.18}$$

Equations (2.17) and (2.18) imply that if $2\alpha+\beta<1$, then for any $\epsilon\in(0,1-2\alpha-\beta)$ there exists $0<\delta_2=\delta_2(\epsilon,\alpha,\beta)<1$, such that $M_{2\alpha+\beta-1+\epsilon}(1,1+x)>A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x\in(0,\delta_2)$.

Finally, we prove that $M_0(a,b)$ is the best possible upper power mean bound for the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ if $2\alpha+\beta<1$.

For any $\epsilon>0$ and $t>1$, one has

$$\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{-\epsilon}(t,1)}=\frac{\left((1+t^{-1})/2\right)^{2\alpha+\beta-1}}{\left((1+t^{-\epsilon})/2\right)^{-1/\epsilon}}t^{\alpha+(\beta/2)}. \tag{2.19}$$

From (2.19) and $\alpha+(\beta/2)>0$ we clearly see that

$$\lim_{t\to+\infty}\frac{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)}{M_{-\epsilon}(t,1)}=+\infty. \tag{2.20}$$

Equation (2.20) implies that if $2\alpha + \beta < 1$, then for any $\epsilon > 0$ there exists $T_2 = T_2(\epsilon, \alpha, \beta) > 1$, such that $A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1) > M_{-\epsilon}(t,1)$ for $t \in (T_2, +\infty)$.

Acknowledgments

This work is partly supported by the National Natural Science Foundation of China (Grant no. 60850005) and the Natural Science Foundation of Zhejiang Province (Grant no. D7080080, Y607128).

References

[1] S. Wu, "Generalization and sharpness of the power means inequality and their applications," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 2, pp. 637–652, 2005.

[2] K. C. Richards, "Sharp power mean bounds for the Gaussian hypergeometric function," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 1, pp. 303–313, 2005.

[3] W. L. Wang, J. J. Wen, and H. N. Shi, "Optimal inequalities involving power means," *Acta Mathematica Sinica*, vol. 47, no. 6, pp. 1053–1062, 2004 (Chinese).

[4] P. A. Hästö, "Optimal inequalities between Seiffert's mean and power means," *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.

[5] H. Alzer, "A power mean inequality for the gamma function," *Monatshefte für Mathematik*, vol. 131, no. 3, pp. 179–188, 2000.

[6] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.

[7] C. D. Tarnavas and D. D. Tarnavas, "An inequality for mixed power means," *Mathematical Inequalities & Applications*, vol. 2, no. 2, pp. 175–181, 1999.

[8] J. Bukor, J. Tóth, and L. Zsilinszky, "The logarithmic mean and the power mean of positive numbers," *Octogon Mathematical Magazine*, vol. 2, no. 1, pp. 19–24, 1994.

[9] J. E. Pečarić, "Generalization of the power means and their inequalities," *Journal of Mathematical Analysis and Applications*, vol. 161, no. 2, pp. 395–404, 1991.

[10] J. Chen and B. Hu, "The identric mean and the power mean inequalities of Ky Fan type," *Facta Universitatis. Series: Mathematics and Informatics*, no. 4, pp. 15–18, 1989.

[11] C. O. Imoru, "The power mean and the logarithmic mean," *International Journal of Mathematics and Mathematical Sciences*, vol. 5, no. 2, pp. 337–343, 1982.

[12] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.

[13] H. Alzer and W. Janous, "Solution of problem 8*," *Crux Mathematicorum*, vol. 13, pp. 173–178, 1987.

[14] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31 of *Mathematics and Its Applications*, D. Reidel, Dordrecht, The Netherlands, 1988.

[15] Q.-J. Mao, "Power mean, logarithmic mean and Heronian dual mean of two positive number," *Journal of Suzhou College of Education*, vol. 16, no. 1-2, pp. 82–85, 1999 (Chinese).

[16] Y.-M. Chu and W.-F. Xia, "Two sharp inequalities for power mean, geometric mean, and harmonic mean," *Journal of Inequalities and Applications*, vol. 2009, Article ID 741923, 6 pages, 2009.