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Research Article

On Ostrowski-Type Inequalities for Higher-Order Partial Derivatives

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We establish some new Ostrowski-type integral inequalities involving higher-order partial derivatives. As applications, we get some interrelated results. Our results provide new estimates on inequalities of this type.

1. Introduction

The following inequality is well known in the literature as Ostrowski's integral inequality (see [1, page 468]).

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b); that is, $\|f'\|_\infty = \sup_{t\in(a,b)}|f'(t)| < \infty$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x \in [a,b]$.

Many generalizations, extensions and variations of this inequality have appeared in the literature; see [1–10] and the references given therein. In particular, in 2009, Wang and Zhao [11] established a new Ostrowski-type inequality for higher-order derivatives as follows (see [11] for definitions and notations):

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{\|f^{(n)}\|_\infty}{n!}\left(\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^n+\frac{(b-a)^n}{(n+1)}\right).\tag{1.2}$$

The main purpose of the present paper is to establish the following Ostrowski-type inequality involving higher-order partial derivatives (see next section for definitions and notations):

$$\begin{aligned}&\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|\\&\quad\leq\frac{K}{n!}\left\{\sum_{m=0}^{n}C_n^m\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^m\left(\left|y-\frac{c+d}{2}\right|+\frac{d-c}{2}\right)^{n-m}\right.\\&\qquad\left.+\frac{1}{(b-a)(d-c)}\sum_{m=0}^{n}\frac{C_n^m(b-a)^{m+1}(d-c)^{n-m+1}}{(m+1)(n-m+1)}\right\},\end{aligned}\tag{1.3}$$

where $K>0$ is a constant, $C_n^m=n!/m!(n-m)!$, $m,n\in\mathbb{N}$, and $0\leq m\leq n$.

This is a generalization of inequality (1.2).

Moreover, as applications, we get some interrelated results. Our results provide new estimates on such type of inequalities.

2. Main Results

Theorem 2.1. *Suppose that*

(1) $f(x,y):[a,b]\times[c,d]\to\mathbb{R}$ *is continuous on* $[a,b]\times[c,d]$;

(2) $f(x,y):[a,b]\times[c,d]\to\mathbb{R}$ *is differentiable in* $(a,b)\times(c,d)$ *up to order* n, *with bounded* n*th-order mixed partial derivatives* $\partial^n f(x,y)/\partial x^m\partial y^{n-m}$ (m,n *are natural numbers, and* $0\leq m\leq n$), *that is,*

$$\sup_{\substack{(s,t)\in(a,b)\times(c,d)\\0\leq m\leq n}}\left|\frac{\partial^n}{\partial s^m\partial t^{n-m}}f(s,t)\right|=K<\infty;\tag{2.1}$$

(3) *there exists* $(x_0,y_0)\in(a,b)\times(c,d)$ *such that* $\partial^k f(s,t)/\partial s^m\partial t^{k-m}|_{(x_0,y_0)}=0$, $k=1,2,\ldots,n-1$; $m=0,1,\ldots,n$,

then for any $(x,y) \in [a,b] \times [c,d]$,

$$\left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\,ds\,dt \right| \le \frac{K}{n!} \left\{ \sum_{m=0}^{n} C_n^m \left(\left| x - \frac{a+b}{2} \right| + \frac{b-a}{2} \right)^m \left(\left| y - \frac{c+d}{2} \right| + \frac{d-c}{2} \right)^{n-m} + \frac{1}{(b-a)(d-c)} \sum_{m=0}^{n} \frac{C_n^m (b-a)^{m+1} (d-c)^{n-m+1}}{(m+1)(n-m+1)} \right\}, \tag{2.2}$$

where $C_n^m = n!/m!(n-m)!$.

Proof. From the hypotheses and using the n-dimensional Taylor expansion of $f(x,y)$ at (x_0, y_0), we have, for some $0 < \theta < 1$,

$$f(x,y) = f(x_0,y_0) + \frac{1}{n!} \sum_{m=0}^{n} C_n^m (x-x_0)^m (y-y_0)^{n-m} \times \left. \frac{\partial^n f}{\partial x^m \partial y^{n-m}} \right|_{(x_0+\theta(x-x_0),\, y_0+\theta(y-y_0))}. \tag{2.3}$$

Dividing both sides of (2.3) by $(b-a)(d-c)$, then integrating over y from c to d first, and then integrating the resulting inequality over x from a to b, we observe that

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\,ds\,dt = f(x_0,y_0) + \frac{1}{n!(b-a)(d-c)} \times \int_a^b \int_c^d \sum_{m=0}^{n} C_n^m (s-x_0)^m (t-y_0)^{n-m} \cdot \left. \frac{\partial^n f}{\partial x^m \partial y^{n-m}} \right|_{(x_0+\theta(s-x_0),\, y_0+\theta(t-y_0))} ds\,dt. \tag{2.4}$$

Note that we have replaced the dummy variables x, y by s, t, respectively. From (2.3) and (2.4), we have

$$f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)\,ds\,dt = \frac{1}{n!} \sum_{m=0}^{n} C_n^m (x-x_0)^m (y-y_0)^{n-m} \cdot \left. \frac{\partial^n f}{\partial x^m \partial y^{n-m}} \right|_{(x_0+\theta(x-x_0),\, y_0+\theta(y-y_0))} - \frac{1}{n!(b-a)(d-c)} \times \int_a^b \int_c^d \sum_{m=0}^{n} C_n^m (s-x_0)^m (t-y_0)^{n-m} \cdot \left. \frac{\partial^n f}{\partial x^m \partial y^{n-m}} \right|_{(x_0+\theta(s-x_0),\, y_0+\theta(t-y_0))} ds\,dt. \tag{2.5}$$

Hence

$$
\begin{aligned}
&\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)\,ds\,dt\right| \\
&\quad\le \left|\frac{1}{n!}\sum_{m=0}^{n}C_n^m(x-x_0)^m(y-y_0)^{n-m}\cdot\left.\frac{\partial^n f}{\partial x^m\partial y^{n-m}}\right|_{(x_0+\theta(x-x_0),y_0+\theta(y-y_0))}\right| + \frac{1}{n!(b-a)(d-c)} \\
&\qquad\times\left|\int_a^b\int_c^d\sum_{m=0}^{n}C_n^m(s-x_0)^m(t-y_0)^{n-m}\cdot\left.\frac{\partial^n f}{\partial x^m\partial y^{n-m}}\right|_{(x_0+\theta(s-x_0),y_0+\theta(t-y_0))}ds\,dt\right| \\
&\quad\le \frac{K}{n!}\left\{\sum_{m=0}^{n}\left|C_n^m(x-x_0)^m(y-y_0)^{n-m}\right|\right. \\
&\qquad\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\sum_{m=0}^{n}\left|C_n^m(s-x_0)^m(t-y_0)^{n-m}\right|ds\,dt\right\} \\
&\quad= \frac{K}{n!}\left\{\sum_{m=0}^{n}C_n^m|x-x_0|^m|y-y_0|^{n-m}\right. \\
&\qquad\left.+\frac{1}{(b-a)(d-c)}\sum_{m=0}^{n}C_n^m\int_a^b\int_c^d|s-x_0|^m|t-y_0|^{n-m}ds\,dt\right\} \\
&\quad= \frac{K}{n!}\left\{\sum_{m=0}^{n}C_n^m|x-x_0|^m|y-y_0|^{n-m}+\frac{1}{(b-a)(d-c)}\sum_{m=0}^{n}\frac{C_n^m}{(m+1)(n-m+1)}\right. \\
&\qquad\left.\times\left[(b-x_0)^{m+1}+(x_0-a)^{m+1}\right]\left[(d-y_0)^{n-m+1}+(y_0-c)^{n-m+1}\right]\right\}.
\end{aligned}
\tag{2.6}
$$

On the other hand, by applying the following two elementary inequalities [11]:

$$
\begin{gathered}
(x-t)^2\le\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2,\quad a\le x\le b,\ a\le t\le b, \\
(b-t)^n+(t-a)^n\le(b-a)^n,\quad a\le t\le b,\ n\ge 1
\end{gathered}
\tag{2.7}
$$

to the right-hand side of (2.6), we obtain

$$\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt\right|$$
$$\leq\frac{K}{n!}\left\{\sum_{m=0}^{n}C_n^m\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^m\left(\left|y-\frac{c+d}{2}\right|+\frac{d-c}{2}\right)^{n-m}\right. \tag{2.8}$$
$$\left.+\frac{1}{(b-a)(d-c)}\sum_{m=0}^{n}\frac{C_n^m(b-a)^{m+1}(d-c)^{n-m+1}}{(m+1)(n-m+1)}\right\}.$$

This completes the proof. □

Remark 2.2. With suitable modifications, it is easy to see that (2.2) reduces to the following inequality in the 1-dimensional situation:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{\|f^{(n)}\|_\infty}{n!}\left(\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^n+\frac{(b-a)^n}{n+1}\right), \tag{2.9}$$

where $f(x):[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, with nth-order derivative $f^{(n)}:(a,b)\to\mathbb{R}$ bounded on (a,b), that is, $\|f^{(n)}\|_\infty=\sup_{t\in(a,b)}|f^{(n)}(t)|<\infty$, and $f^{(k)}(x_0)=0, k=1,2,\ldots,n-1$.

Observe that this is a recent result of Wang and Zhao [11].

Theorem 2.3. *Suppose that*

(1) $f(x,y):[a,b]\times[c,d]\to\mathbb{R}$ *is continuous on* $[a,b]\times[c,d]$;

(2) $f(x,y):[a,b]\times[c,d]\to\mathbb{R}$ *is twice differentiable in* $(a,b)\times(c,d)$ *with bounded second-order partial derivatives* $\partial^2 f(x,y)/\partial x^m\partial y^{2-m}(m=0,1,2)$, *that is,*

$$\sup_{\substack{(s,t)\in(a,b)\times(c,d)\\ m=0,1,2}}\left|\frac{\partial^2}{\partial s^m\partial t^{2-m}}f(s,t)\right|=L<\infty; \tag{2.10}$$

(3) *there exists* $(x_0,y_0)\in(a,b)\times(c,d)$ *such that* $\partial f(s,t)/\partial s|_{(x_0,y_0)}=\partial f(s,t)/\partial t|_{(x_0,y_0)}=0$.

Then, one has

$$\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dsdt\right|$$
$$\leq\frac{L}{2}\left\{\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}+\left|y-\frac{c+d}{2}\right|+\frac{d-c}{2}\right)^2\right. \tag{2.11}$$
$$\left.+\frac{(b-a)^2}{3}+\frac{(d-c)^2}{3}+\frac{(b-a)(d-c)}{2}\right\}.$$

Proof. From the hypotheses and in view of the 2-dimensional Taylor expansion, it easily follows that for some $0 < \theta < 1$,

$$\begin{aligned}
&f(x,y) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)\,ds\,dt \\
&= \frac{1}{2}(x-x_0)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0+\theta(x-x_0),y_0+\theta(y-y_0))} + (x-x_0)(y-y_0)\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(x_0+\theta(x-x_0),y_0+\theta(y-y_0))} \\
&\quad + \frac{1}{2}(y-y_0)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0+\theta(x-x_0),y_0+\theta(y-y_0))} \\
&\quad - \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d \Bigg((s-x_0)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0+\theta(s-x_0),y_0+\theta(t-y_0))} \\
&\qquad + 2(s-x_0)(t-y_0)\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(x_0+\theta(s-x_0),y_0+\theta(t-y_0))} \\
&\qquad + (t-y_0)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0+\theta(s-x_0),y_0+\theta(t-y_0))} \Bigg) ds\,dt.
\end{aligned} \tag{2.12}$$

Hence,

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)\,ds\,dt \right| \\
&\leq L\Bigg\{ \frac{1}{2}(x-x_0)^2 + |(x-x_0)(y-y_0)| + \frac{1}{2}(y-y_0)^2 \\
&\quad + \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d \left[(s-x_0)^2 + 2|(s-x_0)(t-y_0)| + (t-y_0)^2\right] ds\,dt \Bigg\} \\
&= L\Bigg\{ \frac{1}{2}(x-x_0)^2 + |(x-x_0)(y-y_0)| + \frac{1}{2}(y-y_0)^2 \\
&\quad + \frac{1}{2(b-a)(d-c)}\Bigg\{ \frac{1}{3}\left[(b-x_0)^3 + (x_0-a)^3\right](d-c) + \frac{1}{3}\left[(d-y_0)^3 + (y_0-c)^3\right](b-a) \\
&\qquad + \frac{1}{2}\left[\left[(b-x_0)^2 + (x_0-a)^2\right]\left[(d-y_0)^2 + (y_0-c)^2\right]\right]\Bigg\}\Bigg\}
\end{aligned}$$

$$\leq \frac{L}{2}\left\{\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}+\left|y-\frac{c+d}{2}\right|+\frac{d-c}{2}\right)^2 + \frac{(b-a)^2}{3}+\frac{(d-c)^2}{3}+\frac{(b-a)(d-c)}{2}\right\}. \tag{2.13}$$

This proves Theorem 2.3. □

Let $f(x,y)$ and $\partial^2 f(x,y)/\partial x^m \partial y^{2-m}(m = 0,1,2)$ change to $f(x)$ and $f''(x)$, respectively, and with suitable modifications, Theorem 2.3 reduces to the following.

Theorem 2.4. *Suppose that*

(1) $f(x) : [a,b] \to \mathbb{R}$ *is continuous on* $[a,b]$;

(2) $f(x) : [a,b] \to \mathbb{R}$ *is twice differentiable in* (a,b) *with bounded second -order derivative, that is,* $\|f''\|_\infty = \sup_{t\in(a,b)}|f(t)| < \infty$;

(3) *there exists* $x_0 \in (a,b)$ *such that* $f'(x_0) = 0$ *(or* $f(a) = f(b)$*),*

then, one has

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{2}\|f''\|_\infty (b-a)^2\left\{\frac{(x-(a+b)/2)^2}{(b-a)^2}+\frac{|x-(a+b)/2|}{b-a}+\frac{7}{12}\right\}. \tag{2.14}$$

This is another recent result of Wang and Zhao in [11].

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References

[1] D. S. Mtrinović, J. E. Pečarić, and A. M. Fink, *Inequalities for Functions and Their Integrals and Dervatives*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.

[2] B. G. Pachpatte, "On an inequality of Ostrowski type in three independent variables," *Journal of Mathematical Analysis and Applications*, vol. 249, no. 2, pp. 583–591, 2000.

[3] B. G. Pachpatte, "On a new Ostrowski type inequality in two independent variables," *Tamkang Journal of Mathematics*, vol. 32, no. 1, pp. 45–49, 2001.

[4] G. A. Anastassiou, "Multivariate Ostrowski type inequalities," *Acta Mathematica Hungarica*, vol. 76, no. 4, pp. 267–278, 1997.

[5] S. S. Dragomir and S. Wang, "A new inequality of Ostrowski's type in L_1 norm and applications to some special means and to some numerical quadrature rules," *Tamkang Journal of Mathematics*, vol. 28, no. 3, pp. 239–244, 1997.

[6] S. S. Dragomir and S. Wang, "An inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules," *Computers & Mathematics with Applications*, vol. 33, no. 11, pp. 15–20, 1997.

[7] S. S. Dragomir and S. Wang, "Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules," *Applied Mathematics Letters*, vol. 11, no. 1, pp. 105–109, 1998.

[8] N. S. Barnett and S. S. Dragomir, "An Ostrowski type inequality for double integrals and applications for cubature formulae," *RGMIA Research Report Collection*, vol. 1, pp. 13–23, 1998.

[9] D. Baĭnov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[10] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, vol. 53 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[11] M. Wang and X. Zhao, "Ostrowski type inequalities for higher-order derivatives," *Journal of Inequalities and Applications*, vol. 2009, Article ID 162689, 8 pages, 2009.